Lecture 13

# The Stokes operator

The Stokes operator arises in the context of the (non-linear!) Navier-Stokes equation and acts in a subspace of a $\mathbb{K}^n$-valued $L_2$-space. In our context we define it using a variant of the classical Dirichlet form. One of the features appearing in the description of the Stokes operator is the use of a Sobolev space of negative order, which is introduced at the beginning. Another important feature is the appearance of divergence free vector fields. This extra condition of vanishing divergence has interesting implications for the theory of the related Sobolev spaces, and a large part of the lecture is devoted to the investigation of these properties.

## 13.1 Interlude: the Sobolev space $H^{-1}(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be open. The space $H_0^1(\Omega)$ is a Hilbert space, and by the Riesz-Fréchet theorem, each continuous antilinear functional on $H_0^1(\Omega)$ is represented by an element of $H_0^1(\Omega)$. For some purposes, however, it is more convenient to work with antilinear functionals directly, i.e., to consider the antidual $H_0^1(\Omega)^*$ without this identification. The basic idea is to use that an element $f \in L_2(\Omega)$ acts on $H_0^1(\Omega)$ in a natural way as a continuous antilinear functional by

$$H_0^1(\Omega) \ni u \mapsto (f \,|\, u)_{L_2(\Omega)} =: \langle f, u \rangle_{H^{-1},H_0^1} .$$

The operator $L_2(\Omega) \ni f \mapsto \langle f, \cdot \rangle_{H^{-1},H_0^1} \in H_0^1(\Omega)^*$ is injective, and because the subspace $\big\{ \langle f, \cdot \rangle_{H^{-1},H_0^1} \,;\, f \in L_2(\Omega) \big\}$ of $H_0^1(\Omega)^*$ separates the points of $H_0^1(\Omega)$, it is dense in $H_0^1(\Omega)^*$. In this context the antidual $H_0^1(\Omega)^*$ is denoted by $H^{-1}(\Omega)$.

**13.1 Remark.** We note that in the situation described above one has

$$H_0^1(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow H^{-1}(\Omega) = H_0^1(\Omega)^*,$$

with dense embeddings, and the embeddings are dual to each other. In such a situation one calls $\big(H_0^1(\Omega), L_2(\Omega), H^{-1}(\Omega)\big)$ a **Gelfand triple**; this will be treated in more generality in Section 14.1.

If $f \in L_2(\Omega)$ and $j \in \{1, \dots, n\}$, then the mapping

$$H_0^1(\Omega) \ni u \mapsto \langle \partial_j f, u \rangle_{H^{-1},H_0^1} := -\, (f \,|\, \partial_j u)_{L_2(\Omega)}$$

belongs to $H_0^1(\Omega)^*$. This definition of $\partial_j f$ is consistent with the definition of the distributional derivative in Section 4.1 because $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, and the mapping $\partial_j\colon L_2(\Omega) \to H^{-1}(\Omega)$ is linear and continuous. With these definitions the differential operator $\Delta = \sum_{j=1}^n \partial_j\partial_j$ acts as a continuous operator $\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$. This interpretation of the Laplace operator implies that

$$\langle \Delta u, \varphi\rangle_{H^{-1},H_0^1} = (u \mid \Delta\varphi)_{L_2(\Omega)}$$

for all $u \in H_0^1(\Omega)$, $\varphi \in C_c^\infty(\Omega)$.

It turns out that the mapping $I - \Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism. If $\Omega$ is bounded and one provides $H_0^1(\Omega)$ with the scalar product $(u,v) \mapsto \int \nabla u \cdot \overline{\nabla v}$, then $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism. See Exercise 13.1 for these properties. (Strictly speaking, the notation '$I - \Delta$' is not correct: the identy is meant to be the embedding $H_0^1(\Omega) \hookrightarrow H^{-1}(\Omega)$.)

## 13.2 The Stokes operator

Let $\Omega \subseteq \mathbb{R}^n$ be open. The Stokes operator is an operator in a subspace $H$ of $L_2(\Omega;\mathbb{K}^n)$. It is defined as the operator associated with the classical Dirichlet form on

$$V := H_{0,\sigma}^1(\Omega;\mathbb{K}^n) := \bigl\{u \in H_0^1(\Omega;\mathbb{K}^n) = H_0^1(\Omega)^n;\ \operatorname{div} u = 0\bigr\}.$$

Note that $V$ is a closed subspace of $H_0^1(\Omega;\mathbb{K}^n)$. The Hilbert space $H$ is defined as the closure of $V$ in $L_2(\Omega;\mathbb{K}^n)$,

$$H := L_{2,\sigma,0}(\Omega;\mathbb{K}^n) := \overline{V}^{L_2(\Omega;\mathbb{K}^n)}.$$

**13.2 Remarks.** (a) The space $L_{2,\sigma}(\Omega;\mathbb{K}^n) := \{f \in L_2(\Omega;\mathbb{K}^n);\ \operatorname{div} f = 0\}$ is a closed subspace of $L_2(\Omega;\mathbb{K}^n)$; this is because $\operatorname{div}\colon L_2(\Omega;\mathbb{K}^n) \to H^{-1}(\Omega)$ is continuous. Therefore $H$ is a (closed) subspace of $L_{2,\sigma}(\Omega;\mathbb{K}^n)$.

(b) In previous instances when working with forms, we always had the Hilbert space $H$ given beforehand, and then had to make sure that the domain of the form is dense (or that $j\colon V \to H$ has dense range). In contrast to this procedure, here we have a situation where $V$ is given and $H$ is adapted to $V$. We point out that $C_c^\infty(\Omega;\mathbb{K}^n) \cap H$ need not be dense in $H$, so that the notation $L_{2,\sigma,0}$ is not entirely consistent with previous notation. (There is no such reservation for sets $\Omega$ that are bounded and have Lipschitz boundary; cf. Theorem 13.13.)

(c) A comment on the notation: the index '$\sigma$' should be remindful of 'solenoidal', which is the classical term for divergence free vectors fields.

We define the form $a\colon V \times V \to \mathbb{R}$ by

$$a(u,v) := \sum_{j=1}^n \int_\Omega \nabla u_j \cdot \overline{\nabla v_j}.$$

In each component of $H_0^1(\Omega)^n$, the form $a$ is the classical Dirichlet form. Therefore we conclude from Section 5.4 that $a$ is symmetric, accretive and $H$-elliptic; for bounded $\Omega$ it is coercive. The (accretive self-adjoint) operator associated with $a$ is therefore given by

$$A = \Big\{(u, f) \in V \times H\,;\ \sum_{j=1}^{n} \int \nabla u_j \cdot \overline{\nabla v_j} = (f \,|\, v)_H \ \ (v \in V)\Big\}. \tag{13.1}$$

In view of Section 13.1, the equality appearing in the description of $A$ can be rewritten as

$$0 = -\sum_{j=1}^{n} \int \nabla u_j \cdot \overline{\nabla v_j} + (f \,|\, v)_H = \sum_{j=1}^{n} \langle \Delta u_j + f_j, v_j \rangle_{H^{-1},H_0^1} \qquad (v \in V),$$

where $f_j$ is considered as an element of $H^{-1}(\Omega)$ via the injection $L_2(\Omega) \hookrightarrow H^{-1}(\Omega)$, and $\Delta$ is the operator $\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ from Section 13.1.

Note that the antidual of $H_0^1(\Omega)^n$ is given by $H^{-1}(\Omega)^n$, with the dual pairing

$$\langle \eta, u \rangle_{H^{-1},H_0^1} := \sum_{j=1}^{n} \langle \eta_j, u_j \rangle_{H^{-1},H_0^1} \qquad \big(\eta \in H^{-1}(\Omega)^n,\ u \in H_0^1(\Omega)^n\big).$$

It will be convenient to have a notation for the subset of $H^{-1}(\Omega)^n$ "orthogonal" to $V$. In order to avoid confusion with the orthogonal complement in Hilbert spaces we will use the notation as the polar,

$$V^{\circ} := \big\{\eta \in H^{-1}(\Omega)^n;\ \langle \eta, v \rangle_{H^{-1},H_0^1} = 0 \ (v \in V)\big\}.$$

It is immediate from the definition of the derivative that for all $f \in L_2(\Omega)$ the gradient $\nabla f \in H^{-1}(\Omega)^n$ belongs to $V^{\circ}$.

After these preliminaries we get a description of the Stokes operator. Abbreviating, we will use the notation $\Delta u = (\Delta u_1, \ldots, \Delta u_n)$ for $u \in H_0^1(\Omega)^n$.

**13.3 Theorem.** *The operator $A$ in $H$ associated with the form $a$ is given by*

$$A = \big\{(u, f) \in V \times H\,;\, \exists \eta \in V^{\circ}\colon -\Delta u + \eta = f\big\}$$

*(where the equality '$-\Delta u + \eta = f$' is an equality in $H^{-1}(\Omega)^n$, with $f \in H \hookrightarrow H^{-1}(\Omega)^n$). Written differently,*

$$\operatorname{dom}(A) = \big\{u \in V\,;\, \exists \eta \in V^{\circ}\colon -\Delta u + \eta \in H\big\},$$
$$Au = -\Delta u + \eta \quad \textit{(with } \eta \textit{ as in } \operatorname{dom}(A)\textit{)}.$$

*Proof.* According the previous discussion, the condition appearing in (13.1) can be rephrased as

$$\langle \Delta u + f, v \rangle_{H^{-1},H_0^1} = 0 \qquad (v \in V),$$

which means that $\eta = \Delta u + f \in V^{\circ}$. □

As mentioned above, the **Stokes operator** $A$ in Theorem 13.3 is self-adjoint and accretive.

For the physical interpretation one wants to express the element $\eta \in V^{\circ}$ appearing in the description of $A$ in a more explicit way. In order to do this we need an additional property:

$$\text{for all } \eta \in V^{\circ} \text{ there exists } p \in L_2(\Omega) \text{ with } \eta = \nabla p. \tag{H}$$

We will comment on this hypothesis in Remarks 13.5. In Section 13.4 it will be shown that (H) holds if $\Omega$ is bounded with Lipschitz boundary. With this hypothesis we get another description of the Stokes operator $A$.

**13.4 Theorem.** *Assume that $\Omega$ satisfies hypothesis (H). Then*

$$A = \big\{(u, f) \in V \times H\,;\, \exists p \in L_2(\Omega)\colon -\Delta u + \nabla p = f\big\}$$

*is the operator associated with the form $a$. Expressed differently,*

$$\operatorname{dom}(A) = \big\{u \in V\,;\, \exists p \in L_2(\Omega)\colon -\Delta u + \nabla p \in H\big\},$$
$$Au = -\Delta u + \nabla p \quad (\textit{with } p \textit{ as in } \operatorname{dom}(A)).$$

In the application in fluid dynamics one would like to interpret $p$ in the statement of Theorem 13.4 as a pressure – the reason for the notation. However, we will not endeavour to enter the physical interpretation of the Stokes operator.

**13.5 Remarks.** Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $u \in L_2(\Omega; \mathbb{K}^n)$ satisfy

$$\int u \cdot \varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C_{\mathrm{c}}^1(\Omega; \mathbb{K}^n) \text{ with } \operatorname{div} \varphi = 0. \tag{13.2}$$

(a) Then $\partial_j u_k - \partial_k u_j = 0$ in $H^{-1}(\Omega)$, for all $j, k \in \{1, \dots, n\}$.

Indeed, let $\psi \in C_{\mathrm{c}}^\infty(\Omega)$, and define $\varphi_j := \partial_k \psi$, $\varphi_k := -\partial_j \psi$, $\varphi_\ell := 0$ for all other components. Then $\operatorname{div} \varphi = 0$, and therefore

$$\int (\partial_j u_k - \partial_k u_j)\psi = \int (u_j \partial_k \psi - u_k \partial_j \psi) = \int u \cdot \varphi = 0.$$

(b) Assume additionally that $u$ is a continuous vector field. In view of (a), the condition (13.2) implies that $u$ satisfies the 'compatibility conditions' for a vector field to be locally the gradient of a potential.

However, the condition (13.2) is not only local, and in fact one can show that it implies that the potential exists also globally. For the idea of the proof we mention that it is sufficient to treat the case that $\Omega$ is connected. Then, fixing an "initial point" $x^0 \in \Omega$, one defines $p(x) := \int_0^1 u(\gamma(t)) \cdot \gamma'(t) \,\mathrm{d}t$, where $\gamma\colon [0,1] \to \Omega$ is a continuously differentiable path connecting $x^0$ with $x$. Using (13.2) it can be shown that this is well-defined – this is the main issue – and that then $u = \nabla p$.

We will not carry out this proof but refer to the proof of Theorem 13.13, where an analogous problem is treated.

## 13.3 Interlude: the Bogovskiĭ formula

This section could also run under the heading "some functions are divergences". For $u \in C_c^\infty(\mathbb{R}^n;\mathbb{K}^n)$ one has $\int \operatorname{div} u \,\mathrm{d}x = 0$, and it is well-known that the converse holds as well: for each $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\int \varphi \,\mathrm{d}x = 0$ there exists $u \in C_c^\infty(\mathbb{R}^n;\mathbb{K}^n)$ with $\operatorname{div} u = \varphi$. In this section we will show that this holds in a more general and more precise version: for suitable bounded $\Omega$ one has that for any $f \in L_2(\Omega)$ with $\int f \,\mathrm{d}x = 0$ there exists $u \in H_0^1(\Omega)^n$ such that $\operatorname{div} u = f$; see Theorem 13.9.

Let $v \in C_c^\infty(\mathbb{R}^n;\mathbb{K}^n)$ be a vector field satisfying the compatibility conditions $\partial_j v_k = \partial_k v_j$ for all $j,k = 1,\dots,n$. Then it is well-known and easy to show that, for any $y \in \mathbb{R}^n$, a potential for $v$ is given by

$$p(x) := \int_0^1 v(y + t(x-y)) \cdot (x-y) \,\mathrm{d}t,$$

i.e., $v = \nabla p$. We smooth this formula out with the help of a function $\rho \in C_c^\infty(\mathbb{R}^n)_+$ satisfying $\int \rho \,\mathrm{d}x = 1$: we define

$$Av(x) := \int \rho(y) \int_0^1 v(ty + (1-t)x) \cdot (x-y) \,\mathrm{d}t \,\mathrm{d}y \tag{13.3}$$

and obtain $Av \in C^\infty(\mathbb{R}^n)$, $\nabla(Av) = v$. In order to write $A$ as an integral operator, we substitute $z = ty + (1-t)x$ and $r = \frac{1}{t}$ to obtain

$$\begin{aligned} Av(x) &= \int_0^1 \int \rho\big(\tfrac{1}{t}(z - (1-t)x)\big) v(z) \cdot \frac{x-z}{t}\, t^{-n} \,\mathrm{d}z \,\mathrm{d}t \\ &= \int_1^\infty \int \rho(x + r(z-x)) v(z) \cdot (x-z) \,\mathrm{d}z \, r^{n-1} \,\mathrm{d}r. \end{aligned}$$

This means that one can write $Av(x) = \int k(x,y) \cdot v(y) \,\mathrm{d}y$, with

$$k(x,y) = \int_1^\infty \rho(x + r(y-x)) r^{n-1} \,\mathrm{d}r \,(x-y).$$

Let $\ell$ be the negative transposed kernel of $k$, i.e.,

$$\ell(x,y) := -k(y,x) = (x-y) \int_1^\infty \rho(y + r(x-y)) r^{n-1} \,\mathrm{d}r \qquad (x,y \in \mathbb{R}^n).$$

It will be shown in the following theorem that then the definition

$$Bf(x) := \int \ell(x,y) f(y) \,\mathrm{d}y \tag{13.4}$$

$$= \int f(y)(x-y) \int_1^\infty \rho(y + r(x-y)) r^{n-1} \,\mathrm{d}r \,\mathrm{d}y, \tag{13.5}$$

for $x \in \mathbb{R}^n$, $f \in C_c^\infty(\mathbb{R}^n)$, yields a mapping $B\colon C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n;\mathbb{K}^n)$. This definition is such that for all $v \in C_c^\infty(\mathbb{R}^n;\mathbb{K}^n)$, $f \in C_c^\infty(\mathbb{R}^n)$ one has $\int (Av) f \,\mathrm{d}x = -\int v \cdot Bf \,\mathrm{d}x$.

**13.6 Theorem.** *For all* $f \in C_c^\infty(\mathbb{R}^n)$ *one has* $Bf \in C_c^\infty(\mathbb{R}^n)$,

$$\operatorname{spt}(Bf) \subseteq \bigl\{\lambda z_1 + (1-\lambda)z_2\,;\ z_1 \in \operatorname{spt} f,\ z_2 \in \operatorname{spt}\rho,\ 0 \leqslant \lambda \leqslant 1\bigr\} =: E. \tag{13.6}$$

*If* $\int f\,\mathrm{d}x = 0$, *then* $\operatorname{div} Bf = f$.

*Proof.* Let $f \in C_c^\infty(\mathbb{R}^n)$.

(i) First we show that $Bf = 0$ on $\mathbb{R}^n \setminus E$. Note that $E$ is a compact set (in general a proper subset of the convex hull of $\operatorname{spt} f \cup \operatorname{spt}\rho$). Let $x \in \mathbb{R}^n \setminus E$. If $y \in \operatorname{spt} f$ and $r \geqslant 1$ then $y + r(x-y) \notin \operatorname{spt}\rho$ (because $z = y + r(x-y) \in \operatorname{spt}\rho$ would lead to $x = \frac{1}{r}z + (1-\frac{1}{r})y \in E$ – a contradiction), and therefore $\rho(y + r(x-y)) = 0$. Hence (13.5) implies $Bf(x) = 0$.

(ii) By the variable transformation $z = x - y$ and then $r = 1 + \frac{s}{|z|}$ in the inner integral we obtain $Bf$ in the form

$$Bf(x) = \int f(x-z)\frac{z}{|z|^n}\int_0^\infty \rho\Bigl(x + s\frac{z}{|z|}\Bigr)(s+|z|)^{n-1}\,\mathrm{d}s\,\mathrm{d}z.$$

In this form one can differentiate under the integral to obtain $Bf \in C^\infty(\mathbb{R}^n)$. (If $R > 0$ is such that $\rho$, $f$ have their supports in $B(0,R)$, then we know from (i) that $Bf = 0$ on $\mathbb{R}^n \setminus B(0,R)$, and for $x \in B(0,R)$ we can use $g(s,z) := \mathbf{1}_{(0,2R)}(s)\mathbf{1}_{B(0,2R)}(z)|z|^{1-n}$ as a dominating function.)

(iii) Let $f, \varphi \in C_c^\infty(\mathbb{R}^n)$, $\int f\,\mathrm{d}x = 0$. Then

$$\int (\operatorname{div} Bf)\varphi\,\mathrm{d}x = -\int Bf\cdot\nabla\varphi\,\mathrm{d}x = \int fA(\nabla\varphi)\,\mathrm{d}x,$$

and this implies

$$0 = \int(\operatorname{div} Bf)\varphi\,\mathrm{d}x - \int fA(\nabla\varphi)\,\mathrm{d}x = \int(\operatorname{div} Bf - f)\varphi\,\mathrm{d}x + \int f(\varphi - A(\nabla\varphi))\,\mathrm{d}x. \tag{13.7}$$

Note that in the above computation $\nabla\varphi$ satisfies the compatibility conditions, therefore $\nabla A(\nabla\varphi) = \nabla\varphi$, and as a consequence $\nabla(\varphi - A(\nabla\varphi)) = 0$, i.e., $\varphi - A(\nabla\varphi)$ is constant. From $\int f\,\mathrm{d}x = 0$ we therefore conclude that the last term in (13.7) vanishes, which implies that $\int(\operatorname{div} Bf - f)\varphi\,\mathrm{d}x = 0$. As this holds for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ we obtain $\operatorname{div} Bf = f$. □

**13.7 Remarks.** (a) The formula (13.5) is the **Bogovskiĭ formula**.

(b) If $n \geqslant 2$, then there exist vector fields $0 \neq v \in C_c^\infty(\mathbb{R}^n;\mathbb{K}^n)$ satisfying $\operatorname{div} v = 0$. Therefore the vector field obtained by the Bogovskiĭ formula is not the unique solution of $\operatorname{div} v = f$.

Next we give reasons why, for suitable bounded $\Omega$, the Bogovskiĭ formula provides a continuous linear operator $B\colon L_2^0(\Omega) \to H_0^1(\Omega)$, where $L_2^0(\Omega) := \bigl\{f \in L_2(\Omega);\ \int f\,\mathrm{d}x = 0\bigr\}$. Let us note immediately that $L_2^0(\Omega)$ is a closed subspace of $L_2(\Omega)$ and that $C_c^\infty \cap L_2^0(\Omega)$ is dense in $L_2^0(\Omega)$; see Exercise 13.2(a).

In the following considerations all functions on $\Omega$ should be considered as extended by zero to $\mathbb{R}^n$.

**13.8 Theorem.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set containing a ball $B(x_0, r)$ such that $\Omega$ is star-shaped with respect to every point of $B(x_0, r)$. Let $\rho \in C_c^\infty(\mathbb{R}^n)_+$ with* $\operatorname{spt}\rho \subseteq B(x_0, r)$, $\int \rho \, \mathrm{d}x = 1$.

*Then $B\colon C_c^\infty \cap L_2^0(\Omega) \to C_c^\infty(\Omega; \mathbb{K}^n)$, defined by* (13.5)*, has a continuous (linear) extension $B\colon L_2^0(\Omega) \to H_0^1(\Omega)$. (The formula* (13.6) *implies that indeed $Bf \in C_c^\infty(\Omega; \mathbb{K}^n)$ for all $f \in C_c^\infty \cap L_2^0(\Omega)$.) For all $f \in L_2^0(\Omega)$ one has* $\operatorname{div} Bf = f$.

Unfortunately we cannot give the proof of this theorem because it relies on a piece of Analysis and Operator Theory that is beyond the scope of the Internet Seminar. In fact, the problem consists in two parts: one needs the Calderón-Zygmund theory of singular integral operators, and then one has to check the applicability to the operator at hand.

Showing that there exists $c > 0$ such that $\|Bf\|_2 \leqslant \|f\|_2$ is not the problem; see Exercise 13.5. The problem is to find the corresponding estimate for the derivatives of $Bf$. For the Calderón-Zygmund theorem needed for this purpose we refer to the original paper [CZ56; Theorem 2] and to [Gal11; Theorem II.11.4]. The application to the Bogovskiĭ operator is treated in [Gal11; proof of Lemma III.3.1, pp. 164, 165].

We emphasise that the main interest in the mapping $B$ in Theorem 13.8 consists in the circumstance that it implies the surjectivity of the map $\operatorname{div}\colon H_0^1(\Omega) \to L_2^0(\Omega)$. (Note that $\operatorname{div}(C_c^\infty(\Omega; \mathbb{K}^n)) \subseteq L_2^0(\Omega)$, hence $\operatorname{div}(H_0^1(\Omega)) \subseteq L_2^0(\Omega)$, for *any* bounded open $\Omega \subseteq \mathbb{R}^n$.)

We show that Theorem 13.8 implies the surjectivity of $\operatorname{div}\colon H_0^1(\Omega) \to L_2^0(\Omega)$ for more general $\Omega$.

**13.9 Theorem.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, connected and with Lipschitz boundary. Let $f \in L_2^0(\Omega)$. Then there exists $v \in H_0^1(\Omega; \mathbb{K}^n)$ such that* $\operatorname{div} v = f$.

*Proof.* (i) It is not difficult to see that for all $x \in \overline{\Omega}$ there exists an open neighbourhood $U_x$ such that $U_x \cap \Omega$ is star-shaped with respect to the points of a ball in $U_x \cap \Omega$. This is obvious for $x \in \Omega$, and for $x \in \partial\Omega$ it results from the Lipschitz property of $\partial\Omega$. Compactness of $\overline{\Omega}$ implies that there exists a finite open covering $(\Omega_j)_{j=1,\dots,m}$ of $\Omega$ by sets to which Thereom 13.8 can be applied.

(ii) It is not too difficult to show that there exist functions $f_1, \dots, f_m \in L_2^0(\Omega)$ such that $[f_j \neq 0] \subseteq \Omega_j$ for all $j \in \{1, \dots, m\}$ and $f = \sum_{j=1}^m f_j$. (Let us illustrate this for $m = 2$. In this case choose $g \in L_2(\Omega)$ with $[g \neq 0] \subseteq \Omega_1 \cap \Omega_2$, $\int g \, \mathrm{d}x = \int \mathbf{1}_{\Omega_1} f \, \mathrm{d}x$, and define $f_1 := \mathbf{1}_{\Omega_1} f - g$, $f_2 := \mathbf{1}_{\Omega_2 \setminus \Omega_1} f + g$. See Exercise 13.2(b) for the general case.) Then for all $j \in \{1, \dots, m\}$ there exists $v^j \in H_0^1(\Omega_j; \mathbb{K}^n)$ with $\operatorname{div} v^j = f_j$, and $v := \sum_{j=1}^m v^j$ has the required properties. □

**13.10 Remark.** We mention that in Theorem 13.9 one can also construct a continuous linear operator $B\colon L_2^0(\Omega) \to H_0^1(\Omega; \mathbb{K}^n)$ such that $\operatorname{div}(Bf) = f$ for all $f \in L_2^0(\Omega)$.

## 13.4 The hypothesis (H) and the Bogovskiĭ formula

In this section we show that the hypothesis (H) is satisfied if $\Omega \subseteq \mathbb{R}^n$ is open, bounded and has Lipschitz boundary.

**13.11 Theorem.** *Let $\Omega$ be as stated above. Then for any $\eta \in H^1_{0,\sigma}(\Omega;\mathbb{K}^n)^\circ$, i.e., $\eta \in H^{-1}(\Omega)^n$ with $\langle \eta, v\rangle_{H^{-1},H^1_0} = 0$ for all $v \in H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$, there exists $p \in L_2(\Omega)$ such that $\eta = \nabla p$. In other words, $\Omega$ satisfies hypothesis* (H).

For the proof we need the following special case of the 'closed range theorem'.

**13.12 Theorem.** *Let $G, H$ be Hilbert spaces, $A \in \mathcal{L}(G,H)$ with the property that* $\operatorname{ran}(A)$ *is closed.*

*Then* $\operatorname{ran}(A^*)$ *is closed.*

*Proof.* (i) First we treat the case that $A$ is bijective. Then the closed graph theorem implies that $A^{-1}$ is continuous, $(A^*)^{-1} = (A^{-1})^*$ is in $\mathcal{L}(G,H)$, and therefore $\operatorname{ran}(A^*) = \operatorname{dom}\big((A^*)^{-1}\big) = G$ is closed.

(ii) We can apply part (i) to $A_1 := A|_{\ker(A)^\perp} \colon \ker(A)^\perp \to \operatorname{ran}(A)$ and obtain that $\operatorname{ran}(A_1^*)$ is closed. Let $J\colon \ker(A)^\perp \hookrightarrow G$, and let $P$ be the orthogonal projection from $H$ onto $\operatorname{ran}(A) = \ker(A^*)^\perp$. Then $J^*$ is the orthogonal projection from $G$ onto $\ker(A)^\perp = \overline{\operatorname{ran}(A^*)}$, and $P^*\colon \ker(A^*)^\perp \hookrightarrow H$; see Exercise 8.2(d). Then $A_1^* = (PAJ)^* = J^*A^*P^*$ shows that $\operatorname{ran}(A^*) = \operatorname{ran}(A_1)$ is closed. □

*Proof of Theorem 13.11.* Without loss of generality we assume that $\Omega$ is connected.

The fundamental observation for the proof is that the continuous linear operators $\operatorname{div}\colon H^1_0(\Omega;\mathbb{K}^n) \to L_2(\Omega)$ and $\nabla\colon L_2(\Omega) \to H^{-1}(\Omega)^n$ are negative adjoints of each other. Indeed, for $v \in H^1_0(\Omega;\mathbb{K}^n)$, $f \in L_2(\Omega)$ one has

$$(f \mid \operatorname{div} v) = \sum_{j=1}^n \int f\overline{\partial_j v_j}\,\mathrm{d}x = -\sum_{j=1}^n \langle \partial_j f, v_j\rangle_{H^{-1},H^1_0} = -\langle \nabla f, v\rangle_{H^{-1},H^1_0}.$$

We know from Lemma 6.7 that $\operatorname{ran}(\nabla)^\circ = \ker(\operatorname{div})$, and this implies $\overline{\operatorname{ran}(\nabla)} = \ker(\operatorname{div})^\circ$. (Note that for this argument no special properties of $\Omega$ are required.) It is shown in Theorem 13.9 that $\operatorname{ran}(\operatorname{div}) = L_2^0(\Omega)$, and clearly $L_2^0(\Omega)$ is a closed subspace of $L_2(\Omega)$. Therefore Theorem 13.12 implies that $\operatorname{ran}(\nabla)$ is closed, $\operatorname{ran}(\nabla) = \overline{\operatorname{ran}(\nabla)} = \ker(\operatorname{div})^\circ$. As $\ker(\operatorname{div}) = H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$ by definition, we obtain the assertion of the theorem. □

## 13.5 Supplement: the space $H^1_{\sigma,0}(\Omega;\mathbb{K}^n)$ and the hypothesis (H′)

The final issue of the lecture will be to investigate a slightly stronger version of hypothesis (H). Using the space $H^1_{\sigma,0}(\Omega;\mathbb{K}^n) := \overline{C_c^\infty(\Omega)^n \cap H^1_{0,\sigma}(\Omega;\mathbb{K}^n)}^{H^1_0}$ we can formulate it as follows:

$$\text{for all } \eta \in H^1_{\sigma,0}(\Omega;\mathbb{K}^n)^\circ \text{ there exists } p \in L_2(\Omega) \text{ with } \eta = \nabla p. \tag{H′}$$

The space $H^1_{\sigma,0}(\Omega;\mathbb{K}^n)$ is a closed subspace of $H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$, and (H′) can be rephrased as the property that $H^1_{\sigma,0}(\Omega;\mathbb{K}^n)^\circ = \operatorname{ran}(\nabla)$, where $\nabla\colon L_2(\Omega) \to H^{-1}(\Omega)^n,\ f \mapsto \nabla f$.

We insert the important observation that for any open set $\Omega \subseteq \mathbb{R}^n$ one has $\operatorname{ran}(\nabla) \subseteq H^1_{0,\sigma}(\Omega;\mathbb{K}^n)^\circ \subseteq H^1_{\sigma,0}(\Omega;\mathbb{K}^n)^\circ$ $(= C^\infty_{c,\sigma}(\Omega;\mathbb{K}^n)^\circ)$. Recall that (H) is equivalent to $H^1_{0,\sigma}(\Omega;\mathbb{K}^n)^\circ = \operatorname{ran}(\nabla)$.

**13.13 Theorem.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and with Lipschitz boundary. Then* (H$'$) *is satisfied, and $H^1_{\sigma,0}(\Omega;\mathbb{K}^n) = H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$, i.e., $C^\infty_{\mathrm{c},\sigma}(\Omega;\mathbb{K}^n)$ is dense in $H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$.*

*Proof.* (i) Assume additionally that $\Omega$ is star-shaped with respect to every point of some ball $B(x^0,r)$ contained in $\Omega$. Without loss of generality assume that $x^0 = 0$. Then one can easily see that $\lambda\overline{\Omega} \subseteq \Omega$ for all $\lambda \in (0,1)$. Let $u \in H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$. Extend $u$ to $\mathbb{R}^n$ by 0. Then it follows from Exercise 4.2 that $\operatorname{div} u = 0$ also for the extended function. For $\lambda \in (0,1)$ define $u_\lambda := u(\lambda^{-1}\cdot)$; then $\operatorname{spt} u_\lambda \subseteq \lambda\overline{\Omega}$ is a compact subset of $\Omega$. If $(\rho_k)_{k\in\mathbb{N}}$ is a $\delta$-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, then one concludes that $\operatorname{div}(\rho_k * u_\lambda) = 0$ for all $k \in \mathbb{N}$, $\rho_k * u_\lambda \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{K}^n)$ for large $k$, and $\rho_k * u_\lambda \to u_\lambda$ in $H^1_0(\Omega;\mathbb{K}^n)$ as $k \to \infty$. As a consequence, $u_\lambda \in H^1_{\sigma,0}(\Omega;\mathbb{K}^n)$ for all $\lambda \in (0,1)$. Taking $\lambda \to 1$ one obtains $u_\lambda \to u$; hence $u \in H^1_{\sigma,0}(\Omega;\mathbb{K}^n)$.

So we have shown that $H^1_{\sigma,0}(\Omega;\mathbb{K}^n) = H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$. Theorem 13.11 shows that (H) is satisfied; hence (H$'$) is satisfied.

(ii) Without loss of generality we assume that $\Omega$ is connected. From the proof of Theorem 13.9 we recall that there exists an open covering $(\Omega_j)_{j=1,\dots,m}$ of $\Omega$, where each $\Omega_j$ is star-shaped with respect to the points in a ball $B(x^j,r_j) \subseteq \Omega_j$.

Let $\eta \in H^1_{\sigma,0}(\Omega;\mathbb{K}^n)^\circ$. Then clearly $\eta^j := \eta|_{H^1_{\sigma,0}(\Omega_j;\mathbb{K}^n)} \in H^1_{\sigma,0}(\Omega_j;\mathbb{K}^n)^\circ$, and from (i) we conclude that there exists $f_j \in L_2(\Omega_j)$ such that $\nabla f_j = \eta^j$, for all $j = 1,\dots,m$.

Note that each function $f_j$ is only determined up to a constant (see Remark 8.3), and we have to "glue together" suitable versions of these functions.

(iii) We show that for all $1 \leqslant j,k \leqslant m$, $j \neq k$, with the property that $\Omega_j \cap \Omega_k \neq \varnothing$ there exists $c_{kj} \in \mathbb{K}$ such that $f_k = f_j + c_{kj}$ on $\Omega_j \cap \Omega_k$. (Note that $\nabla(f_k - f_j) = 0$ on $\Omega_j \cap \Omega_k$ alone does not imply that $f_k - f_j$ is constant since $\Omega_j \cap \Omega_k$ need not be connected.)

Let $y \in \Omega_j \cap \Omega_k$, $r_y > 0$ be such that $B(y,r_y) \subseteq \Omega_j \cap \Omega_k$. Then $\nabla(f_k - f_j) = 0$ on $B(y,r_y)$ implies that there exists $c_{kj} \in \mathbb{K}$ such that $f_k - f_j = c_{kj}$ on $B(y,r_y)$. We are going to show that each $y' \in \Omega_j \cap \Omega_k$ has a neighbourhood where $f_k - f_j = c_{kj}$; then it follows that $f_k - f_j = c_{kj}$ on $\Omega_j \cap \Omega_k$.

Let $y' \in \Omega_j \cap \Omega_k$, and let $0 < r \leqslant \min\{r_j, r_k, r_y\}$ be such that $B(y',r) \subseteq \Omega_j \cap \Omega_k$. Then there exists $c'_{kj} \in \mathbb{K}$ such that $f_k - f_j = c'_{kj}$ on $B(y',r)$. Let $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)_+$, $\operatorname{spt}\varphi \subseteq B(0,r)$, $\int \varphi \,\mathrm{d}x = 1$. Then $\varphi(\cdot - y) - \varphi(\cdot - x^j)$ is the divergence of a function $\Phi_{yx^j} \in C^\infty_{\mathrm{c}}(\Omega_j)^n$. (See Exercise 13.3; in fact $\operatorname{spt}\Phi_{yx^j}$ is a subset of the convex hull $\operatorname{co}\bigl(B(x^j,r) \cup B(y,r)\bigr) \subseteq \Omega_j$ of $B(x^j,r) \cup B(y,r)$.) Similarly one obtains $\Phi_{yx^k} \in C^\infty_{\mathrm{c}}(\Omega_k)^n$ such that $\operatorname{div}\Phi_{yx^k} = \varphi(\cdot - y) - \varphi(\cdot - x^k)$. Note that $\nabla f_j = \eta^j$ implies

$$\bigl\langle \eta, \Phi_{yx^j} \bigr\rangle = \bigl\langle \eta^j, \Phi_{yx^j} \bigr\rangle = -\bigl(f_j \,\big|\, \operatorname{div}\Phi_{yx^j}\bigr) = \bigl(f_j \,\big|\, \varphi(\cdot - x^j) - \varphi(\cdot - y)\bigr),$$

and similarly for $k$. This implies

$$\begin{aligned}\bigl\langle \eta, \Phi_{yx^j} - \Phi_{yx^k} \bigr\rangle &= \bigl(f_j \,\big|\, \varphi(\cdot - x^j) - \varphi(\cdot - y)\bigr) - \bigl(f_k \,\big|\, \varphi(\cdot - x^k) - \varphi(\cdot - y)\bigr) \\ &= \bigl(f_j \,\big|\, \varphi(\cdot - x^j)\bigr) - \bigl(f_k \,\big|\, \varphi(\cdot - x^k)\bigr) + c_{kj}.\end{aligned} \tag{13.8}$$

Similarly, "connecting" $x^j$ with $x^k$ via $y'$ and using the corresponding functions, one obtains

$$\bigl\langle \eta, \Phi_{y'x^j} - \Phi_{y'x^k} \bigr\rangle = \bigl(f_j \,\big|\, \varphi(\cdot - x^j)\bigr) - \bigl(f_k \,\big|\, \varphi(\cdot - x^k)\bigr) + c'_{kj}.$$

The above definitions imply that $\operatorname{div}\bigl((\Phi_{yx^j} - \Phi_{yx^k}) - (\Phi_{y'x^j} - \Phi_{y'x^k})\bigr) = 0$, and therefore

$$0 = \bigl\langle \eta, \Phi_{yx^j} - \Phi_{yx^k} \bigr\rangle - \bigl\langle \eta, \Phi_{y'x^j} - \Phi_{y'x^k} \bigr\rangle = c_{kj} - c'_{kj}.$$

So, we obtain $f_k - f_j = c_{kj}$ on $B(x'', r)$.

(iv) From step (iii) we conclude that there exists $0 < r \leqslant \min\{r_1, \dots, r_m\}$ such that, given a function $\varphi \in C_c^\infty(\mathbb{R}^n)_+$ with $\operatorname{spt}\varphi \subseteq B(0,r)$ and $\int \varphi = 1$, the following property holds: for all $1 \leqslant j,k \leqslant m$ with $j \neq k$, $\Omega_j \cap \Omega_k \neq \varnothing$ there exists a function $\Phi_{kj} \in C_c^\infty(\Omega_j \cup \Omega_k)^n$ with $\operatorname{div}\Phi_{kj} = \varphi(\cdot - x^k) - \varphi(\cdot - x^j)$, and for this function one has

$$\langle \eta, \Phi_{kj} \rangle = \int f_j \varphi(\cdot - x^j) - \int f_k \varphi(\cdot - x^k) + c_{kj}.$$

(Indeed, use $\Phi_{kj} := \Phi_{yx^j} - \Phi_{yx^k}$ from (iii) and recall (13.8).)

Now let $(j_0, j_1, \dots, j_\ell = j_0)$ be a 'closed chain' in $\{1, \dots, m\}$ of 'neighbouring indices' (i.e., $j_{k-1} \neq j_k$ and $\Omega_{j_{k-1}} \cap \Omega_{j_k} \neq \varnothing$ for all $1 \leqslant k \leqslant \ell$). Then $\operatorname{div}\bigl(\sum_{k=1}^\ell \Phi_{j_k j_{k-1}}\bigr) = 0$, hence

$$0 = \Bigl\langle \eta, \sum_{k=1}^{\ell} \Phi_{j_k j_{k-1}} \Bigr\rangle = \sum_{k=1}^{\ell} c_{j_k j_{k-1}}.$$

Thus we obtain a well-defined function $f \in L_2(\Omega)$ if we define $f := f_1$ on $\Omega_1$, and for $j \in \{2, \dots, m\}$ we choose a 'chain' $(j_0 = 1, j_1, \dots, j_{\ell-1}, j_\ell = j)$ of neighbouring indices (which can be found because $\Omega$ is connected) and define $f := f_j - \sum_{k=1}^\ell c_{j_k j_{k-1}}$ on $\Omega_j$. It is clear that then $\eta = \nabla f$ locally on $\Omega$, and this implies $\eta = \nabla f$.

(v) From what is shown above we know that $\Omega$ satisfies condition (H$'$), i.e., $\operatorname{ran}(\nabla) = H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^\circ$. Together with the property $\operatorname{ran}(\nabla) \subseteq H^1_{0,\sigma}(\Omega; \mathbb{K}^n)^\circ \subseteq H^1_{\sigma,0}(\Omega; \mathbb{K}^n)^\circ$ we obtain equality in the last inclusions. Therefore the reflexivity of $H^1_0(\Omega; \mathbb{K}^n)$ implies $H^1_{0,\sigma}(\Omega; \mathbb{K}^n) = H^1_{\sigma,0}(\Omega; \mathbb{K}^n)$. □

Note that Theorem 13.13 strengthens Theorem 13.11. In step (i) of the proof we used Theorem 13.11, but only for the case of 'strongly star-shaped' $\Omega$.

**13.14 Remarks.** In these remarks we sketch how Theorems 13.11 and 13.13 are obtained in [Tem77; I, § 1].

(a) The basis is Nečas' inequality, stating that there exists $c > 0$ such that

$$\|f\|_{L_2} \leqslant c\Bigl(\sum_{j=1}^n \|\partial_j f\|_{H^{-1}} + \|f\|_{H^{-1}}\Bigr) \qquad (f \in L_2(\Omega)),$$

if $\Omega$ has Lipschitz boundary, asserted in [Neč12; Lemma 7.1, p. 186]. Using the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L_2(\Omega)$ one can show that this inequality implies

$$\|f\|_2 \leqslant c \sum_{j=1}^n \|\partial_j f\|_{H^{-1}} \qquad \Bigl(f \in L_2(\Omega) \text{ with } \int_\Omega f \, \mathrm{d}x = 0\Bigr) \tag{13.9}$$

if $\Omega$ is bounded with Lipschitz boundary. From (13.9) one obtains that $\operatorname{ran}(\nabla)$ is closed (where $\nabla \colon L_2(\Omega) \to H^{-1}(\Omega)^n$); see Exercise 13.4. As also $\operatorname{ran}(\nabla)$ is dense in $H^1_{0,\sigma}(\Omega; \mathbb{K}^n)^\circ$, Theorem 13.11 is proved.

(b) Theorem 13.13 is also derived in [Tem77; I, Remark 1.4]. The argument there ist based on a stronger version of (13.9), where $f$ is not a priori in $L_2(\Omega)$. The source [Neč66] for this version, quoted in [Tem77], was not available to us.

## Notes

Our introduction and presentation of the Stokes operator follows [Mon06] and [AE12]. Working out details we realised that we needed properties of Sobolev spaces of divergence free vector fields. These spaces are also of importance for the treatment of the Navier-Stokes equation.

The treatment given for the space $H^1_{0,\sigma}(\Omega;\mathbb{K}^n)$ in Section 13.3 can be found to a large part in [Gal11; Section III.3]. The *derivation* of the Bogovskiĭ formula presented at the beginning of Section 13.3 was found by the authors, and the same holds for the proof of Theorem 13.13. We mention that Bogovskiĭ's operator is also treated in the $L_p$-context in [Gal11] and in Sobolev spaces of negative order in [GHH06]. The treatment in 13.4 uses ideas contained in [Tem77; Ch. I, § 1].

## Exercises

**13.1** Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Show that the mapping $I - \Delta\colon H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism.

(b) Assume additionally that $\Omega$ is bounded, and provide $H^1_0(\Omega)$ with the scalar product $(u, v) \mapsto \int \nabla u \cdot \overline{\nabla v}$. Show that $-\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism.

**13.2** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded.

(a) Show that $L^0_2(\Omega)$ is a closed subspace of $L_2(\Omega)$ and that $L^0_{2,\mathrm{c}}(\Omega) := \{f \in L^0_2(\Omega);\ \operatorname{spt} f \text{ compact}\}$ is dense in $L_2(\Omega)$.

(b) Assume additionally that $\Omega$ is connected. Let $(\Omega_j)_{j=1,\dots,m}$ be a finite open covering of $\Omega$. Let $f \in L^0_2(\Omega)$.

Show that there exist functions $f_1, \dots, f_m \in L^0_2(\Omega)$, $[f_j \neq 0] \subseteq \Omega_j$ for all $1 \leqslant j \leqslant m$ such that $f = \sum_{j=1}^m f_j$.

**13.3** Let $\varphi \in C^\infty_\mathrm{c}(\mathbb{R}^n)$, $y, z \in \mathbb{R}^n$. Define $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x) := \int_{-\infty}^0 \big(\varphi(\cdot - z) - \varphi(\cdot - y)\big)(x + t(z - y))\,\mathrm{d}t\,(z - y).$$

Show that $\Phi \in C^\infty_\mathrm{c}(\mathbb{R}^n)$, $\operatorname{div}\Phi = \varphi(\cdot - z) - \varphi(\cdot - y)$ and that $\operatorname{spt}\Phi$ is contained in the convex hull of $(\operatorname{spt}\varphi + y) \cup (\operatorname{spt}\varphi + z)$.

**13.4** Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and with Lipschitz boundary. Show that the inequality (13.9) implies that $\operatorname{ran}(\nabla)$ is closed, for the mapping $\nabla\colon L_2(\Omega) \to H^{-1}(\Omega)^n$.

**13.5** Let $\Omega$ and the Bogovskiĭ operator $B$ be as in Theorem 13.8. Show that there exists $c > 0$ such that $\|Bf\|_2 \leqslant c\|f\|_2$ for all $f \in C^\infty_\mathrm{c} \cap L^0_2(\Omega)$.

Hint: Recall the dominating function indicated in the proof of Theorem 13.6. Then either use Proposition 4.3, or else show an $L_1$-$L_1$ bound and an $L_\infty$-$L_\infty$ bound for $B$, and then use Riesz-Thorin.

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