Lecture 12

Sectorial forms

In this lecture we study sectorial forms. If the form is closed, then the approach presented here is equivalent to the approach via elliptic forms. We point out that even if the form is not closed, we can always associate an m-sectorial operator. Two examples are given that illustrate very well the theory: the Robin Laplacian and the Dirichlet-to-Neumann operator are revisited, but now on rough domains.

12.1 Closed sectorial forms

In this section we want to establish a slightly different formulation of the generation theorems of Section 5.3. As defined in Lecture 5, a form a is a sesquilinear mapping $a\colon V \times V \to \mathbb{K}$, for some $\mathbb{K}$-vector space V. In contrast to previous sections, V will carry no additional stucture. For our purposes it will be convenient to simply call V the **domain of** a and set $\operatorname{dom}(a) := V$. Evidently, this is a misuse of the symbol 'dom', since actually the domain of a, in the usual sense, is the cartesian product $V \times V$. However, this notation is useful, has a long history, and should not lead to (too much) confusion.

Let $a\colon \operatorname{dom}(a) \times \operatorname{dom}(a) \to \mathbb{C}$ be a form. We recall that, by definition, a is sectorial if there exists $\theta \in [0, \pi/2)$ such that $a(u) \in \{z \in \mathbb{C} \setminus \{0\};\ |\operatorname{Arg} z| \leqslant \theta\} \cup \{0\}$ for all $u \in \operatorname{dom}(a)$. This is the same as saying that there exists $c \geqslant 0$ such that

$$|\operatorname{Im} a(u)| \leqslant c \operatorname{Re} a(u) \tag{12.1}$$

for all $u \in \operatorname{dom}(a)$. We further recall that $\operatorname{Re} a, \operatorname{Im} a\colon \operatorname{dom}(a) \times \operatorname{dom}(a) \to \mathbb{C}$ are symmetric forms such that $a = \operatorname{Re} a + \mathrm{i} \operatorname{Im} a$ and that $\operatorname{Re} a(u) = (\operatorname{Re} a)(u)$ and $\operatorname{Im} a(u) = (\operatorname{Im} a)(u)$ for all $u \in \operatorname{dom}(a)$. Using Proposition 5.2 we deduce from (12.1) the inequality

$$|a(u, v)| \leqslant (1 + c)(\operatorname{Re} a(u))^{1/2}(\operatorname{Re} a(v))^{1/2}. \tag{12.2}$$

This is a sort of intrinsic continuity of a which we will use throughout.

Let H be a complex Hilbert space. A **sectorial form in** H is a couple (a, j) where $a\colon \operatorname{dom}(a) \times \operatorname{dom}(a) \to \mathbb{C}$ is a sectorial form and $j\colon \operatorname{dom}(a) \to H$ is linear. We say that (a, j) is **densely defined** if j has dense range.

12.1 Remark. Later we want to use the results of this section also for the case $\mathbb{K} = \mathbb{R}$, if a is a symmetric form $a\colon \operatorname{dom}(a) \times \operatorname{dom}(a) \to \mathbb{R}$. Note that then the inequality (12.2) holds with $c = 0$; in fact the validity of (12.2) is a key point for the results of this section. We ask the reader to keep this in mind and to read the section under this aspect, observing that the results also hold in this case.

12.2 Proposition. *Let (a,j) be a densely defined sectorial form in H. Then*

$$A_0 := \bigl\{(x,y) \in H \times H\,;\ \exists u \in \operatorname{dom}(a)\colon j(u) = x,\ \ a(u,v) = (y\,|\,j(v))_H\ \ (v \in \operatorname{dom}(a))\bigr\} \tag{12.3}$$

defines a sectorial operator in H.

Proof. (i) Let $(0,y) \in A_0$. We show that $y = 0$; this implies that A_0 is an operator. Since $(0,y) \in A_0$ there exists $u \in \operatorname{dom}(a)$ such that $j(u) = 0$ and $a(u,v) = (y\,|\,j(v))_H$ for all $v \in \operatorname{dom}(a)$. In particular, $a(u) = 0$. By (12.2) this implies that $a(u,v) = 0$ for all $v \in \operatorname{dom}(a)$. Consequently $(y\,|\,j(v))_H = 0$ for all $v \in \operatorname{dom}(a)$. Since j has dense range it follows that $y = 0$.

(ii) Let $x \in \operatorname{dom}(A_0)$. There exists $u \in \operatorname{dom}(a)$ such that $j(u) = x$ and $a(u,v) = (A_0x\,|\,j(v))_H$ for all $v \in \operatorname{dom}(a)$. In particular, $(A_0x\,|\,x) = a(u)$. Thus A_0 is sectorial. □

In general, A_0 is not m-sectorial, but later we will construct an m-sectorial extension. We will give a condition which implies that A_0 is m-sectorial.

Let (a,j) be a densely defined sectorial form in H. Then

$$(u\,|\,v)_a := (\operatorname{Re} a)(u,v) + (j(u)\,|\,j(v))_H \tag{12.4}$$

defines a semi-inner product, i.e., a symmetric, accretive sesquilinear form on $\operatorname{dom}(a)$. Thus

$$\|u\|_a := \sqrt{(u\,|\,u)_a} = \bigl(\operatorname{Re} a(u) + \|j(u)\|_H^2\bigr)^{1/2}$$

defines a seminorm on $\operatorname{dom}(a)$.

12.3 Remark. (Completion of a semi-inner product space) Let E be a vector space over $\mathbb{K}$ and $(\cdot\,|\,\cdot)$ a **semi-inner product** on E (i.e, the form $(\cdot\,|\,\cdot)$ is symmetric and accretive), with associated semi-norm $\|\cdot\|$.

(a) There exist a Hilbert space $\tilde{E}$ and a linear mapping $q\colon E \to \tilde{E}$ which

(i) is **isometric**, i.e., $(q(u)\,|\,q(v))_{\tilde{E}} = (u\,|\,v)$ for all $u,v \in E$, and

(ii) has dense range.

Indeed, note that $F := \{u \in E\,;\ \|u\| = 0\}$ is a subspace of E, and let $q\colon E \to E/F =: G$ denote the quotient map. Since $|(u\,|\,v)| \leqslant \|u\|\|v\|$, by Proposition 5.2, $\|u\| = 0$ implies $(u\,|\,v) = 0$ for all $v \in E$. Thus, setting $(q(u)\,|\,q(v))_G = (u\,|\,v)$ one obtains a well-defined scalar product on G such that q is isometric. We define $\tilde{E}$ as the completion of G.

(b) The space $\tilde{E}$ is unique up to unitary equivalence. We call the pair $(\tilde{E}, q)$ the **completion** of E.

(c) The mapping q is injective if and only if $(\cdot\,|\,\cdot)$ is positive definite, and q is surjective if and only if $(E, \|\cdot\|)$ is complete. If $(E, \|\cdot\|)$ is complete, but $(\cdot\,|\,\cdot)$ is not positive definite, then its completion $(\tilde{E}, q) = (E/F, q)$ is different from E. (Recall that a semi-normed space is called complete if every Cauchy sequence is convergent.)

(d) If H is a Hilbert space, and $j\colon E \to H$ is a continuous linear mapping, then there exists a unique continuous linear mapping $\tilde{\jmath}\colon \tilde{E} \to H$ such that $\tilde{\jmath} \circ q = j$. Similarly, if $a\colon E \times E \to \mathbb{K}$ is a bounded form (i.e., $|a(u,v)| \leqslant M\|u\|\|v\|$ for all $u,v \in E$), then there exists a uniquely determined bounded form $\tilde{a}\colon \tilde{E} \times \tilde{E} \to \mathbb{K}$ such that $\tilde{a}(q(u), q(v)) = a(u,v)$ for all $u,v \in E$.

The form (a,j) is called **closed** if the semi-inner product space $(\operatorname{dom}(a), (\cdot\,|\,\cdot)_a)$ is complete.

12.4 Theorem. *Let (a,j) be a densely defined closed sectorial form in H. Then the operator A_0 defined in Proposition 12.2 is m-sectorial. We call $A := A_0$ **the operator associated with** (a,j) and write $A \sim (a,j)$.*

Proof. Denote the completion of $(\operatorname{dom}(a), (\cdot\,|\,\cdot)_a)$ by (V,q). Since $\operatorname{dom}(a)$ is complete, q is surjective. From Remark 12.3(d) we recall the existence and continuity of $\tilde{\jmath}\colon V \to H$ and $\tilde{a}\colon V \times V \to \mathbb{C}$. Moreover,

$$\operatorname{Re} \tilde{a}(q(u)) + \|\tilde{\jmath}(q(u))\|_H^2 = \operatorname{Re} a(u) + \|j(u)\|_H^2 = \|u\|_a^2 = \|q(u)\|_V^2$$

for all $u \in \operatorname{dom}(a)$. Thus $\tilde{a}$ is $\tilde{\jmath}$-elliptic. By Corollary 5.11 the operator $\tilde{A}$ associated with $(\tilde{a}, \tilde{\jmath})$ is m-sectorial.

We show that $\tilde{A} = A_0$. Indeed, due to the surjectivity of q one has $(x,y) \in \tilde{A}$ if and only if there exists $u \in \operatorname{dom}(a)$ such that $\tilde{\jmath}(q(u)) = x$ and $\tilde{a}(q(u), q(v)) = (y\,|\,\tilde{\jmath}(q(v)))_H$ for all $v \in \operatorname{dom}(a)$. This is the same as saying that there exists $u \in \operatorname{dom}(a)$ such that $j(u) = x$ and $a(u,v) = (y\,|\,j(v))_H$ for all $v \in \operatorname{dom}(a)$, which is equivalent to $(x,y) \in A_0$. □

12.2 The Friedrichs extension

Let H be a complex Hilbert space. In this section we consider the situation that a is a form whose domain is a subspace of H and that $j\colon \operatorname{dom}(a) \hookrightarrow H$ is the embedding. In that case we drop the j in our notation and call a **a form in** H. If we want to emphasise that we are in this situation we will sometimes call a an **embedded** form. If a is a sectorial form in H and $\operatorname{dom}(a)$ is dense, the operator A_0 of Proposition 12.2 is described by

$$A_0 = \bigl\{(u,y) \in \operatorname{dom}(a) \times H\,;\, a(u,v) = (y\,|\,v)_H \ \ (v \in \operatorname{dom}(a))\bigr\}.$$

The form a is closed if and only if $\operatorname{dom}(a)$ is complete for the norm $\|\cdot\|_a$ given by

$$\|u\|_a^2 = \operatorname{Re} a(u) + \|u\|_H^2.$$

In that case the associated operator $A = A_0$ is m-sectorial, by Theorem 12.4.

12.5 Theorem. *(Friedrichs extension) Let B be a densely defined sectorial operator in H. Then there exists a unique densely defined embedded closed sectorial form a in H such that* $\operatorname{dom}(B) \subseteq \operatorname{dom}(a)$, $\operatorname{dom}(B)$ *dense in* $(\operatorname{dom}(a), \|\cdot\|_a)$*, and*

$$a(u,v) = (Bu\,|\,v)_H$$

for all $u \in \operatorname{dom}(B)$, $v \in \operatorname{dom}(a)$.

Let $A \sim a$. Then $B \subseteq A$.

The operator A is called the **Friedrichs extension** of B. Note that the theorem also holds for accretive symmetric operators in real Hilbert spaces; see Remark 12.1.

Proof of Theorem 12.5. Define $b\colon \operatorname{dom}(B)\times\operatorname{dom}(B)\to\mathbb{C}$ by $b(u,v):=(Bu\,|\,v)_H$. Then b is densely defined and sectorial. We use the scalar product $(\cdot\,|\,\cdot)_b$ analogous to (12.4) on $\operatorname{dom}(b)=\operatorname{dom}(B)$ and the embedding $j\colon \operatorname{dom}(b)\hookrightarrow H$. Let (V,q) be the completion of $(\operatorname{dom}(b),\|\cdot\|_b)$; then q is injective by Remark 12.3(c). We consider $\operatorname{dom}(b)$ as a subset of V and drop the notation q. We show that $\tilde{\jmath}\in\mathcal{L}(V,H)$ – from Remark 12.3(d) – is injective. Let $u\in V$ such that $\tilde{\jmath}(u)=0$. There exists a sequence (u_n) in $\operatorname{dom}(b)$ such that $u_n\to u$ in V. Then $u_n=\tilde{\jmath}(u_n)\to\tilde{\jmath}(u)=0$ in H. Using (12.2) one obtains

$$\begin{aligned}\operatorname{Re} b(u_n) &= \operatorname{Re} b(u_n,u_n-u_k)+\operatorname{Re} b(u_n,u_k)\\ &\leqslant \|u_n\|_b\|u_n-u_k\|_b+|(Bu_n\,|\,u_k)_H| \qquad (k,n\in\mathbb{N}).\end{aligned}$$

Sending $k\to\infty$ we deduce that $\operatorname{Re} b(u_n)\leqslant\|u_n\|_V\|u_n-u\|_V$ for all $n\in\mathbb{N}$. It follows that $\|u_n\|_V^2=\operatorname{Re} b(u_n)+\|u_n\|_H^2\to 0$ as $n\to\infty$, $\|u\|_V=\lim_{n\to\infty}\|u_n\|_V=0$.

Because $\tilde{\jmath}$ is injective we now can consider V as a subspace of H. Then $a:=\tilde{b}$ with $\operatorname{dom}(a):=V$ (with the notation of Remark 12.3(d)) is a closed sectorial form in H with the required properties, and the associated operator A is an extension of B.

In order to show uniqueness let a and b be two forms in H with the required properties. Then $\operatorname{dom}(B)\subseteq\operatorname{dom}(a)\cap\operatorname{dom}(b)$, $a(u,v)=(Bu\,|\,v)=b(u,v)$ for all $u,v\in\operatorname{dom}(B)$, and $\operatorname{dom}(B)$ is dense in $(\operatorname{dom}(a),\|\cdot\|_a)$ and in $(\operatorname{dom}(b),\|\cdot\|_b)$. Since $\|\cdot\|_a=\|\cdot\|_b$ on $\operatorname{dom}(B)$ it follows that $a=b$. ☐

We now prove a converse version of Theorem 12.4.

12.6 Corollary. *Let A be an m-sectorial operator in H. Then there exists a unique densely defined embedded closed sectorial form a in H such that A is associated with a.*

Proof. The existence follows from Theorem 12.5 since m-sectorial operators do not have proper m-sectorial extensions.

Let a be as asserted. Then $\operatorname{dom}(A)\subseteq\operatorname{dom}(a)$ and $a(u,v)=(Au\,|\,v)$ for all $u,v\in\operatorname{dom}(A)$. It follows from Lemma 9.19(a) that $\operatorname{dom}(A)$ is dense in $(\operatorname{dom}(a),\|\cdot\|_a)$. Therefore the uniqueness follows from the uniqueness in Theorem 12.5. ☐

12.3 Sectorial versus elliptic

We now have two different generation results: on the one hand based on the notion of sectoriality in Section 12.1, on the other hand based on the notion of ellipticity in Lecture 5. We will show that they are basically the same. Before making this precise we first apply the usual rescaling procedure.

Let H be a complex Hilbert space. A **quasi-sectorial** form is a couple (a,j) where $a\colon\operatorname{dom}(a)\times\operatorname{dom}(a)\to\mathbb{C}$ is a sesquilinear form whose domain is a complex vector space and $j\colon\operatorname{dom}(a)\to H$ is linear, with the property that there exists $\omega\in\mathbb{R}$ such that the form

$$a_\omega(u,v)=a(u,v)+\omega\,(j(u)\,|\,j(v))_H$$

is sectorial. If a is densely defined (i.e., j has dense range) we define the operator A_0 corresponding to (a,j) as in Proposition 12.2. One easily sees that $A_0+\omega$ corresponds to a_ω. Now assume that a is **closed**, i.e., a_ω is closed. Then $A_\omega \sim (a_\omega, j)$ is m-sectorial, $A_\omega = A_0 + \omega$, and thus $A := A_0$ is quasi-m-sectorial.

In the following remark we will make it clear that the notions of 'densely defined closed quasi-sectorial form' and 'j-elliptic form' are equivalent.

12.7 Remarks. (a) 'Closed sectorial' implies 'continuous j-elliptic'.

Let (a, j) be a densely defined closed sectorial form in H. Then the completion (V, q) of $(\mathrm{dom}(a), \|\cdot\|_a)$ is a Hilbert space for the norm given by $\|u\|_V := \|u\|_{\tilde a} = (\operatorname{Re}\tilde a(u) + \|\tilde\jmath(u)\|_H^2)^{1/2}$. The form $\tilde a$ is continuous for this norm and $\tilde\jmath$-elliptic.

(b) 'Continuous j-elliptic' implies 'closed quasi-sectorial, $\|\cdot\|_V \sim \|\cdot\|_{a_\omega}$'.

Let V be a Hilbert space, and assume that $j \in \mathcal{L}(V, H)$ has dense range. Let $a\colon V \times V \to \mathbb{C}$ be continuous and j-elliptic, say

$$\begin{aligned} |a(u,v)| &\leqslant M\|u\|_V\|v\|_V, \\ \operatorname{Re} a(u) + \omega\|u\|_H^2 &\geqslant \alpha\|j(u)\|_V^2. \end{aligned}$$

Then a_ω is sectorial (see Theorem 5.9) and the norm $\|\cdot\|_{a_\omega}$ is equivalent to the given norm $\|\cdot\|_V$ on V. The latter shows that a_ω and hence a is closed.

(c) Sectorial on a complete domain.

Let V, H be Hilbert spaces, $V \subseteq H$ dense, with continuous embedding, and let $a\colon V \times V \to \mathbb{C}$ be a closed sectorial form in H. Observe that the norm of V is not used for these properties. They imply that $\|u\|_a = (\operatorname{Re} a(u) + \|u\|_H^2)^{1/2}$ defines a complete norm on V. Since also $(V, \|\cdot\|_a) \hookrightarrow H$, it follows from the closed graph theorem that both norms, $\|\cdot\|_a$ and $\|\cdot\|_V$, are equivalent. Thus the form a is H-elliptic also with respect to the given norm on V.

12.4 The non-complete case

Here we consider densely defined sectorial forms that are not closed. Let H be a Hilbert space over $\mathbb{C}$. Let (a, j) be a densely defined sectorial form in H. Let (V, q) be the completion of $(\mathrm{dom}(a), \|\cdot\|_a)$, and let $\tilde\jmath, \tilde a$ be as explained in Remark 12.3(d). Then $(\tilde a, \tilde\jmath)$ is a closed sectorial form in H. Let A be the m-sectorial operator associated with $(\tilde a, \tilde\jmath)$ according to Theorem 12.4. We call A the **operator associated with** (a, j). Recall that

$$A = \bigl\{(x,y) \in H \times H\,;\ \exists u \in V\colon \tilde\jmath(u) = x,\ \tilde a(u,v) = (y \,|\, \tilde\jmath(v))_H \ \ (v \in V)\bigr\}.$$

It follows from the proof of Theorem 12.4 that this definition is consistent with the definition given in Theorem 12.4.

12.8 Theorem. *For the operator A defined above one has the description*

$$\begin{aligned} A = \bigl\{(x,y) \in H \times H\,;\ &\textit{there exists } (u_k) \textit{ in } \mathrm{dom}(a) \textit{ such that} \\ &\text{(a) } j(u_k) \to x \textit{ as } k \to \infty, \\ &\text{(b) } \sup_{k\in\mathbb{N}} \operatorname{Re} a(u_k) < \infty, \\ &\text{(c) } a(u_k, v) \to (y \,|\, j(v))_H \ \ (v \in \mathrm{dom}(a))\bigr\}. \end{aligned}$$

In this description the condition (b) can be replaced by

(b') $\lim_{k,\ell\to\infty} \operatorname{Re} a(u_k - u_\ell) = 0$.

The operator A is an extension of A_0 defined in Proposition 12.2.

Proof. Let $(x, y) \in A$. Then there exists $w \in V$ such that $\tilde{\jmath}(w) = x$ and $\tilde{a}(w, v) = (y \,|\, \tilde{\jmath}(v))_H$ for all $v \in V$. Since q has dense range there exists a sequence (u_k) in $\operatorname{dom}(a)$ such that $q(u_k) \to w$ in V. By the continuity of $\tilde{\jmath}$ it follows that $j(u_k) = \tilde{\jmath}(q(u_k)) \to \tilde{\jmath}(w) = x$. Since q is isometric, we obtain

$$\operatorname{Re} a(u_k - u_\ell) + \|j(u_k) - j(u_\ell)\|_H^2 = \|u_k - u_\ell\|_a^2 = \|q(u_k) - q(u_\ell)\|_V^2 \to 0$$

as $k, \ell \to \infty$. Finally, for $v \in \operatorname{dom}(a)$ we have

$$a(u_k, v) = \tilde{a}(q(u_k), q(v)) \to \tilde{a}(w, q(v)) = (y \,|\, \tilde{\jmath}(q(v)))_H = (y \,|\, j(v))_H\,.$$

This shows the existence of a sequence (u_k) as asserted, and for this sequence one even has the stronger property (b').

Conversely, let $(x, y) \in H \times H$ such that there exists a sequence (u_k) in $\operatorname{dom}(a)$ satisfying (a), (b), (c). It follows from (a) and (b) that $\sup_{k\in\mathbb{N}} \|u_k\|_a < \infty$. Thus, taking a subsequence if necessary, we can assume that $(q(u_k))$ converges weakly to some $w \in V$. Hence $j(u_k) = \tilde{\jmath}(q(u_k)) \to \tilde{\jmath}(w)$ weakly in H, and $\tilde{\jmath}(w) = x$ by (a). Property (c) implies

$$\tilde{a}(w, q(v)) = \lim_{n\to\infty} \tilde{a}(q(u_k), q(v)) = \lim_{n\to\infty} a(u_k, v) = (y \,|\, j(v))_H$$

for all $v \in \operatorname{dom}(a)$. Since q has dense range and $\tilde{a}, \tilde{\jmath}$ are continuous, it follows that $\tilde{a}(w, \tilde{v}) = (y \,|\, \tilde{\jmath}(\tilde{v}))_H$ for all $\tilde{v} \in V$. This implies that $(x, y) \in A$.

If $(x, y) \in A_0$ and $u \in \operatorname{dom}(a)$ is as in (12.3), then the constant sequence $(u_k) = (u)$ satisfies the conditions required in the description of A, so $(x, y) \in A$. □

12.9 Remarks. In this remark we assume that a is an embedded densely defined sectorial form in H. Then the description of the operator associated with a is as in Theorem 12.8, but without 'j' in conditions (a) and (c).

(a) Let $(\tilde{a}, \tilde{\jmath})$ be the 'completion' of (a, j) as described at the beginning of the section. We point out that in general $\tilde{\jmath}$ is not injective. We call the form a **closable** if $\tilde{\jmath}$ is injective. Then we may identify $\operatorname{dom}(\tilde{a})$ with a subspace of H.

An example of a closable form has been given in Section 12.2, leading to the Friedrichs extension.

(b) We have seen that we may associate an m-sectorial operator A with a, no matter whether a is closable or not. Moreover, from Theorem 12.6 we know that there exists a unique closed form $\bar{a}$ in H that is associated with A. In general there is no simple relation between a and $\bar{a}$.

(c) We add a comment on the terminology. The notation 'closed form' (in the situation of embedded forms) was forged by Kato – see, for instance, [Kat80; VI, §1.3] – in a vague analogy to 'closed operators'. However, whereas closed operators *are* closed in the product topology, for forms there is no visible closed set. Concerning 'closable' – see [Kat80; VI, §1.4] – we emphasise that our procedure associates a closed form with *any* sectorial form.

12.10 Remark. Coming back to accretive symmetric forms in a real Hilbert space – recall Remark 12.1 – we note that in this case one obtains an associated accretive self-adjoint operator A and the same description as in Theorem 12.8.

12.5 The Robin Laplacian for rough domains

We choose $\mathbb{K} = \mathbb{R}$ thoughout this section. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. On $\partial\Omega$ we consider the $(n-1)$-dimensional Hausdorff measure σ; we refer to the end of this section for the definition. If Ω has C^1-boundary, then σ coincides with the surface measure. We assume that $\sigma(\partial\Omega) < \infty$. Let $0 < c \leqslant \beta \in L_\infty(\partial\Omega)$.

Let $u \in H^1(\Omega)$. We call

$$\operatorname{tr} u := \bigl\{g \in L_2(\partial\Omega)\,;\ \text{there exists } (u_k) \text{ in } C(\overline{\Omega}) \cap H^1(\Omega) \text{ such that} \\ u_k \to u \text{ in } H^1(\Omega), u_k|_{\partial\Omega} \to g \text{ in } L_2(\partial\Omega)\bigr\}$$

the **trace** of u. Expressed differently, tr is the closure in $H^1(\Omega) \times L_2(\partial\Omega)$ of the operator $u \mapsto u|_{\partial\Omega} \colon C(\overline{\Omega}) \cap H^1(\Omega) \to L_2(\partial\Omega)$. In general $\operatorname{tr} u$ is a set, but if Ω has Lipschitz boundary, then $\operatorname{tr} u$ has precisely one element for each $u \in H^1(\Omega)$; for the case of C^1-boundary this has been shown in Theorem 7.9, and for the case of Lipschitz boundary we refer to [Alt85; A 5.7].

Let $u \in H^1(\Omega)$ such that $\Delta u \in L_2(\Omega)$. Let $h \in L_2(\partial\Omega)$. We say that $\partial_\nu u = h$ if

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\Delta u) v \,\mathrm{d}x = \int_{\partial\Omega} hv \,\mathrm{d}\sigma$$

for all $v \in C(\overline{\Omega}) \cap H^1(\Omega)$. We say that $\partial_\nu u \in L_2(\partial\Omega)$ if there exists $h \in L_2(\partial\Omega)$ such that $\partial_\nu u = h$. Note that this is analogous to the definition of the weak normal derivative in Section 7.3. The uniqueness of $\partial_\nu u$ is obtained from the denseness of $\bigl\{u|_{\partial\Omega}\,;\ u \in C^1(\overline{\Omega})\bigr\}$ in $L_2(\partial\Omega)$.

Now we can define the **Robin Laplacian** Δ_β under our present general hypotheses.

12.11 Theorem. *The relation*

$$\Delta_\beta = \bigl\{(u, f) \in H^1(\Omega) \times L_2(\Omega)\,;\ -\Delta u = f,\ \exists g \in \operatorname{tr} u \colon \partial_\nu u = -\beta g\bigr\}$$

is an accretive self-adjoint operator.

It is part of the assertion in Theorem 12.11 that the relation Δ_β defines an operator even though the trace may be multi-valued, i.e., it can happen that there exists $0 \neq g \in \operatorname{tr} 0$; see Exercise 12.4.

Proof of Theorem 12.11. Let $\operatorname{dom}(a) := C(\overline{\Omega}) \cap H^1(\Omega)$ and

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta uv \,\mathrm{d}\sigma.$$

Then a is symmetric, densely defined and accretive. Let A be the operator associated with a; see Section 12.4. Then A is self-adjoint and accretive by Remark 12.10. Using Theorem 12.8 and Remark 12.10 we are going to show that $\Delta_\beta = A$.

'$A \subseteq \Delta_\beta$'. Let $(u, f) \in A$. Then there exists a sequence (u_k) in $\operatorname{dom}(a)$ such that

(a) $u_k \to u$ in $L_2(\Omega)$,

(b') $\lim_{k,\ell\to\infty} a(u_k - u_\ell) = 0$,

(c) $\int_\Omega \nabla u_k \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta u_k v \,\mathrm{d}\sigma \to \int_\Omega f v \,\mathrm{d}x$ for all $v \in \operatorname{dom}(a)$.

It follows from (a) and (b') that $u \in H^1(\Omega)$ and $u_k \to u$ in $H^1(\Omega)$ and that $g := \lim_{k\to\infty} u_k|_{\partial\Omega}$ exists in $L_2(\partial\Omega)$. Now (c) implies that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta g v \,\mathrm{d}\sigma = \int_\Omega f v \,\mathrm{d}x$$

for all $v \in \operatorname{dom}(a)$. Taking $v \in C_\mathrm{c}^\infty(\Omega)$ we conclude that $-\Delta u = f$. Thus

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega (\Delta u) v \,\mathrm{d}x = -\int_{\partial\Omega} \beta g v \,\mathrm{d}\sigma$$

for all $v \in \operatorname{dom}(a)$. This means by our definition that $\partial_\nu u = -\beta g$.

'$\Delta_\beta \subseteq A$'. Let $(u, f) \in \Delta_\beta$. Then $u \in H^1(\Omega)$, $f = -\Delta u$, and there exists $g \in \operatorname{tr} u$ such that $\partial_\nu u = -\beta g$. Thus, there exists a sequence (u_k) in $\operatorname{dom}(a)$ such that $u_k \to u$ in $H^1(\Omega)$ and $u_k|_{\partial\Omega} \to g$ in $L_2(\partial\Omega)$ as $k \to \infty$. Thus (a) and (b') hold, and

$$a(u_k, v) = \int_\Omega \nabla u_k \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta u_k v \,\mathrm{d}\sigma \to \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \beta g v \,\mathrm{d}\sigma = \int_\Omega f v \,\mathrm{d}x$$

for all $v \in \operatorname{dom}(a)$ since $-\Delta u = f$ and $\partial_\nu u = -\beta g$. It follows that $(u, f) \in A$, by Theorem 12.8. □

We close the section by a short introduction to the d-dimensional Hausdorff measure on $\mathbb{R}^n$, for $d \geqslant 0$. Let $A \subseteq \mathbb{R}^n$. For $\varepsilon > 0$ and coverings $(C_j)_{j\in\mathbb{N}}$ of A by sets $C_j \subseteq \mathbb{R}^n$ satisfying $\sup_{j\in\mathbb{N}} \operatorname{diam} C_j \leqslant \varepsilon$ we let

$$\sigma_{d,\varepsilon}(A) := \omega_d \inf \sum_{j=1}^\infty \Big(\frac{\operatorname{diam} C_j}{2}\Big)^d,$$

where the infimum is taken over all coverings as above, and

$$\omega_d := \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)},$$

which is the volume of the unit ball in $\mathbb{R}^d$ if $d \in \mathbb{N}$. Then

$$\sigma_d^*(A) := \lim_{\varepsilon\to 0} \sigma_{d,\varepsilon}(A)$$

is the **outer d-dimensional Hausdorff measure** of A, where the limit exists because $\sigma_{d,\varepsilon}(A)$ is decreasing in ε. Carathéodory's construction of measurable sets yields a measure σ_d, the **d-dimensional Hausdorff measure**, and it turns out that Borel sets are measurable. If $d \in \mathbb{N}$, and $E = \mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^n$, then σ_d is the Lebesgue measure on $\mathbb{R}^d \cong E$. For all of these properties (and more) we refer to [EG92; Chapter 2].

12.6 The Dirichlet-to-Neumann operator for rough domains

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. We keep the definitions concerning σ and ∂_ν from Section 12.5. Again we use $\mathbb{K} = \mathbb{R}$ and assume that $\sigma(\partial\Omega) < \infty$. Now we define the **Dirichlet-to-Neumann operator** D_0.

12.12 Theorem. *The relation*

$$D_0 = \bigl\{(g,h) \in L_2(\partial\Omega) \times L_2(\partial\Omega)\,;\ \exists u \in H^1(\Omega)\colon \Delta u = 0,\ g \in \operatorname{tr} u,\ h = \partial_\nu u\bigr\}.$$

is an accretive self-adjoint operator in $L_2(\partial\Omega)$.

For the proof we need a remarkable inequality due to Maz'ja. Let $q := \frac{2n}{n-1}$. There exists a constant $c_{\mathrm{M}} > 0$ such that

$$\|u\|_{L_q(\Omega)}^2 \leqslant c_{\mathrm{M}}\Bigl(\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma\Bigr) \tag{12.5}$$

for all $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ (see [Maz85; Example 3.6.2/1 and Theorem 3.6.3]).

Proof of Theorem 12.12. Let $\operatorname{dom}(a) = C(\overline{\Omega}) \cap H^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$. Then a is an accretive symmetric form. Let $j\colon \operatorname{dom}(a) \to L_2(\partial\Omega)$ be given by $j(u) := u|_{\partial\Omega}$. Then j has dense range because $C^1(\overline{\Omega}) \subseteq \operatorname{dom}(a)$. Let A be the operator associated with (a,j); see Section 12.4. Then A is self-adjoint and accretive by Remark 12.10. We show that $A = D_0$.

Let $(g,h) \in A$. Then there exists a sequence (u_k) in $C(\overline{\Omega}) \cap H^1(\Omega)$ such that $u_k|_{\partial\Omega} \to g$ in $L_2(\partial\Omega)$, $\lim_{k,\ell\to\infty} \int_\Omega |\nabla(u_k - u_\ell)|^2 = 0$ and $\lim_{k\to\infty} a(u_k,v) = \int_{\partial\Omega} hv \,\mathrm{d}\sigma$ for all $v \in C(\overline{\Omega}) \cap H^1(\Omega)$. Now Maz'ja's inequality implies that (u_k) is a Cauchy sequence in $H^1(\Omega)$. (Here we use that $q > 2$ and Ω has finite measure, and therefore $L_q(\Omega) \subseteq L_2(\Omega)$, with continuous embedding.) Let $u := \lim_{k\to\infty} u_k$ in $H^1(\Omega)$. Then $g \in \operatorname{tr} u$. Moreover,

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\partial\Omega} hv \,\mathrm{d}\sigma \qquad (v \in C(\overline{\Omega}) \cap H^1(\Omega)).$$

Taking test functions $v \in C_{\mathrm{c}}^\infty(\Omega)$ one obtains $\Delta u = 0$. Thus

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega (\Delta u) v = \int_{\partial\Omega} hv \,\mathrm{d}\sigma$$

for all $v \in C(\overline{\Omega}) \cap H^1(\Omega)$. Hence $\partial_\nu u = h$ by the definition in Section 12.5. We have shown that $A \subseteq D_0$.

Conversely, let $(g,h) \in D_0$. Then there exists $u \in H^1(\Omega)$ such that $g \in \operatorname{tr} u$, $\Delta u = 0$, $\partial_\nu u = h$. Hence there exists a sequence (u_k) in $\operatorname{dom}(a)$ such that $u_k|_{\partial\Omega} \to g$ in $L_2(\partial\Omega)$ and $u_k \to u$ in $H^1(\Omega)$. Thus $\sup_{k\in\mathbb{N}} a(u_k) < \infty$ and

$$a(u_k,v) = \int_\Omega \nabla u_k \cdot \nabla v \to \int_\Omega \nabla u \cdot \nabla v = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega (\Delta u) v = \int_{\partial\Omega} hv \,\mathrm{d}\sigma$$

for all $v \in \operatorname{dom}(a)$. This shows that $(g,h) \in A$. $\square$

12.13 Remark. Regrettably, it is completely beyond the scope of the Internet Seminar to provide a proof of Maz'ja's inequality. We refer to Maz'ja's book [Maz85] as well as to a self-contained presentation in a manuscript of Daners [Daned].

We mention that for our purpose it would have been sufficient to have (12.5) with $q = 2$. In this case, and for Ω with C^1-boundary, we can show (12.5). Indeed, in the proof of Proposition 8.1 we have shown the inequality

$$\int_\Omega |u|^2 \,\mathrm{d}x = \frac{1}{2}\int_\Omega |u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + c\int_\Omega |\nabla u|^2 \,\mathrm{d}x + c\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma,$$

and this inequality implies

$$\|u\|_2^2 \leqslant (2c+1)\int_\Omega |\nabla u|^2 \,\mathrm{d}x + 2c\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma.$$

However, this inequality is deficient in comparison to (12.5) with respect to several features: Maz'ja's inequality holds for arbitrary open sets – this is the salient point in our application – and on the left hand side one has the stronger q-norm.

Notes

The material of this lecture comes from [AE11a] and [AE11b]. The trace may be multi-valued if Ω does not have Lipschitz boundary; see Exercise 12.4. Much more information on the Dirichlet-to-Neumann operator is contained in [AE11b].

For the case of embedded non-closable accretive *symmetric* forms, a procedure to define an associated closed form has been presented in [Sim78].

Exercises

12.1 Let H be a complex Hilbert space, a an embedded sectorial form in H. Show the following criteria for a being closed or closable:

(a) a is closed if and only if for any Cauchy sequence (u_n) in $(\mathrm{dom}(a), \|\cdot\|_a)$ with $u_n \to u$ in H one has $\|u_n - u\|_a \to 0$.

(b) a is closable if and only if for any Cauchy sequence (u_n) in $(\mathrm{dom}(a), \|\cdot\|_a)$ with $u_n \to 0$ in H one has $\|u_n\|_a \to 0$.

12.2 Let $H := L_2(-1,1)$, a_1, a_2 in H defined by $\mathrm{dom}(a_j) = C_\mathrm{c}^\infty(-1,1)$,

$$\begin{aligned} a_1(u,v) &:= u(0)\overline{v(0)}, \\ a_2(u,v) &:= \int_{-1}^{1} u'(x)\overline{v'(x)}\,\mathrm{d}x + u(0)\overline{v(0)} \end{aligned}$$

for all $u, v \in C_\mathrm{c}^\infty(-1,1)$.

For $j = 1,2$ determine whether a_j is closable. Find the completion of $(\mathrm{dom}(a_j), (\cdot\,|\,\cdot)_{a_j})$ and the operator associated with a_j.

12.3 Let $\Omega := (-1,0) \cup (0,1)$.

(a) Determine the relation tr of Section 12.5, and show that dom(tr) is not dense.

(b) Find $\partial_\nu u$ for those $u \in H^1(\Omega)$ with $\Delta u \in L_2(\Omega)$ for which $\partial_\nu u \in L_2(\partial\Omega)$.

(c) Determine the Robin-Laplacian for $\beta = 1$.

12.4 Let $S := [0,1] \times \{0\} \subseteq \mathbb{R}^2$, and let (x_n) be a bounded sequence in $\mathbb{R}^2 \setminus S$ having the set S as its cluster values. Let (r_n) be a sequence in $(0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$, $B[x_n, r_n] \cap S = \varnothing$ $(n \in \mathbb{N})$ and $B[x_n, r_n] \cap B[x_m, r_m] = \varnothing$ $(m, n \in \mathbb{N},\ m \neq n)$. Let $\Omega := \bigcup_{n\in\mathbb{N}} B(x_n, r_n)$.

(a) Determine $\partial\Omega$ and $\sigma_1(\partial\Omega)$ (1-dimensional Hausdorff measure).

(b) Show that dom(tr) is dense in $H^1(\Omega)$ and that $\operatorname{tr} 0 = L_2(S)$ (where $L_2(S) \subseteq L_2(\partial\Omega)$ is the natural embedding).

(c) Let D_0 be the Dirichlet-to-Neumann operator for Ω. Show that $L_2(S) \subseteq \operatorname{dom}(D_0)$ and that $D_0|_{L_2(S)} = 0$.

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