Lecture 11

Elliptic operators

Elliptic operators with measurable coefficients are a classical topic in partial differential equations. Their realisations under diverse boundary conditions generate semigroups and thus lead to solutions of parabolic initial boundary value problems. Form methods are most efficient to treat these problems and to derive properties of the solutions of the equations. It will become apparent that a large amount of the topics presented so far enter the treatment of these equations. In particular the properties of the Sobolev space $H^1(\Omega)$ and the invariance criteria will play a decisive role in the treatment. The latter lead to positivity and to sub-Markovian and substochastic behaviour of the generated semigroups.

In order to achieve these goals it turns out that again additional order properties of $H^1(\Omega)$ – treated in an interlude – are needed.

11.1 Perturbation of continuous forms

We start with a perturbation result. We refer to the proof of Theorem 7.15 and to Exercise 7.3 for related methods.

11.1 Lemma. *Let V, H be Hilbert spaces with $V \overset{\mathrm{d}}{\hookrightarrow} H$. Let $a\colon V \times V \to \mathbb{K}$ be an H-elliptic continuous form. Let $b\colon V \times V \to \mathbb{K}$ be a continuous form such that*

$$|b(u)| \leqslant M\|u\|_V\|u\|_H \qquad (u, v \in V).$$

Then $a + b\colon V \times V \to \mathbb{K}$ is H-elliptic.

Proof. By the H-ellipticity of a there exist $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\operatorname{Re} a(u) + \omega\|u\|_H^2 \geqslant \alpha\|u\|_V^2$$

for all $u \in V$.

By the "Peter-Paul inequality" (i.e., Young's inequality, $ab \leqslant \frac{1}{2}\big(\gamma a^2 + \frac{1}{\gamma}b^2\big)$ for all $a, b \geqslant 0,\ \gamma > 0$) one has

$$\begin{aligned}\operatorname{Re} a(u) + \operatorname{Re} b(u) + \omega\|u\|_H^2 &\geqslant \alpha\|u\|_V^2 - M\|u\|_V\|u\|_H \\ &\geqslant \alpha\|u\|_V^2 - \frac{1}{2}\Big(\alpha\|u\|_V^2 + \frac{1}{\alpha}M^2\|u\|_H^2\Big).\end{aligned}$$

This implies

$$\operatorname{Re}\big(a(u) + b(u)\big) + \Big(\omega + \frac{M^2}{2\alpha}\Big)\|u\|_H^2 \geqslant \frac{\alpha}{2}\|u\|_V^2 \qquad (u \in V). \qquad \square$$

11.2 Elliptic operators

Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $a_{jk} \in L_\infty(\Omega)$ $(j,k = 1,\dots,n)$ be coefficient functions satisfying the **ellipticity condition**

$$\operatorname{Re} \sum_{j,k=1}^{n} a_{jk}(x)\xi_k\overline{\xi_j} \geqslant \alpha|\xi|^2 \qquad (\xi \in \mathbb{K}^n) \tag{11.1}$$

for all $x \in \Omega$, with $\alpha > 0$, and let $b_j, c_j \in L_\infty(\Omega)$ $(j = 1,\dots,n)$, $d \in L_\infty(\Omega)$. Our aim is to define operators in $L_2(\Omega)$ corresponding to the "elliptic operator in divergence form" $\mathcal{A}$ written formally as

$$\begin{aligned}\mathcal{A}u &= \sum_{j,k=1}^{n} -\partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n}(b_j\partial_j u - \partial_j(c_j u)) + du \\ &= -\operatorname{div}\bigl((a_{jk})\nabla u\bigr) + b\cdot\nabla u - \operatorname{div}(cu) + du.\end{aligned} \tag{11.2}$$

We consider the form

$$a\colon H^1(\Omega)\times H^1(\Omega) \to \mathbb{K}$$

given by

$$a(u,v) = \int_\Omega \Bigl(\sum_{j,k=1}^{n} a_{jk}\partial_k u\overline{\partial_j v} + \sum_{j=1}^{n}(b_j\partial_j u\overline{v} + c_j u\overline{\partial_j v}) + du\overline{v}\Bigr)\,\mathrm{d}x.$$

11.2 Proposition. *The form a is continuous and $L_2(\Omega)$-elliptic.*

Proof. Let the form $a_0\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{K}$ be given by

$$a_0(u,v) = \int_\Omega \sum_{j,k=1}^{n} a_{jk}\partial_k u\overline{\partial_j v}\,\mathrm{d}x.$$

Then the boundedness of the coefficients a_{jk} implies that a_0 is continuous. The ellipticity condition (11.1) implies that $\operatorname{Re} a_0(u) + \alpha\|u\|_{L_2}^2 \geqslant \alpha\|\nabla u\|_{L_2}^2 + \alpha\|u\|_{L_2}^2 = \alpha\|u\|_{H^1}^2$ for all $u \in H^1(\Omega)$. Thus a_0 is $L_2(\Omega)$-elliptic.

Define $a_1\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{K}$ by

$$a_1(u,v) = \int_\Omega \Bigl(\sum_{j=1}^{n}(b_j\partial_j u\overline{v} + c_j u\overline{\partial_j v}) + du\overline{v}\Bigr)\,\mathrm{d}x.$$

Then $a = a_0 + a_1$.

The boundedness of the coefficient functions b_j, c_j and d implies that a_1 is continuous. It also implies that there exists $M \geqslant 0$ such that

$$\begin{aligned}|a_1(u)| &\leqslant M\int_\Omega |\nabla u||u|\,\mathrm{d}x + \|d\|_{L_\infty}\|u\|_{L_2}^2 \\ &\leqslant M\Bigl(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Bigr)^{\frac12}\Bigl(\int_\Omega |u|^2\,\mathrm{d}x\Bigr)^{\frac12} + \|d\|_{L_\infty}\|u\|_{L_2}^2 \\ &\leqslant (M + \|d\|_{L_\infty})\|u\|_{H^1}\|u\|_{L_2}\end{aligned}$$

for all $u \in H^1(\Omega)$. Now the claim follows from Lemma 11.1. □

Now let V be a closed subspace of $H^1(\Omega)$ containing $H^1_0(\Omega)$. Then the restriction of a to $V \times V$ is also continuous and $L_2(\Omega)$-elliptic. Denote by A_V the operator associated with $a|_{V\times V}$. Then $-A_V$ generates a C_0-semigroup T_V on $L_2(\Omega)$, and if $\mathbb{K} = \mathbb{C}$, then $-A_V$ generates a holomorphic C_0-semigroup T_V on $L_2(\Omega)$; see Section 5.3.

Coming back to (11.2), let $u \in H^1(\Omega)$ and multiply (11.2) by a test function $v \in C_c^\infty(\Omega)$. A formal integration by parts gives

$$\int_\Omega (\mathcal{A}u)\bar{v}\,\mathrm{d}x = a(u,v).$$

So we might say that for $u \in H^1(\Omega)$ one has $\mathcal{A}u = f \in L_2(\Omega)$ if

$$\int_\Omega f\bar{v}\,\mathrm{d}x = a(u,v) \qquad (v \in C_c^\infty(\Omega)).$$

This leads to the definition of the maximal operator

$$A_{\max} := \Big\{(u,f) \in H^1(\Omega) \times L_2(\Omega)\,;\ a(u,v) = \int f\bar{v}\,\mathrm{d}x\ (v \in C_c^\infty(\Omega))\Big\}.$$

11.3 Remark. For each closed subspace $V \subseteq H^1(\Omega)$ with $H^1_0(\Omega) \subseteq V$ the operator A_V is a restriction of $A_{\max}$. For $(u,f) \in A_{\max}$ to be in A_V it is needed that
(i) $u \in V$ (and not merely for $u \in H^1(\Omega)$),
(ii) $a(u,v) = (f \,|\, v)_{L_2}$ for all $v \in V$ (and not merely for $v \in C_c^\infty(\Omega)$).
These conditions can be interpreted as a boundary condition.

11.4 Examples. (a) Let us first consider the case $V = H^1_0(\Omega)$. Then $A_{H^1_0}$ is just the restriction of $A_{\max}$ to $\mathrm{dom}(A_{\max}) \cap H^1_0(\Omega)$. We write $A_\mathrm{D} := A_{H^1_0}$ and call A_D the realisation of the elliptic operator $\mathcal{A}$ with **Dirichlet boundary conditions**. We define $T_\mathrm{D} := T_{H^1_0}$.

(b) Next we consider $V = H^1(\Omega)$. We define $T_\mathrm{N} := T_{H^1}$ and call $A_\mathrm{N} := A_{H^1}$ the realisation of the elliptic operator $\mathcal{A}$ with **Neumann boundary conditions**. However, it is not the normal derivative of u which is 0 at the boundary, but rather the **conormal derivative**

$$\sum_{j,k=1}^n (a_{jk}\partial_k u)\nu_j + \sum_{j=1}^n c_j u \nu_j$$

on $\partial\Omega$, which is defined in terms of the coefficients of $\mathcal{A}$.

Clearly, the conormal derivative has no meaning for general domains, but even for nice domains there is the problem that the partial derivatives of u occur, and there is no reason why they should have a trace on $\partial\Omega$. We will motivate the Neumann boundary condition involving the conormal derivative under suitable "smooth hypotheses".

Assume that Ω is bounded and has C^1-boundary. Assume that $a_{jk} \in C^1(\overline{\Omega})$, $c_j \in C^1(\overline{\Omega})$, $b_j \in C(\overline{\Omega})$, $d \in C(\overline{\Omega})$. Let $u \in \mathrm{dom}(A_\mathrm{N})$, $A_\mathrm{N}u = f$, and assume addititionally that $u \in C^2(\overline{\Omega})$. Since $A_{\max}u = f$ by Remark 11.3 and since $a_{jk}\partial_k u \in C^1(\overline{\Omega})$, $c_j u \in C^1(\overline{\Omega})$, partial integration yields

$$f = -\sum_{j,k=1}^n \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^n b_j\partial_j u - \sum_{j=1}^n \partial_j(c_j u) + du.$$

By the definition of the associated operator A_{N} we have

$$\begin{aligned}\int_\Omega f\overline{v}\,\mathrm{d}x &= a(u,v)\\ &= \int_\Omega\Big(\sum_{j,k=1}^n a_{jk}\partial_k u\overline{\partial_j v}+\sum_{j=1}^n b_j\partial_j u\overline{v}+\sum_{j=1}^n c_j u\overline{\partial_j v}+du\overline{v}\Big)\,\mathrm{d}x\end{aligned}$$

for all $v\in C^1(\overline{\Omega})\subseteq H^1(\Omega)$. Applying Gauss' theorem (Theorem 7.4) we deduce that

$$\int_\Omega f\overline{v}\,\mathrm{d}x = \int_\Omega f\overline{v}\,\mathrm{d}x + \int_{\partial\Omega}\Big(\sum_{j,k=1}^n (a_{jk}\partial_k u)\nu_j\Big)\overline{v}\,\mathrm{d}\sigma + \int_{\partial\Omega}\Big(\sum_{j=1}^n c_j u\nu_j\Big)\overline{v}\,\mathrm{d}\sigma$$

for all $v\in C^1(\overline{\Omega})\subseteq H^1(\Omega)$, where $\nu=(\nu_1,\dots,\nu_n)$ is the outer normal. (These computations correspond to the argument given at the beginning of Section 7.4 concerning the Neumann boundary condition for the Laplacian.) This implies that

$$\sum_{j=1}^n\Big(\sum_{k=1}^n a_{jk}\partial_k u + c_j u\Big)\nu_j = \big((a_{jk})\nabla u + cu\big)\cdot\nu = 0 \quad \text{on } \partial\Omega. \tag{11.3}$$

Thus (11.3) is the Neumann boundary condition we are realising by the form a on $H^1(\Omega)\times H^1(\Omega)$.

(c) There are other possible choices of V. For example, assume that Ω has C^1-boundary, and let $\Gamma\subseteq\partial\Omega$ be a Borel set. Then we may consider

$$V=\big\{u\in H^1(\Omega);\ \operatorname{tr}u=0 \text{ on } \Gamma\big\}.$$

We call A_V the realisation of the elliptic operator $\mathcal{A}$ with **mixed boundary conditions** (i.e., Dirichlet on Γ, Neumann on $\partial\Omega\setminus\Gamma$).

11.5 Remarks. Assume that the coefficient matrix $(a_{jk}(x))$ is self-adjoint, i.e., $a_{jk}(x)=\overline{a_{kj}(x)}$, for all $x\in\Omega$.

(a) Then the ellipticity condition (11.1) says that the smallest eigenvalue of each matrix $(a_{jk}(x))$ is $\geqslant\alpha$.

(b) If $b_j=\overline{c_j}$ $(j=1,\dots,n)$ and d is real-valued, then a is symmetric and the operator A_V is self-adjoint.

We conclude this section with comments on the interplay between the real and complex cases. Obviously, in order to obtain a holomorphic C_0-semigroup one has to work with the field $\mathbb{C}$.

11.6 Remark. Assume that all the coefficient functions a_{jk},b_j,c_j,d are real-valued. Assume that the ellipticity condition (11.1) is satisfied. (Take notice of Exercise 11.1!)

(a) Let V be a closed subspace of $H^1(\Omega;\mathbb{C})$ containing $H^1_0(\Omega;\mathbb{C})$ with the property that $u\in V$ implies $\operatorname{Re}u\in V$. Then $T_V(t)$ leaves $L_2(\Omega;\mathbb{R})$ invariant, for $t\geqslant 0$; this follows from Theorem 10.12(a).

Let $V_{\mathbb{R}} = V \cap L_2(\Omega; \mathbb{R})$. It is clear that the restriction of a to $V_{\mathbb{R}} \times V_{\mathbb{R}}$ is again continuous and elliptic and that minus the generator of T_V restricted to $L_2(\Omega; \mathbb{R})$ is the operator associated with $a|_{V_{\mathbb{R}} \times V_{\mathbb{R}}}$.

(b) Conversely, if V is a closed subspace of $H^1(\Omega; \mathbb{R})$ containing $H_0^1(\Omega; \mathbb{R})$, then $V_{\mathbb{C}} = V \oplus \mathrm{i}V$ is a space as we considered in (a).

In order to study further properties of the semigroup on $L_2(\Omega; \mathbb{R})$ generated by an elliptic operator with real coefficients we need additional properties of the Sobolev space $H^1(\Omega)$.

11.3 Interlude: Further order properties of $H^1(\Omega)$

In this section let $\mathbb{K} = \mathbb{R}$. Let $\Omega \subseteq \mathbb{R}^n$ be open. In Section 9.3 we have seen that $H^1(\Omega)$ is a sublattice of $L_2(\Omega)$. More precisely, if $u \in H^1(\Omega)$, then Theorem 9.14 implies that $u^+, u^- = (-u)^+, |u| = u^+ + u^- \in H^1(\Omega)$ and $\partial_j u^+ = \mathbf{1}_{[u>0]}\partial_j u$, $\partial_j u^- = -\mathbf{1}_{[u<0]}\partial_j u$, $\partial_j |u| = \partial_j u^+ + \partial_j u^- = (\operatorname{sgn} u)\partial_j u$. It follows that

$$\||u|\|_{H^1(\Omega)} = \|u\|_{H^1(\Omega)} \qquad (u \in H^1(\Omega)) \tag{11.4}$$

since $\partial_j u = \partial_j u^+ - \partial_j u^- = \mathbf{1}_{[u\neq 0]}\partial_j u$ $(j = 1, \ldots, n)$. Incidentally, this last equality implies that $\mathbf{1}_{[u=0]}\partial_j u = 0$, i.e., $\partial_j u = 0$ a.e. on $[u = 0]$ ("Stampacchia-Lemma"); see also Remark 9.10.

Next we show that the lattice operations are continuous.

11.7 Proposition. (a) *The mapping $H^1(\Omega) \ni u \mapsto |u| \in H^1(\Omega)$ is continuous.*

(b) *The mappings $(u, v) \mapsto u \wedge v$ and $(u, v) \mapsto u \vee v$ are continuous from $H^1(\Omega) \times H^1(\Omega)$ to $H^1(\Omega)$. In particular, the mappings $H^1(\Omega) \ni u \mapsto u^+, u^- \in H^1(\Omega)$ are continuous.*

Proof. (a) Let (u_k) be a sequence in $H^1(\Omega)$, $u_k \to u$ in $H^1(\Omega)$. Then $|u_k| \to |u|$ in $L_2(\Omega)$, and (11.4) implies that $(|u_k|)$ is bounded in $H^1(\Omega)$. Therefore Remark 9.15 implies that $|u_k| \to |u|$ weakly in $H^1(\Omega)$.

From (11.4) we also obtain

$$\||u_k|\|_{H^1} = \|u_k\|_{H^1} \to \|u\|_{H^1} = \||u|\|_{H^1} \quad (k \to \infty),$$

and this implies that $|u_k| \to |u|$ in $H^1(\Omega)$; see the subsequent Remark 11.8.

(b) follows from $u \wedge v = \frac{1}{2}(u + v - |u - v|)$, $u \vee v = \frac{1}{2}(u + v + |u - v|)$ and part (a). □

11.8 Remark. Let H be a Hilbert space, (u_k) a sequence in H, $u_k \to u$ weakly in H, and $\|u_k\| \to \|u\|$ $(k \to \infty)$. Then

$$\|u_k - u\|^2 = \|u_k\|^2 + \|u\|^2 - 2\operatorname{Re}(u_k \,|\, u) \to 0;$$

so $u_k \to u$ in H $(k \to \infty)$.

For a function $u \colon \Omega \to \mathbb{R}$ we denote by $\tilde{u} \colon \mathbb{R}^n \to \mathbb{R}$ the extension of u by 0,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

From Subsection 4.1.4 we recall the definitions

$$\begin{aligned} H^1_{\mathrm{c}}(\Omega) &:= \big\{ f \in H^1(\Omega);\ \operatorname{spt} f \text{ compact in } \Omega \big\}, \\ H^1_0(\Omega) &:= \overline{H^1_{\mathrm{c}}(\Omega)}^{H^1(\Omega)} \end{aligned}$$

and the denseness property $H^1_0(\Omega) = \overline{C^\infty_{\mathrm{c}}(\Omega)}^{H^1(\Omega)}$. We also recall that for $u \in H^1_0(\Omega)$ one has $\tilde{u} \in H^1(\mathbb{R}^n)$, $\partial_j \tilde{u} = \widetilde{\partial_j u}$ $(j = 1, \dots, n)$; see Remark 4.11(b) and Exercise 4.2. (In fact, the formula for the derivative is not stated explicitly, but is implicit in the proof.) We already know that $H^1_0(\Omega)$ is a sublattice of $H^1(\Omega)$ (see Theorem 9.14). Now we show that it is even an ideal (in the sense of vector lattices).

11.9 Proposition. *Let $u \in H^1_0(\Omega)$, $v \in H^1(\Omega)$, $0 \leqslant v \leqslant u$. Then $v \in H^1_0(\Omega)$.*

Proof. There exists a sequence (u_k) in $H^1_{\mathrm{c}}(\Omega)$ such that $u_k \to u$ in $H^1(\Omega)$. It follows from Proposition 11.7 that $u_k^+ \wedge v \to u^+ \wedge v = v$ in $H^1(\Omega)$ as $k \to \infty$. Since $u_k^+ \wedge v \in H^1_{\mathrm{c}}(\Omega)$ we conclude that $v \in H^1_0(\Omega)$. □

11.10 Proposition. *Let Ω be bounded and with C^1-boundary. Let $\Gamma \subseteq \partial\Omega$ be a Borel subset, and let V be as in Example 11.4(c). Then V is a sublattice of $H^1(\Omega)$, and $u \in V$, $v \in H^1(\Omega)$, $0 \leqslant v \leqslant u$ implies $v \in V$.*

Proof. The fundamental observation is that the mapping $\operatorname{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$ is a continuous lattice homomorphism. This means that additionally to continuity and linearity one also has $\operatorname{tr}(u \vee v) = \operatorname{tr} u \vee \operatorname{tr} v$ $(u, v \in H^1(\Omega))$. This last property is clear for $u, v \in H^1(\Omega) \cap C(\overline{\Omega})$ and carries over to $H^1(\Omega)$ by denseness and continuity; see Theorems 7.7 and 7.9.

Therefore, if $u \in V$, then $\operatorname{tr}(u^+)|_\Gamma = (\operatorname{tr} u)^+|_\Gamma = 0$, i.e., $u^+ \in V$. This implies that V is a sublattice.

Let u, v be as in the statement of the proposition. Then $0 \leqslant \operatorname{tr} v|_\Gamma \leqslant \operatorname{tr} u|_\Gamma = 0$. This shows that $v \in V$. □

We will also need the following denseness properties.

11.11 Lemma. *The set $C^\infty_{\mathrm{c}}(\Omega)_+ := \{\varphi \in C^\infty_{\mathrm{c}}(\Omega);\ \varphi \geqslant 0\}$ is dense in $H^1_0(\Omega)_+ := \{u \in H^1_0(\Omega);\ u \geqslant 0\}$.*

Proof. (i) In this step we show that $H^1_{\mathrm{c}}(\Omega)_+ := \{u \in H^1_{\mathrm{c}}(\Omega);\ u \geqslant 0\}$ is dense in $H^1_0(\Omega)_+$.

Let $u \in H^1_0(\Omega)_+$. There exists a sequence (u_k) in $H^1_{\mathrm{c}}(\Omega)$ such that $u_k \to u$. Then clearly $u_k^+ \in H^1_{\mathrm{c}}(\Omega)_+$ for all $k \in \mathbb{N}$, and Proposition 11.7 implies that $u_k^+ \to u^+ = u$.

(ii) Let $u \in H^1_{\mathrm{c}}(\Omega)_+$. Let (ρ_k) be a δ-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. It was shown in the proof of Theorem 4.12(b) that then $\rho_k * \tilde{u} \in C^\infty_{\mathrm{c}}(\Omega)$ for large k, and $(\rho_k * \tilde{u})|_\Omega \to \tilde{u}$ in $H^1(\Omega)$. Clearly $\rho_k * \tilde{u} \geqslant 0$ for all $k \in \mathbb{N}$, and this shows that $u \in \overline{C^\infty_{\mathrm{c}}(\Omega)_+}^{H^1(\Omega)}$. □

11.12 Proposition. *Let Ω be bounded and with continuous boundary. Then $\check{C}^\infty(\Omega)_+ := \big\{u \in \check{C}^\infty(\Omega);\ u \geqslant 0\big\}$ is dense in $H^1(\Omega)_+ := \{u \in H^1(\Omega);\ u \geqslant 0\}$.*

Proof. We refer to the proof of Theorem 7.7. Following the lines of this proof one can see that, starting with a function $u \in H^1(\Omega)_+$, in all the steps one stays in the realm of positive functions. As a consequence, the approximating function constructed in the proof of Theorem 7.7 will be positive. □

11.4 Elliptic operators with real coefficients

In this section we assume throughout that $\mathbb{K} = \mathbb{R}$. Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $a_{jk}, b_j, c_j, d \in L_\infty(\Omega)$ (all real-valued) be as in Section 11.2; in particular, we assume that the ellipticity condition (11.1) is satisfied.

A **vector sublattice** of $H^1(\Omega)$ is a subspace with the property that $u \in V$ implies $u^+ \in V$.

11.13 Proposition. *Let V be a closed vector sublattice of $H^1(\Omega)$ containing $H_0^1(\Omega)$. Then the semigroup T_V generated by $-A_V$ is positive.*

Proof. Let $u \in H^1(\Omega)$. Then Theorem 9.14 implies $\partial_j u^+ = \mathbf{1}_{[u>0]}\partial_j u$, $\partial_j u^- = -\mathbf{1}_{[u<0]}\partial_j u$; therefore $\partial_k u^+ \partial_j u^- = 0$, $\partial_j u^+ u^- = 0$, $u^+ \partial_j u^- = 0$, $u^+ u^- = 0$. Thus $a(u^+, u^-) = 0$. Now it follows from Theorem 10.12(b) that T_V is positive. □

Proposition 11.13 implies that the semigroup is positive for Dirichlet, Neumann and mixed boundary conditions; see Theorem 9.14 and Proposition 11.10.

Next we show additional properties for Dirichlet boundary conditions.

11.14 Proposition. (a) *Assume that $c = (c_1, \ldots, c_n) \in C^1(\Omega; \mathbb{R}^n)$ and $\operatorname{div} c \leqslant d$. Then T_{D} is sub-Markovian.*

(b) *Let $b = (b_1, \ldots, b_n) \in C^1(\Omega; \mathbb{R}^n)$, $\operatorname{div} b \leqslant d$. Then T_{D} is substochastic.*

Proof. (a) Let $u \in H_0^1(\Omega)$. Then $u \wedge 1 \in H_0^1(\Omega)$ and $\partial_j(u \wedge 1) = \mathbf{1}_{[u<1]}\partial_j u$, by Theorem 9.14. Since $u = u \wedge 1 + (u-1)^+$, it follows that $(u-1)^+ \in H_0^1(\Omega)$ and $\partial_j (u-1)^+ = \mathbf{1}_{[u\geqslant 1]}\partial_j u$. Thus $\partial_k (u \wedge 1)\, \partial_j (u-1)^+ = 0$ and $\partial_j (u \wedge 1)\, (u-1)^+ = 0$. It follows that

$$\begin{aligned} a(u \wedge 1, (u-1)^+) &= \int_\Omega \Big(\sum_{j=1}^n c_j (u \wedge 1)\, \partial_j (u-1)^+ + d(u \wedge 1)(u-1)^+ \Big)\, \mathrm{d}x \\ &= \int_\Omega \Big(\sum_{j=1}^n c_j \partial_j (u-1)^+ + d(u-1)^+ \Big)\, \mathrm{d}x \end{aligned}$$

(where the last equality holds because $u \wedge 1 = 1$ on $[u \geqslant 1]$). From the hypotheses we obtain

$$\int_\Omega \Big(\sum_{j=1}^n c_j \partial_j \varphi + d\varphi \Big)\, \mathrm{d}x = \int_\Omega \Big(-\sum_{j=1}^n \partial_j c_j + d \Big) \varphi \, \mathrm{d}x \geqslant 0$$

for all $0 \leqslant \varphi \in C_\mathrm{c}^\infty(\Omega)$. Since $(u-1)^+$ can be approximated by positive test functions, by Lemma 11.11, it follows that $a(u \wedge 1, (u-1)^+) \geqslant 0$. Now it follows from Theorem 10.12(c) that T_{D} is sub-Markovian.

(b) The proof is analogous to (a) and uses Theorem 10.12(d). □

11.15 Remark. In Proposition 11.14(a) it would be sufficient to require the validity of $\operatorname{div} c \leqslant d$ in the distributional sense, without requiring c to be differentiable, and similarly for b in Proposition 11.14(b). (A corresponding statement for the subsequent result would be somewhat more subtle, because of the occurence of the boundary terms.)

Next we consider boundary conditions which are defined by more general spaces V.

11.16 Proposition. *Assume that* Ω *is bounded and has* C^1*-boundary. Assume that* $u \in V$ *implies* $u \wedge 1 \in V$.

(a) *If* $c \in C^1(\overline{\Omega}; \mathbb{R}^n)$, $\operatorname{div} c \leqslant d$ *on* Ω *and* $c \cdot \nu \geqslant 0$ *on* $\partial\Omega$, *then* T_V *is sub-Markovian.*

(b) *If* $b \in C^1(\overline{\Omega}; \mathbb{R}^n)$, $\operatorname{div} b \leqslant d$ *on* Ω *and* $b \cdot \nu \geqslant 0$ *on* $\partial\Omega$, *then* T_V *is substochastic. (As before,* $\nu(z) = (\nu_1(z), \dots, \nu_n(z))$ *denotes the exterior normal at* $z \in \partial\Omega$.*)*

Proof. (a) As in the proof of Proposition 11.14 one has

$$a(u \wedge 1, (u-1)^+) = \int_\Omega \Big(\sum_{j=1}^n c_j \partial_j (u-1)^+ + d(u-1)^+ \Big) \,\mathrm{d}x \qquad (u \in V).$$

For $0 \leqslant \varphi \in C^1(\overline{\Omega})$ we now have by Gauss' theorem (Theorem 7.4) that

$$\int_\Omega \Big(\sum_{j=1}^n c_j \partial_j \varphi + d\varphi \Big) \,\mathrm{d}x = \int_\Omega \Big(-\sum_{j=1}^n \partial_j c_j + d \Big) \varphi \,\mathrm{d}x + \int_{\partial\Omega} \sum_{j=1}^n \nu_j c_j \varphi \,\mathrm{d}\sigma \geqslant 0.$$

By approximation – applying Proposition 11.12 – we deduce that $a(u \wedge 1, (u-1)^+) \geqslant 0$.

The proof of (b) is analogous. □

11.17 Remarks. Let Ω and V be as in Proposition 11.16, and assume that $\mathbf{1}_\Omega \in V$.

(a) If in Proposition 11.16(a) one has the equalities

$$\operatorname{div} c = d \quad \text{on } \Omega, \qquad c \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

then $T_V(t)\mathbf{1}_\Omega = \mathbf{1}_\Omega$ for all $t \geqslant 0$. This means that T_V is not only sub-Markovian but **Markovian**.

Indeed, the proof of Proposition 11.16 shows that $a(\mathbf{1}_\Omega, v) = 0$ for all $v \in C^1(\overline{\Omega})$ and hence for all $v \in H^1(\Omega)$, by Proposition 11.12. This implies that $\mathbf{1}_\Omega \in \operatorname{dom}(A_V)$ and $A_V \mathbf{1}_\Omega = 0$. Now Theorem 1.12(a) implies the assertion.

(b) Similarly, if in Proposition 11.16(b) one has the equalities

$$\operatorname{div} b = d \quad \text{on } \Omega, \qquad b \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

then T_V is not only substochastic but **stochastic**, i.e., $\|T_V(t)u\|_1 = \|u\|_1$ for all $0 \leqslant u \in L_2 \cap L_1(\Omega)$, $t \geqslant 0$. See Exercise 11.4.

11.18 Remark. Deviating from the initial announcement of this section that only $\mathbb{K} = \mathbb{R}$ is treated, we include a comment on the complex case. It should be noticed that then Theorems 10.8 and 10.15 are applicable to the situations treated in Propositions 11.14 and 11.16 and that they yield L_p-properties and holomorphic extensions for the generated semigroups.

11.5 Domination

We assume $\mathbb{K} = \mathbb{R}$ throughout this section. Let $\Omega \subseteq \mathbb{R}^n$ be open, V a closed vector sublattice of $H^1(\Omega)$ containing $H^1_0(\Omega)$. Recall from Theorem 11.13 that then the semigroup T_V generated by $-A_V$ is positive. We recall the notation $A_{\mathrm{D}}, T_{\mathrm{D}}, A_{\mathrm{N}}, T_{\mathrm{N}}$ from Examples 11.4.

11.19 Theorem. *For all* $t \geqslant 0$ *one has* $0 \leqslant T_{\mathrm{D}}(t) \leqslant T_V(t)$*, i.e.,* $0 \leqslant T_{\mathrm{D}}(t)f \leqslant T_V(t)f$ *for all* $0 \leqslant f \in L_2(\Omega)$*.*

Proof. Because of the exponential formula, Theorem 2.12, it suffices to show the domination property for the resolvents; i.e., for large λ we have to show that

$$(\lambda + A_{\mathrm{D}})^{-1} f \leqslant (\lambda + A_V)^{-1} f$$

for all $0 \leqslant f \in L_2(\Omega)$. Adding λ to the coefficient d we may assume $\lambda = 0$ and also that the form a is coercive on $H^1(\Omega) \times H^1(\Omega)$.

Let $0 \leqslant f \in L_2(\Omega)$, $u_1 = A_{\mathrm{D}}^{-1} f$, $u_2 = A_V^{-1} f$. We know that $u_1 \in H^1_0(\Omega)_+$, $u_2 \in V_+ := V \cap L_2(\Omega)_+$ and

$$\begin{aligned} a(u_1, v) &= (f \mid v)_{L_2} \quad \text{for all } v \in H^1_0(\Omega), \\ a(u_2, v) &= (f \mid v)_{L_2} \quad \text{for all } v \in V. \end{aligned}$$

Thus $a(u_1 - u_2, v) = 0$ for all $v \in H^1_0(\Omega)$. Observe that $0 \leqslant (u_1 - u_2)^+ \leqslant u_1$. Thus $(u_1 - u_2)^+ \in H^1_0(\Omega)$ by the ideal property Proposition 11.9. Taking $v := (u_1 - u_2)^+$ we obtain $a(u_1 - u_2, (u_1 - u_2)^+) = 0$. But $a(w^-, w^+) = 0$ for all $w \in H^1(\Omega)$ as we had seen in the proof of Proposition 11.13. Thus $a((u_1 - u_2)^+) = 0$, which implies $(u_1 - u_2)^+ = 0$ by our coerciveness assumption. Consequently, $u_1 \leqslant u_2$. We have shown that $A_{\mathrm{D}}^{-1} \leqslant A_V^{-1}$. ☐

11.20 Theorem. *Assume additionally that* V *is an ideal in* $H^1(\Omega)$*, i.e.,* $u \in V$*,* $v \in H^1(\Omega)$*,* $0 \leqslant v \leqslant u$ *implies* $v \in V$*. Then* $T_V(t) \leqslant T_{\mathrm{N}}(t)$ *for all* $t \geqslant 0$*.*

The proof is analogous to the proof of Theorem 11.19; see Exercise 11.5.

If A_V is the elliptic operator with mixed boundary conditions, then V is an ideal in $H^1(\Omega)$, by Proposition 11.10, and so $T_V(t) \leqslant T_{\mathrm{N}}(t)$ for all $t \geqslant 0$.

Finally we want to prove domain monotonicity for Dirichlet boundary conditions. Let $\Omega_1 \subseteq \Omega$ be open. We consider the semigroup T_{D} on $L_2(\Omega)$. But we may also restrict the coefficients to Ω_1 and consider the semigroup T^1_{D} on $L_2(\Omega_1)$. We identify $L_2(\Omega_1)$ with a subspace of $L_2(\Omega)$ by extending functions in $L_2(\Omega_1)$ by 0 on $\Omega \setminus \Omega_1$.

11.21 Theorem. *One has* $T^1_{\mathrm{D}}(t)f \leqslant T_{\mathrm{D}}(t)f$ *for all* $0 \leqslant f \in L_2(\Omega_1)$*,* $t \geqslant 0$*.*

Proof. By the exponential formula it suffices to show that $(\lambda + A^1_{\mathrm{D}})^{-1} f \leqslant (\lambda + A_{\mathrm{D}})^{-1} f$ on Ω_1 for large enough λ and $0 \leqslant f \in L_2(\Omega_1)$. As in the proof of Theorem 11.19 we may assume that the form a is coercive and $\lambda = 0$.

Let $0 \leqslant f \in L_2(\Omega_1)$, $u_1 = (A^1_{\mathrm{D}})^{-1} f$, $u_2 = (A_{\mathrm{D}})^{-1} f$. Then $u_1 \in H^1_0(\Omega_1)_+$, $u_2 \in H^1_0(\Omega)_+$ and

$$\begin{aligned} a(u_1, v) &= \int_{\Omega_1} f v \,\mathrm{d}x \quad \text{for all } v \in H^1_0(\Omega_1), \\ a(u_2, v) &= \int_{\Omega} f v \,\mathrm{d}x \quad \text{for all } v \in H^1_0(\Omega). \end{aligned}$$

Observe that for $v \in H_0^1(\Omega_1)$ the extension $\tilde{v}$ is in $H_0^1(\Omega)$ and $\partial_j \tilde{v} = \widetilde{\partial_j v}$ for all $j = 1, \ldots, n$ (see the paragraph before Proposition 11.9). Thus $a(u_1 - u_2, v) = 0$ for all $v \in H_0^1(\Omega_1)$. Since $u_1, u_2 \geqslant 0$ one has $0 \leqslant (u_1 - u_2)^+ \leqslant u_1$. It follows from Proposition 11.9 that $v := (u_1 - u_2)^+ \in H_0^1(\Omega_1)$. Hence $a(u_1 - u_2, (u_1 - u_2)^+) = 0$. Since $a((u_1 - u_2)^-, (u_1 - u_2)^+) = 0$ it follows that $a((u_1 - u_2)^+) = 0$ and hence $(u_1 - u_2)^+ = 0$ by the coerciveness assumption. Thus $u_1 \leqslant u_2$. $\square$

Notes

The application of invariance criteria – in the form of the Beurling-Deny criteria – to symmetric elliptic operators, in particular in connection with the heat equation with potential ("Schrödinger semigroups"), has a longer history; see for instance [RS78], [Dav89]. The application to non-symmetric operators seems to start with [MR92], [Ouh92], [Ouh96]; see [Ouh05] and [Are06] for more recent presentations. Domination is also considered in the above references. It is interesting that the conditions for domination given in Exercise 11.5 are also necessary; see [MVV05; Corollary 4.3]. We refer to [MVV05] and the literature quoted there for the treatment of more general domination results, which are then treated in the context of invariance criteria.

Perturbations as in Lemma 11.1 play an important role for the second order equation (cosine functions), see [ABHN11; Chapter 7 and Section 3.14] as well as the corresponding notes.

We note that the treatment of second order elliptic operators by forms is particularly effective for operators in divergence form, as written in (11.2). This nomenclature concerns the second order part of the operator. Transforming an expression $\sum_{j,k=1}^n a_{jk}\partial_j\partial_k u$ into divergence form would require differentiability properties of a_{jk} and produce first order terms (called drift terms).

Exercises

11.1 Let $(a_{jk}) \in \mathbb{R}^{n\times n}$, $\alpha > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}\xi_k\xi_j \geqslant \alpha|\xi|^2$$

for all $\xi \in \mathbb{R}^n$. Show that

$$\operatorname{Re}\sum_{j,k=1}^{n} a_{jk}\xi_k\overline{\xi_j} \geqslant \alpha|\xi|^2$$

for all $\xi \in \mathbb{C}^n$.

11.2 Let $\Omega \subseteq \mathbb{R}^2$ be open, $(a_{jk}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $b = c = 0$, $d = 0$.

(a) Show that $A_{\mathrm{D}} = -\Delta_{\mathrm{D}}$.

(b) Assume that Ω is bounded with C^1-boundary. Find the conormal derivative corresponding to A_{N}; cf. Example 11.4(b). Find Ω with $A_{\mathrm{N}} \neq -\Delta_{\mathrm{N}}$. Can one see that for all of these $\Omega \neq \varnothing$ one has $A_{\mathrm{N}} \neq -\Delta_{\mathrm{N}}$?

11.3 Let a_{jk}, b_j, c_j, d be as in Section 11.2.

(a) Assume additionally that $b \in C_{\mathrm{b}}^1(\Omega; \mathbb{K}^n)$ (bounded derivatives!). Let the formal elliptic operators $\mathcal{A}_1, \mathcal{A}_2$, in the sense of (11.2), be defined by

$$\mathcal{A}_1 u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + b \cdot (\nabla u), \quad \mathcal{A}_2 u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \operatorname{div}(bu) - (\operatorname{div} b)u.$$

Show that $A_{1,\mathrm{D}} = A_{2,\mathrm{D}}$.

(b) Assume additionally that $b, c \in C_{\mathrm{b}}^1(\Omega; \mathbb{K}^n)$, $c = b$, and let $\mathcal{A}$ be defined by

$$\mathcal{A} u := -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + b \cdot (\nabla u) - \operatorname{div}(cu).$$

Show that A_{D} is associated with a formal elliptic operator without drift terms.

11.4 Prove Remark 11.17(b).

11.5 (a) Let $\mathbb{K} = \mathbb{R}$, (Ω, μ) a measure space, $H := L_2(\mu)$, V, W Hilbert spaces, $V \overset{\mathrm{d}}{\hookrightarrow} H$, $W \overset{\mathrm{d}}{\hookrightarrow} H$ and let $a\colon V \times V \to \mathbb{R}$, $b\colon W \times W \to \mathbb{R}$ be continuous bilinear forms, both H-elliptic. Denote by A the operator associated with a and by B the operator associated with b. Assume that the semigroups T generated by $-A$ and S generated by $-B$ are both positive. Assume

(i) $V \subseteq W$, and if $v \in V$, $w \in W$, $0 \leqslant w \leqslant v$, then $w \in V$;

(ii) $a(u,v) \geqslant b(u,v)$ for all $0 \leqslant u, v \in V$.

Show that $T(t) \leqslant S(t)$ $(t \geqslant 0)$.

(b) Prove Theorem 11.20.

References

[ABHN11] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander: *Vector-valued Laplace transforms and Cauchy problems.* Second edition. Birkhäuser, Basel, 2011.

[Are06] W. Arendt: Heat Kernels. Internet Seminar 2005/06. 2006. URL: `www.uni-ulm.de/mawi/iaa/members/professors/arendt.html`.

[Dav89] E. B. Davies: *Heat kernels and spectral theory.* Cambridge University Press, Cambridge, 1989.

[MR92] Z.-M. Ma and M. Röckner: *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Springer-Verlag, Berlin, 1992.

[MVV05] A. Manavi, H. Vogt, and J. Voigt: Domination of semigroups associated with sectorial forms. *J. Operator Theory* **54**, 9–25 (2005).

[Ouh05] E. M. Ouhabaz: *Analysis of Heat Equations on Domains.* Princeton University Press, Princeton, 2005.

[Ouh92] E. M. Ouhabaz: L^∞-contractivity of semigroups generated by sectorial forms. *J. London Math. Soc.* **(2) 46**, 529–542 (1992).

[Ouh96] E. M. Ouhabaz: Invariance of closed convex sets and domination criteria for semigroups. *Potential Anal.* **5**, 611–625 (1996).

[RS78] M. Reed and B. Simon: *Methods of Modern Mathematical Physics IV: Analyis of Operators.* Academic Press, New York, 1978.