Lecture 10

Interpolation of holomorphic semigroups

In the first section of this lecture we will present an extremely powerful tool of Functional Analysis, important in many areas: complex interpolation. It should be looked upon as the surprising fact that (elementary) complex methods are a useful tool for deriving inequalities. The main result is the Stein interpolation theorem; as a particular case one also obtains the famous and important Riesz-Thorin interpolation theorem.

For us, the important consequence will be that a holomorphic semigroup on some L_{p_1} that is bounded on some L_{p_0} for real times can be 'interpolated' holomorphically to other L_p's. In the last section we demonstrate the interplay of invariance, interpolation and duality in applications to C_0-semigroups on L_2.

10.1 Interlude: the Stein interpolation theorem

Throughout this section the scalar field will be $\mathbb{K} = \mathbb{C}$.

10.1.1 The three lines theorem

The content of this section is a version of the maximum principle for holomorphic functions on an unbounded set. First we recall the maximum principle.

If $\Omega \subseteq \mathbb{C}$ is a bounded open set, and $h\colon \overline{\Omega} \to \mathbb{C}$ is continuous and holomorphic on Ω, then $\|h\|_{\overline{\Omega}} \leqslant \|h\|_{\partial\Omega}$. This is an easy consequence of Cauchy's integral formula. Here and in the following we denote by $\|\cdot\|_M$ the supremum norm taken over the set M.

In this and the following section the set $S \subseteq \mathbb{C}$ will be the open strip

$$S := \{z \in \mathbb{C};\, 0 < \operatorname{Re} z < 1\}.$$

10.1 Theorem. *(Three lines theorem) Let $h\colon \overline{S} \to \mathbb{C}$ be continuous and bounded, and $h|_S$ holomorphic. Then*

$$\|h\|_{\overline{S}} \leqslant \|h\|_{\partial S}.$$

Proof. For $n \in \mathbb{N}$, the function $\psi_n(z) := \frac{n}{z+n}$ is continuous on $\overline{S}$ and holomorphic on S. From the maximum principle one obtains

$$\|\psi_n h\|_{\overline{S_k}} \leqslant \|\psi_n h\|_{\partial S_k} \qquad (k \in \mathbb{N}),$$

where $S_k := \{x \in S;\ |\mathrm{Im}\, z| < k\}$. As $\|\psi_n h\|_{\{z\in\partial S_k;\ |\mathrm{Im}\, z|=k\}} \to 0$ $(k \to \infty)$, we conclude that

$$\|\psi_n h\|_{\overline{S}} \leqslant \|\psi_n h\|_{\partial S} \leqslant \|h\|_{\partial S} \qquad (n \in \mathbb{N}).$$

Letting $n \to \infty$ we obtain the assertion. □

10.2 Remark. Taking more astute functions ψ_n one may weaken the assumption that h is bounded; it suffices that $|h(z)| \leqslant c\, \mathrm{e}^{\mathrm{e}^{\alpha|\mathrm{Im}\, z|}}$ with $c \geqslant 0$ and $\alpha < \pi$ (see Exercise 10.1).

It is of interest to distinguish between the suprema of h at $[\mathrm{Re} = 0]$ and $[\mathrm{Re} = 1]$. This is expressed in the following statement.

10.3 Corollary. *Let h be as in Theorem 10.1. Then*

$$\|h\|_{[\mathrm{Re}=\tau]} \leqslant \|h\|_{[\mathrm{Re}=0]}^{1-\tau} \|h\|_{[\mathrm{Re}=1]}^{\tau}$$

for all $0 \leqslant \tau \leqslant 1$.

Proof. Let $b_j > \|h\|_{[\mathrm{Re}=j]}$ for $j = 0, 1$. We choose $\lambda \in \mathbb{R}$ such that $\mathrm{e}^{\lambda} b_1 = b_0$. Then we apply Theorem 10.1 to the function $z \mapsto \mathrm{e}^{\lambda z} h(z)$ and obtain

$$\mathrm{e}^{\lambda\tau} \|h\|_{[\mathrm{Re}=\tau]} \leqslant b_0 = \mathrm{e}^{\lambda} b_1 \qquad (0 \leqslant \tau \leqslant 1);$$

hence

$$\|h\|_{[\mathrm{Re}=\tau]} \leqslant \mathrm{e}^{-\lambda\tau} b_0 = \Big(\frac{b_1}{b_0}\Big)^{\tau} b_0 = b_0^{1-\tau} b_1^{\tau}.$$

Taking the infima over the b_j's one obtains the assertion. □

10.4 Remark. It follows that $\tau \mapsto \|h\|_{[\mathrm{Re}=\tau]}$ is a log-convex function, i.e., $\tau \mapsto \ln \|h\|_{[\mathrm{Re}=\tau]}$ is convex.

10.1.2 The Stein interpolation theorem

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{A}_{\mathrm{c}} \subseteq \{A \in \mathcal{A};\ \mu(A) < \infty\}$ be a ring of subsets of Ω, with the property that the space of simple functions over $\mathcal{A}_{\mathrm{c}}$,

$$S(\mathcal{A}_{\mathrm{c}}) := \mathrm{lin}\{\mathbf{1}_A;\ A \in \mathcal{A}_{\mathrm{c}}\}$$

is dense in $L_1(\mu)$. We will also use the notation

$$L_1(\mathcal{A}_{\mathrm{c}}) := \Big\{u\colon \Omega \to \mathbb{C};\ \mathbf{1}_A u \in L_1(\mu) \text{ for all } A \in \mathcal{A}_{\mathrm{c}},\ \exists (A_n) \text{ in } \mathcal{A}_{\mathrm{c}}\colon [u \neq 0] \subseteq \bigcup_{n\in\mathbb{N}} A_n\Big\},$$

where we understand the elements of $L_1(\mathcal{A}_{\mathrm{c}})$ as equivalence classes of a.e. equal functions. With these conventions one has $uv \in L_1(\mu)$ for all $u \in S(\mathcal{A}_{\mathrm{c}}), v \in L_1(\mathcal{A}_{\mathrm{c}})$. We note that for σ-finite measure spaces the requirement on $[u \neq 0]$ in the definition of $L_1(\mathcal{A}_{\mathrm{c}})$ can be dispensed with. (The index 'c' should be remindful of 'compact': if $\Omega \subseteq \mathbb{R}^n$ is an open set, then one can choose $\mathcal{A}_{\mathrm{c}}$ as the system of relatively compact measurable subsets of Ω.)

Let the strip $S \subseteq \mathbb{C}$ be defined as in Subsection 10.1.1, and let $p_0, p_1, q_0, q_1 \in [1,\infty]$, $M_0, M_1 \geqslant 0$. For $\tau \in (0,1)$ we denote

$$\frac{1}{p_\tau} := \frac{1-\tau}{p_0} + \frac{\tau}{p_1}, \quad \frac{1}{q_\tau} := \frac{1-\tau}{q_0} + \frac{\tau}{q_1}, \quad M_\tau := M_0^{1-\tau} M_1^\tau.$$

Finally, let $L(S(\mathcal{A}_\mathrm{c}), L_1(\mathcal{A}_\mathrm{c}))$ denote the linear operators from $S(\mathcal{A}_\mathrm{c})$ to $L_1(\mathcal{A}_\mathrm{c})$ (without continuity requirement), and let $\Phi\colon \overline{S} \to L(S(\mathcal{A}_\mathrm{c}), L_1(\mathcal{A}_\mathrm{c}))$ be a mapping satisfying the following two conditions.

(i) $\|\Phi(j+\mathrm{i}s)u\|_{q_j} \leqslant M_j \|u\|_{p_j}$ for all $u \in S(\mathcal{A}_\mathrm{c})$, all $s \in \mathbb{R}$ and $j = 0,1$. (The estimate means, in particular, that $\Phi(j+\mathrm{i}s)u \in L_{q_j}(\mu)$ for all the indicated terms.)

(ii) For all $A, B \in \mathcal{A}_\mathrm{c}$ the function $\overline{S} \ni z \mapsto \int (\Phi(z)\mathbf{1}_A)\mathbf{1}_B \,\mathrm{d}\mu$ is continuous and bounded, and its restriction to S is holomorphic.

After these preparations we can state the Stein interpolation theorem, the main result of this section.

10.5 Theorem. *(Stein) In the context described above it follows that*

$$\|\Phi(\tau+\mathrm{i}s)u\|_{q_\tau} \leqslant M_\tau \|u\|_{p_\tau}$$

for all $u \in S(\mathcal{A}_\mathrm{c})$, $s \in \mathbb{R}$ and $\tau \in (0,1)$.

Before proceeding to the proof we mention the important application to the situation where the function Φ is constant.

10.6 Corollary. *(Riesz-Thorin) Let $(\Omega,\mu), p_0, p_1, q_0, q_1, M_0, M_1$ be as before, and let $B \in L(S(\mathcal{A}_\mathrm{c}), L_1(\mathcal{A}_\mathrm{c}))$ be such that $\|Bu\|_{q_j} \leqslant M_j \|u\|_{p_j}$ $(u \in S(\mathcal{A}_\mathrm{c}),\ j = 0,1)$.*

Then for all $\tau \in (0,1)$ one has $\|Bu\|_{q_\tau} \leqslant M_\tau \|u\|_{p_\tau}$ $(u \in S(\mathcal{A}_\mathrm{c}))$.

As an application we recall Example 9.8, where it was shown that the operators $\mathrm{e}^{t\Delta_\mathrm{D}}$ are sub-Markovian. Since we know that $\mathrm{e}^{t\Delta_\mathrm{D}}$ is contractive in $L_2(\Omega)$, we conclude from Corollary 10.6 that $\mathrm{e}^{t\Delta_\mathrm{D}}$ is contractive in $L_p(\Omega)$ for all $p \in [2,\infty]$.

The following fact will be needed in the proof of Theorem 10.5.

10.7 Lemma. *Let the notation be as above. Let $p, p' \in [1,\infty]$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u \in L_1(\mathcal{A}_\mathrm{c})$, and assume that there exists $c \geqslant 0$ such that*

$$\left| \int uv \,\mathrm{d}\mu \right| \leqslant c\|v\|_{p'} \tag{10.1}$$

for all $v \in S(\mathcal{A}_\mathrm{c})$. Then $u \in L_p(\mu)$, $\|u\|_p \leqslant c$.

Proof. (i) In the first step we show that (10.1) carries over to all

$$v \in L_{\infty,\mathrm{c}}(\mathcal{A}_\mathrm{c}) := \{w\colon \Omega \to \mathbb{C};\ w \text{ measurable, bounded, } \exists A \in \mathcal{A}_\mathrm{c}\colon [w \neq 0] \subseteq A\}.$$

Let $v \in L_{\infty,\mathrm{c}}(\mathcal{A}_\mathrm{c})$, $A \in \mathcal{A}_\mathrm{c}$ such that $[v \neq 0] \subseteq A$. The hypotheses imply that there exists a sequence (v_n) in $S(\mathcal{A}_\mathrm{c})$, such that $v_n \to v$ a.e. Since $\mathcal{A}_\mathrm{c}$ is a ring, we can choose (v_n)

such that $[v_n \neq 0] \subseteq A$ and $\|v_n\|_\infty \leqslant \|v\|_\infty$ for all $n \in \mathbb{N}$. Then $\int uv_n \,\mathrm{d}\mu \to \int uv \,\mathrm{d}\mu$ as well as $v_n \to v$ in $L_{p'}(\mu)$ $(n \to \infty)$, and this implies (10.1).

(ii) Recall that $[u \neq 0] \subseteq \bigcup_{n\in\mathbb{N}} A_n$ for a suitable sequence (A_n) in $\mathcal{A}_\mathrm{c}$.

If $p = 1$, then $\int_A |u| \,\mathrm{d}\mu = \int u(\overline{\operatorname{sgn} u}\,\mathbf{1}_A)\,\mathrm{d}\mu \leqslant c$ for all $A \in \mathcal{A}_\mathrm{c}$, and therefore $\|u\|_1 \leqslant c$.

In the case $1 < p < \infty$ we use that $|u|$ can be approximated pointwise by an increasing sequence $(v_k)_{k\in\mathbb{N}}$ in $L_{\infty,\mathrm{c}}(\mathcal{A}_\mathrm{c})_+$. We estimate

$$\|v_k\|_p^p = \int v_k^p \,\mathrm{d}\mu \leqslant \int u(\overline{\operatorname{sgn} u}\, v_k^{p-1})\,\mathrm{d}\mu \leqslant c\big\|v_k^{p-1}\big\|_{p'} = c\|v_k\|_p^{p-1};$$

hence $\|v_k\|_p \leqslant c$. Now the monotone convergence theorem implies $u \in L_p(\mu)$, $\|u\|_p \leqslant c$.

If $p = \infty$ and $[|u| > c]$ is not a null set, then there exists $A \in \mathcal{A}_\mathrm{c}$ with $\mu(A) > 0$ and $A \subseteq [|u| > c]$. Then

$$\int u(\overline{\operatorname{sgn} u}\,\mathbf{1}_A)\,\mathrm{d}\mu = \int |u|\mathbf{1}_A \,\mathrm{d}\mu > c\mu(A) = c\|\mathbf{1}_A\|_1$$

leads to a contradiction. $\square$

Proof of Theorem 10.5. Let $\tau \in (0,1)$, and let $u, v \in S(\mathcal{A}_\mathrm{c})$, $\|u\|_{p_\tau} = 1$, $\|v\|_{q'_\tau} = 1$ (where $\frac{1}{q_\tau} + \frac{1}{q'_\tau} = 1$). We are going to show that then $\left|\int (\Phi(\tau)u)v \,\mathrm{d}\mu\right| \leqslant M_\tau$.

For $z \in \overline{S}$ define $\alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}$, $\beta(z) := \frac{1-z}{q'_0} + \frac{z}{q'_1}$,

$$F(z) := \begin{cases} |u|^{\alpha(z)p_\tau} \operatorname{sgn} u & \text{if } p_\tau \neq \infty, \\ u & \text{if } p_\tau = \infty, \end{cases}$$

$$G(z) := \begin{cases} |v|^{\beta(z)q'_\tau} \operatorname{sgn} v & \text{if } q'_\tau \neq \infty, \\ v & \text{if } q'_\tau = \infty. \end{cases}$$

Then $F(\tau) = u$, $G(\tau) = v$. Note that $F(z), G(z) \in S(\mathcal{A}_\mathrm{c})$ for all $z \in \overline{S}$. Indeed, u can be written as $u = \sum_{j=1}^n c_j \mathbf{1}_{A_j}$, with $A_1, \dots, A_n \in \mathcal{A}_\mathrm{c}$ pairwise disjoint, and then $F(z) = \sum_{j=1}^n |c_j|^{\alpha(z)p_\tau}(\operatorname{sgn} c_j)\mathbf{1}_{A_j}$, if $p_\tau \neq \infty$, and similarly for $G(z)$.

Finally we define

$$h(z) := \int \big(\Phi(z)F(z)\big)G(z)\,\mathrm{d}\mu \qquad (z \in \overline{S}).$$

Then h is continuous, bounded, and holomorphic on S. Indeed, it is sufficient to show this for the case $u = c\mathbf{1}_A$, $v = d\mathbf{1}_B$, with $c, d \in \mathbb{C}$, $A, B \in \mathcal{A}_\mathrm{c}$. If $p_\tau, q'_\tau < \infty$, then

$$\int \big(\Phi(z)F(z)\big)G(z)\,\mathrm{d}\mu = |c|^{\alpha(z)p_\tau}(\operatorname{sgn} c)|d|^{\beta(z)q'_\tau}(\operatorname{sgn} d)\int (\Phi(z)\mathbf{1}_A)\mathbf{1}_B \,\mathrm{d}\mu \qquad (z \in \overline{S}),$$

and this function has the required properties, by condition (ii). An analogous – easier – computation shows that this also holds if one or both of p_τ, q'_τ are equal to ∞.

The definition of F is such that $\|F(\sigma + \mathrm{i}s)\|_{p_\sigma} = 1$ for all $\sigma \in [0,1]$, $s \in \mathbb{R}$. For the proof recall that $\alpha(\sigma) = 1/p_\sigma$. If $p_\sigma < \infty$, then $p_\tau < \infty$, and therefore

$$\|F(\sigma + \mathrm{i}s)\|_{p_\sigma}^{p_\sigma} = \int \big(|u|^{\alpha(\sigma)p_\tau}\big)^{p_\sigma}\,\mathrm{d}\mu = \|u\|_{p_\tau}^{p_\tau} = 1.$$

If $p_\sigma = \infty$, then $\alpha(\sigma) = 0$, and therefore

$$\|F(\sigma + \mathrm{i}s)\|_{p_\sigma} = \left\| |u|^0 \mathbf{1}_{[u\neq 0]} \right\|_\infty = 1.$$

An analogous observation applies to G. This shows that

$$\|\Phi(\mathrm{i}s)F(\mathrm{i}s)\|_{q_0} \leqslant M_0 \|F(\mathrm{i}s)\|_{p_0} = M_0,$$

$$|h(\mathrm{i}s)| = \left| \int (\Phi(\mathrm{i}s)F(\mathrm{i}s))G(\mathrm{i}s)\,\mathrm{d}\mu \right| \leqslant \|\Phi(\mathrm{i}s)F(\mathrm{i}s)\|_{q_0} \|G(\mathrm{i}s)\|_{q_0'} \leqslant M_0$$

for all $s \in \mathbb{R}$. In the same way one obtains $|h(1 + \mathrm{i}s)| \leqslant M_1$ for all $s \in \mathbb{R}$.

At this point we can apply Corollary 10.3 and obtain $\left|\int (\Phi(\tau)u)v\,\mathrm{d}\mu\right| = |h(\tau)| \leqslant M_\tau$. So, we have shown that

$$\left| \int (\Phi(\tau)u)v\,\mathrm{d}\mu \right| \leqslant M_\tau \|u\|_{p_\tau} \|v\|_{q_\tau'} \qquad (u, v \in S(\mathcal{A}_\mathrm{c})).$$

In view of Lemma 10.7 this implies the assertion for $s = 0$.

If $s \in \mathbb{R}$, then the result proved so far can be applied to the function $z \mapsto \Phi(z + \mathrm{i}s)$, and this yields the asserted inequality for general s. □

10.2 Interpolation of semigroups

As in Section 10.1, the scalar field in this section will be $\mathbb{K} = \mathbb{C}$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $p_1 \in [1, \infty)$, $\theta \in (0, \pi/2]$, and let T be a bounded holomorphic C_0-semigroup on $L_{p_1}(\mu)$ of angle θ, $M_1 := \sup_{z\in\Sigma_\theta} \|T(z)\|_{\mathcal{L}(L_{p_1}(\mu))} < \infty$. Let $p_0 \in [1, \infty]$, $p_0 \neq p_1$, and assume that $T|_{[0,\infty)}$ is L_{p_0}-bounded; by this we mean that there exists $M_0 \geqslant 0$ such that

$$\|T(t)u\|_{p_0} \leqslant M_0 \|u\|_{p_0} \qquad (u \in L_{p_1} \cap L_{p_0}(\mu),\ t \geqslant 0).$$

10.8 Theorem. *Let the hypotheses be as above, and let $\tau \in (0, 1)$, $\theta_\tau := (1 - \tau)\theta$, $\frac{1}{p_\tau} := \frac{1-\tau}{p_0} + \frac{\tau}{p_1}$, $M_\tau := M_0^{1-\tau} M_1^\tau$.*

Then for all $z \in \Sigma_{\theta_\tau}$ the operator $T(z)|_{L_{p_1}\cap L_{p_\tau}(\mu)}$ extends (uniquely) to an operator $T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$, and T_τ is a bounded holomorphic C_0-semigroup of angle θ_τ, $\|T_\tau(z)\| \leqslant M_\tau$ for all $z \in \Sigma_{\theta_\tau}$.

Proof. We define $\mathcal{A}_\mathrm{c} := \{A \in \mathcal{A};\, \mu(A) < \infty\}$ and use the notation $S(\mathcal{A}_\mathrm{c}), L_1(\mathcal{A}_\mathrm{c})$ from the beginning of Subsection 10.1.2.

(i) The essential part of the theorem is the boundedness statement for $T(z)|_{S(\mathcal{A}_\mathrm{c})}$; its proof will require the Stein interpolation theorem.

Let $0 < \theta' < \theta$. The function $\psi(z) := \mathrm{e}^{\mathrm{i}\theta' z}$ maps the strip $\overline{S}$ continuously onto the 'semi-sector' $\Sigma_{\theta'}' := \{z \in \mathbb{C} \setminus \{0\};\, 0 \leqslant \operatorname{Arg} z \leqslant \theta'\}$, and ψ is holomorphic on S. Now one can see that the function $\Phi := T \circ \psi \colon \overline{S} \to L_1(\mathcal{A}_\mathrm{c})$ satisfies the hypotheses of Theorem 10.5 with $q_0 = p_0$, $q_1 = p_1$. Let us just comment on the holomorphy hypothesis: for $A \in \mathcal{A}_\mathrm{c}$ the function $S \ni z \mapsto T(\psi(z))\mathbf{1}_A \in L_{p_1}(\mu)$ is holomorphic, and this implies that for all

$B \in \mathcal{A}_c$ the function $S \ni z \mapsto \int (T(\psi(z))\mathbf{1}_A)\mathbf{1}_B \, d\mu$ is holomorphic. The other properties are checked similarly.

For every $s \in \mathbb{R}$, Theorem 10.5 implies that $T(\psi(\tau + is))|_{S(\mathcal{A}_c)}$ is bounded with respect to the L_{p_τ}-norm, with norm $\leqslant M_\tau$. The points $\psi(\tau + is) = e^{i\theta'(\tau+is)} = e^{-\theta' s}e^{i\theta'\tau}$ are contained in the open semi-sector $\mathring{\Sigma}'_{\theta_\tau}$, and in fact all points of this open semi-sector can be obtained by a suitable choice of s and θ'.

For the complementary open semi-sector $\{\bar{z};\, z \in \mathring{\Sigma}'_{\theta_\tau}\}$ the reasoning is analogous, and for $z \geqslant 0$ the boundedness statement follows from Corollary 10.6.

(ii) As $S(\mathcal{A}_c)$ is a dense subspace of $L_{p_\tau}(\mu)$, the operator $T(z)|_{S(\mathcal{A}_c)}$ has a unique extension $T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$. We show that $T(z)|_{L_{p_1} \cap L_{p_\tau}(\mu)} = T_\tau(z)|_{L_{p_1} \cap L_{p_\tau}(\mu)}$: Let $u \in L_{p_1} \cap L_{p_\tau}(\mu)$. Then there exists a sequence (u_n) in $S(\mathcal{A}_c)$ with $u_n \to u$ in $L_{p_1}(\mu)$ as well as in $L_{p_\tau}(\mu)$. This implies $T(z)u = T_\tau(z)u$.

In order to show that $\Sigma_{\theta_\tau} \ni z \mapsto T_\tau(z)$ is holomorphic we use the results of Section 3.1. For $u, v \in S(\mathcal{A}_c)$ the function $\Sigma_{\theta_\tau} \ni z \mapsto \int (T(z)u)v \, d\mu$ is holomorphic. From the denseness of $S(\mathcal{A}_c)$ in $L_{p_\tau}(\mu)$ and in $L_{p'_\tau}(\mu)$ one concludes that

$$E := \left\{ B \mapsto \int (Bu)v \, d\mu;\, u, v \in S(\mathcal{A}_c) \right\} \subseteq \mathcal{L}(L_{p_\tau}(\mu))'$$

is norming for $\mathcal{L}(L_{p_\tau}(\mu))$. Therefore the implication '(iv) $\Rightarrow$ (i)' in Theorem 3.2 implies that $z \mapsto T_\tau(z) \in \mathcal{L}(L_{p_\tau}(\mu))$ is holomorphic on Σ_{θ_τ}.

(iii) In order to show the strong continuity of T_τ at 0 we use Hölder's inequality

$$\|u\|_{p_\tau} \leqslant \|u\|_{p_0}^{1-\tau} \|u\|_{p_1}^{\tau} \qquad \big(u \in S(\mathcal{A}_c)\big).$$

Let $u \in S(\mathcal{A}_c)$. Then the boundedness of $\{\|T(t)u\|_{p_0};\, t \geqslant 0\}$ together with the continuity of $T(\cdot)u$ at 0 in $L_{p_1}(\mu)$ implies that $\|T(t)u - u\|_{p_\tau} \to 0$ as $t \to 0$. Then the combination of Lemma 1.5 and Proposition 3.11 implies that T_τ is strongly continuous at 0. □

10.3 Adjoint semigroups

In this section we insert some information on adjoint semigroups. Let H be a Hilbert space, T a C_0-semigroup on H. Then clearly $T^* := (T(t)^*)_{t \geqslant 0}$ is a one-parameter semigroup on H, but it is not obvious that T^* is strongly continuous. We will show this by looking at the generator of T. However, we will restrict our treatment to the case of quasi-contractive semigroups.

10.9 Theorem. *Let H be a Hilbert space, let T be a quasi-contractive C_0-semigroup on H, and let A be its generator. Then A^* is the generator of a quasi-contractive C_0-semigroup, and the generated C_0-semigroup is T^* as defined above.*

Proof. By rescaling we can reduce the situation to the case that T is contractive. Then A is an m-accretive operator.

As A is closed, A^* is densely defined; see Theorem 6.3(b). We know from Theorem 3.18 that $(0, \infty) \subseteq \rho(A)$, and that $\|(\lambda - A)^{-1}\| \leqslant \frac{1}{\lambda}$ for all $\lambda > 0$. Similarly as in the proof of

Theorem 6.3(b) one obtains $(\lambda - A^*)^{-1} = ((\lambda - A)^{-1})^*$. It follows that $\lambda \in \rho(A^*)$ and $\|(\lambda - A^*)^{-1}\| \leqslant \frac{1}{\lambda}$ for all $\lambda > 0$. Therefore Lemma 3.16 in combination with Theorem 3.18 implies that A^* generates a contractive C_0-semigroup.

From the exponential formula, Theorem 2.12, we conclude that the C_0-semigroup generated by A^* is the adjoint semigroup T^*. □

10.10 Remark. If the semigroup T in Theorem 10.9 is holomorphic of some angle $\theta \in (0, \pi/2]$, then T^* (defined as the adjoint of $T|_{[0,\infty)}$) has a holomorphic extension to the sector Σ_θ. This extension is given by

$$T^*(z) := T(\bar{z})^* \qquad (z \in \Sigma_\theta). \tag{10.2}$$

Indeed, it is not difficult to show that T^*, defined by (10.2), is holomorphic. Hence T^* is a holomorphic C_0-semigroup.

10.11 Remarks. (a) Using the general Hille-Yosida generation theorem (see Exercise 2.5) one also obtains Theorem 10.9 for general C_0-semigroups on H.

(b) If X is a Banach space and T is a C_0-semigroup on X, then it is not generally true that $T'(t) := T(t)'$ $(t \geqslant 0)$ defines a C_0-semigroup on X', where $T(t)' \in \mathcal{L}(X')$ is the dual operator. It is true, however, if X is reflexive. More generally, if T is a one-parameter semigroup on X that is weakly continuous, then T is a C_0-semigroup.

10.4 Applications of invariance criteria and interpolation

Throughout this section let (Ω, μ) be a measure space.

An operator $S \in \mathcal{L}(L_1(\mu))$ is called **substochastic** if S is positive and contractive. An operator $S \in \mathcal{L}(L_2(\mu))$ is called **L_1-contractive** if $\|Su\|_1 \leqslant \|u\|_1$ for all $u \in L_2 \cap L_1(\mu)$, and S is **substochastic** if S is positive and L_1-contractive. The same notation will be used for semigroups if all the semigroup operators satisfy the corresponding property.

10.12 Theorem. *Let $V \subseteq H := L_2(\mu)$ be continuously and densely embedded, and let $a\colon V \times V \to \mathbb{K}$ be a bounded H-elliptic form. Let A be the operator associated with a, and let T be the C_0-semigroup generated by $-A$. Then one has the following properties.*

(a) *T is real if and only if $\operatorname{Re} u \in V$ for all $u \in V$ and $a(u, v) \in \mathbb{R}$ for all real $u, v \in V$.*

(b) *T is positive if and only if T is real, and $u^+ \in V$, $a(u^+, u^-) \leqslant 0$ for all real $u \in V$.*

(c) *T is sub-Markovian if and only if T is real, and $u \wedge 1 \in V$, $a(u \wedge 1, (u-1)^+) \geqslant 0$ for all real $u \in V$.*

(d) *T is substochastic if and only if T is real, and $u \wedge 1 \in V$, $a((u-1)^+, u \wedge 1) \geqslant 0$ for all real $u \in V$.*

For the proof of (d) we need an auxiliary result.

10.13 Lemma. *Let $S \in \mathcal{L}(L_2(\mu))$. Then S is L_∞-contractive if and only if S^* is L_1-contractive, and S is sub-Markovian if and only if S^* is substochastic.*

Proof. Let S be L_∞-contractive. Let $v \in L_2 \cap L_1(\mu)$, $\|v\|_1 \leqslant 1$. Then

$$\left|\int u\overline{S^*v}\,\mathrm{d}\mu\right| = \left|\int (Su)\bar v\,\mathrm{d}\mu\right| \leqslant 1 \qquad (u \in L_2 \cap L_\infty(\mu),\ \|u\|_\infty \leqslant 1),$$

and from Lemma 10.7 we conclude that $\|S^*v\|_1 \leqslant 1$.

The converse statement is proved in the same way.

For the second statement it is now sufficient to notice that S is positive if and only if S^* is positive. □

Proof of Theorem 10.12. In view of Remarks 9.3 the statements (a), (b), (c) are easy consequences of '(i)⇔(ii)' in Theorem 9.20. For the necessity in (a) we note that only the case $\mathbb{K} = \mathbb{C}$ is of interest and that for real $u, v \in V$ one obtains

$$0 \leqslant \operatorname{Re} a\big(\operatorname{Re}(u \pm \mathrm{i}v), u \pm \mathrm{i}v - \operatorname{Re}(u \pm \mathrm{i}v)\big) = \operatorname{Re} a(u, \pm\mathrm{i}v) = \pm \operatorname{Im} a(u, v),$$

which implies that $a(u, v) \in \mathbb{R}$. For (b) we note that $u - u^+ = -u^-$, and for (c) we note that $u - u \wedge 1 = (u - 1)^+$.

For (d) we note that $-A^*$ is the generator of the C_0-semigroup T^*, by Theorem 10.9, and that A^* is associated with the form a^*, by Theorem 6.10 (which by Remark 5.6 and Lemma 6.9 also holds for H-elliptic forms). It is easy to see that T is real if and only if T^* is real, and therefore Lemma 10.13 implies that T is substochastic if and only if T is real and T^* is sub-Markovian, or equivalently, $u \wedge 1 \in V$ and $a((u-1)^+, u \wedge 1) = a^*(u \wedge 1, (u-1)^+) \geqslant 0$ for all real $u \in V$, by part (c). □

10.14 Remarks. (a) A form a satisfying conditions (c) and (d) of Theorem 10.12 is called a (**non-symmetric**) **Dirichlet form**. The conditions formulated in (b), (c) and (d) are the **Beurling-Deny criteria**.

(b) Using the condition (iii) of Theorem 9.20 one could also formulate the conditions in Theorem 10.12 with a dense subset of V.

10.15 Theorem. *Let T be a C_0-semigroup on $L_2(\mu)$.*

(a) *Assume that T is sub-Markovian and substochastic. Then for all $p \in [1, \infty)$ the operators $T(t)|_{L_2 \cap L_p(\mu)}$ extends to operators $T_p(t) \in \mathcal{L}(L_p(\mu))$, and T_p thus defined is a contractive C_0-semigroup on $L_p(\mu)$. For $1 \leqslant p, q < \infty$ the semigroups T_p, T_q are **consistent**, i.e., $T_p(t)|_{L_p \cap L_q(\mu)} = T_q(t)|_{L_p \cap L_q(\mu)}$ for all $t \geqslant 0$.*

(b) *Assume that $T(t)$ is self-adjoint for all $t \geqslant 0$ and that T is sub-Markovian. Then the assertions of (a) hold. If $\mathbb{K} = \mathbb{C}$, then for all $p \in (1, \infty)$ the semigroup T_p extends to a contractive holomorphic C_0-semigroup of angle*

$$\theta_p = \begin{cases} \big(1 - \frac{1}{p}\big)\pi & \text{if } 1 < p < 2, \\ \frac{1}{p}\pi & \text{if } 2 \leqslant p < \infty. \end{cases}$$

Proof. (a) Let $1 < p < \infty$. For every $t > 0$, Exercise 10.3 (or Corollary 10.6, if $\mathbb{K} = \mathbb{C}$) implies that $T(t)|_{L_1 \cap L_\infty(\mu)}$ extends to a contractive operator $T_p(t)$ on $L_p(\mu)$. It is standard to show that T_p is a one-parameter semigroup. The strong continuity of T_p at 0 is obtained as follows. If $u \in L_p \cap L_2(\mu)$ and (t_n) is a null sequence in $(0, \infty)$, then $T(t_n)u \to u$ in

$L_2(\mu)$ implies that for a subsequence one has $T(t_{n_k})u \to u$ a.e., and the contractivity of T_p in combination with Lemma 10.16, proved subsequently, implies that $T_p(t_{n_k})u \to u$ in $L_p(\mu)$. Applying a standard sub-sub-sequence argument one obtains $T_p(t)u \to u$ in $L_p(\mu)$ as $t \to 0+$. Lemma 1.5 concludes the argument.

The consistency is shown as in (ii) of the proof of Theorem 10.8.

(b) From $T(t)^* = T(t)$ for all $t \geqslant 0$ and Lemma 10.13 it follows that T is also substochastic. Thus (a) is applicable. Also, Theorem 10.9 implies that the generator A of T is self-adjoint, and as T is contractive, $-A$ is accretive.

Now let $\mathbb{K} = \mathbb{C}$. Then it follows that $-A$ is sectorial of angle 0. Hence $-A$ is the generator of a contractive holomorphic C_0-semigroup of angle $\pi/2$; see Theorem 3.22. In view of Theorem 10.8 this implies the remaining assertions. □

10.16 Lemma. *Let $1 \leqslant p < \infty$, and let (f_n) be a sequence in $L_p(\mu)$, $f \in L_p(\mu)$ such that $f_n \to f$ a.e. and $\limsup_{n\to\infty} \|f_n\|_p \leqslant \|f\|_p$. Then $f_n \to f$ in $L_p(\mu)$.*

Proof. For $n \in \mathbb{N}$ let $\check{f}_n := \operatorname{sgn} f_n\,(|f| \wedge |f_n|)$. Then $\check{f}_n \to f$ in $L_p(\mu)$ by the dominated convergence theorem. Moreover $|f_n| = |\check{f}_n| + |f_n - \check{f}_n|$, hence $|f_n|^p \geqslant |\check{f}_n|^p + |f_n - \check{f}_n|^p$ for all $n \in \mathbb{N}$. Therefore $\|f_n - \check{f}_n\|_p^p \leqslant \|f_n\|_p^p - \|\check{f}_n\|_p^p \to 0$, and this implies that $f_n = (f_n - \check{f}_n) + \check{f}_n \to f$ in $L_p(\mu)$. □

Notes

The three lines theorem is generally attributed to J. Hadamard. The Stein interpolation theorem, essentially in the form presented here, is contained in [Ste56]. We refer to this paper for some history of the Riesz-Thorin convexity theorem, finally proved by Thorin by the complex variable method, which initiated a whole new branch of Functional Analysis. In fact, the paper [Ste56] can be considered as the start of interpolation theory, for which we refer to the seminal paper of Calderón [Cal64] as well as to the monographs [BL76], [Lun09].

The application of invariance and interpolation as described in Section 10.4 is well-established in the theory of semigroups for diffusion equations, Schrödinger semigroups and related. Interestingly enough, nice and elegant as the proof of Theorem 10.15 may seem, the angle of holomorphy for the L_p-semigroup is not optimal, in this case, neither for holomorphy nor for contractivity. There exist other methods providing more sophisticated estimates.

Exercises

10.1 For this exercise let S be the strip

$$S := \{z \in \mathbb{C};\ -1/2 < \operatorname{Re} z < 1/2\}$$

(of width 1). Let $h\colon \overline{S} \to \mathbb{C}$ be continuous and holomorphic on S. Assume that h is bounded on ∂S, and that there exists $\alpha < \pi$ such that

$$|h(z)| \leqslant \mathrm{e}^{\mathrm{e}^{\alpha|\operatorname{Im} z|}} \qquad (z \in S).$$

Show that h is bounded by $\|h\|_{\partial S}$. (Hint: Use $\psi_n(z) := \mathrm{e}^{-\frac{1}{n}(\mathrm{e}^{\mathrm{i}\beta z}+\mathrm{e}^{-\mathrm{i}\beta z})}$, with $\alpha < \beta < \pi$.)

10.2 Let (Ω, μ) be a measure space. Show that

$$\Big(\big\|\tfrac{1}{2}(f+g)\big\|_p^p + \big\|\tfrac{1}{2}(f-g)\big\|_p^p\Big)^{1/p} \leqslant 2^{-1/p}\big(\|f\|_p^p + \|g\|_p^p\big)^{1/p} \tag{10.3}$$

for all $f, g \in L_p(\mu)$, $2 \leqslant p \leqslant \infty$.

Hint: Use the mapping

$$T\colon L_p(\mu) \times L_p(\mu) \to L_p(\mu) \times L_p(\mu), (f,g) \mapsto \big(\tfrac{1}{2}(f+g), \tfrac{1}{2}(f-g)\big).$$

Compute the norm of T for $p = 2$ and for $p = \infty$ and use the Riesz-Thorin interpolation theorem. (The inequality (10.3) is one of Clarkson's inequalities. The other inequalities of Clarkson involve p and the conjugate exponent and can also be obtained by interpolation, but this is more complicated.)

10.3 (a) Let $p \in (1, \infty)$, $r \in [0, \infty)$. Show that

$$r = \inf_{\alpha \in (0,\infty)\cap\mathbb{Q}} \big(\tfrac{1}{p}\alpha^{1-p}r^p + (1 - \tfrac{1}{p})\alpha\big).$$

(b) Let (Ω, μ) be a measure space, and let $S \in \mathcal{L}(L_2(\mu))$ be sub-Markovian and substochastic. Show that S is L_p-contractive for all $p \in (1, \infty)$. (The case $\mathbb{K} = \mathbb{C}$ is already covered by Corollary 10.6, but not the case $\mathbb{K} = \mathbb{R}$!)

Hint: Using (a) twice show first that $S|u| \leqslant \big(S|u|^p\big)^{1/p}$ for simple functions.

(c) Let (Ω, μ) be a measure space, let $S \in \mathcal{L}(L_2(\mu))$ be sub-Markovian, and assume that there exists $c > 0$ such that $\frac{1}{c}S$ is substochastic. Show that S interpolates to an operator $S_p \in \mathcal{L}(L_p(\mu))$ with $\|S_p\| \leqslant c^{1/p}$, for all $1 < p < \infty$.

10.4 (Continuation of Exercise 9.5) Let the hypotheses be as in Exercise 9.5, and additionally $b \in C^1(\Omega)$. Assume that $\omega \in \mathbb{R}$ is such that $\operatorname{div} b(x) \leqslant \omega$ for all $x \in \Omega$.

(a) Show that $\|T(t)u\|_1 \leqslant \mathrm{e}^{\omega t}\|u\|_1$ ($u \in L_2 \cap L_1(\Omega)$, $t \geqslant 0$), where T is the C_0-semigroup generated by the operator $-A$.

Hint: Use $C_{\mathrm{c}}^1(\Omega)$ as the dense subset of $V = H_0^1(\Omega)$ for the application of the invariance criterion to the semigroup $\big(e^{-\omega t}T(t)\big)_{t\geqslant 0}$. Observe that on $C_{\mathrm{c}}^1(\Omega)$ one can transform $(b \cdot \nabla u \,|\, v)$ – using integration by parts – to an expression where b only appears in the second argument of the scalar product, and u in the first argument appears without derivative.

(b) Compute estimates for $\|T_p(t)\|$ in terms of $\omega := \sup \operatorname{div} b$ for $t \geqslant 0$, $1 \leqslant p < \infty$, where T_p is the interpolated semigroup on $L_p(\Omega)$, analogous to Theorem 10.15(b).

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