Lecture 9

Invariance of closed convex sets

In this lecture we investigate criteria for a closed convex set to be invariant under a semigroup. To begin with, we present criteria involving properties of the generator. Applying these criteria to the Dirichlet and Neumann Laplacian one realises that further properies of H^1-functions are needed. These will be provided in an interlude on lattice properties of H^1. In the last section we present criteria involving properties of forms. Their applicability to a wide range of problems will be presented in a later lecture on elliptic operators.

9.1 Invariance for semigroups

Let T be a C_0-semigroup on a Banach space X over $\mathbb{K}$, with generator A. Our aim is to characterise when a closed convex subset C of X is **invariant under the semigroup** T, i.e., $T(t)(C) \subseteq C$ for all $t \geqslant 0$. At first we note that invariance under T is equivalent to invariance under the resolvent.

9.1 Proposition. *Let $C \subseteq X$ be closed and convex, $C \neq \varnothing$. Then the following assertions are equivalent.*

(i) *C is invariant under T.*

(ii) *There exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and $\lambda R(\lambda, A)(C) \subseteq C$ for all $\lambda > \omega$.*

Noting that $\lambda R(\lambda, A) = (I - \frac{1}{\lambda}A)^{-1}$ we see that condition (ii) can be expressed equivalently by requiring that there exists $r_0 > 0$ such that $\{1/r;\ 0 < r < r_0\} \subseteq \rho(A)$ and $(I - rA)^{-1}(C) \subseteq C$ for all $0 < r < r_0$. It is this version of condition (ii) that will mostly be used in the following.

For the proof of the implication '(i)⇒(ii)' we need a fact concerning integration. It should be understood as a statement on generalised convex combinations.

9.2 Lemma. *Let C be a closed convex subset of a Banach space X, let $-\infty < a < b < \infty$, $u \in C([a, b]; X)$ with $u(t) \in C$ for all $t \in [a, b]$, and $\varphi \in C[a, b]$, $\varphi \geqslant 0$, $\int_a^b \varphi(t)\, \mathrm{d}t = 1$.*

Then $\int_a^b \varphi(t)u(t)\, \mathrm{d}t \in C$.

Proof. For simplicity of notation (and without loss of generality) we assume that $[a, b] = [0, 1]$.

For $n \in \mathbb{N}$ define

$$u_n := u(0)\mathbf{1}_{\{0\}} + \sum_{k=1}^{n} u(k/n)\mathbf{1}_{((k-1)/n, k/n]}.$$

Then $\|\varphi u_n - \varphi u\|_\infty \to 0$ $(n \to \infty)$; hence $\int_0^1 \varphi u_n \,\mathrm{d}t \to \int_0^1 \varphi u \,\mathrm{d}t$. Moreover $\int_0^1 \varphi u_n \,\mathrm{d}t = \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \varphi(t)\,\mathrm{d}t\, u(k/n) \in C$, as a convex combination of elements of C. As C is closed we obtain the assertion. $\square$

Proof of Proposition 9.1. (i)⇒(ii). Let $\omega \in \mathbb{R}$, $M \geqslant 0$ be such $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ $(t \geqslant 0)$, and let $\lambda > \omega$. Then $\lambda R(\lambda, A) = \int_0^\infty \lambda \mathrm{e}^{-\lambda t} T(t)\,\mathrm{d}t$ (strong improper integral). Let $x \in C$. For $r > 0$ we obtain $(1 - \mathrm{e}^{-\lambda r})^{-1} \int_0^r \lambda \mathrm{e}^{-\lambda t} T(t)x\,\mathrm{d}t \in C$, by Lemma 9.2. Letting $r \to \infty$ we conclude that $\lambda R(\lambda, A)x \in C$.

(ii)⇒(i). This follows from 'Euler's formula' (Theorem 2.12):

$$T(t)x = \lim_{n\to\infty} \Big(I - \frac{t}{n}A\Big)^{-n} x \in C \qquad (x \in C). \qquad \square$$

In order to motivate why one may be interested in the invariance of closed convex sets, we indicate several examples.

9.3 Remarks. Let (Ω, μ) be a measure space, $H := L_2(\mu; \mathbb{K})$.

(a) Let $C \subseteq L_2(\mu)$ be the **positive cone**, $C := L_2(\mu)_+ := \{u \in L_2(\mu)\,;\ u \geqslant 0\}$. Clearly, C is a closed convex subset of $L_2(\mu)$. An operator $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **positive**, or **positivity preserving**, i.e., $Su \geqslant 0$ for all $u \geqslant 0$.

(b) Let $\mathbb{K} = \mathbb{C}$, and let $C := L_2(\mu; \mathbb{R})$ be the subset of real-valued functions. An operator $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is 'real', i.e., Su is real-valued for all real-valued u.

(c) Let $C := \{u \in L_2(\mu)\,;\ \|u\|_\infty \leqslant 1\}$. Then C is convex and closed, and $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **L_∞-contractive**, i.e., $\|Su\|_\infty \leqslant \|u\|_\infty$ for all $u \in L_2(\mu) \cap L_\infty(\mu)$.

(d) Let $C := \{u \in L_2(\mu)\,;\ u \leqslant 1\}$. Then C is convex and closed, and $S \in \mathcal{L}(H)$ leaves C invariant if and only if S is **sub-Markovian**, i.e., S is positive (in particular, real) and L_∞-contractive.

Indeed, let $u \in L_2(\mu)_+$. Then $-\alpha u \leqslant 1$ and therefore $-\alpha Su \leqslant 1$, for all $\alpha \geqslant 0$. This implies that $Su \geqslant 0$. This shows that S is a positive operator.

Let $u \in L_2(\mu) \cap L_\infty(\mu)$, $\|u\|_\infty \leqslant 1$. Then, for any $\gamma \in \mathbb{K}$ with $|\gamma| = 1$ one obtains

$$\operatorname{Re}(\gamma Su) = \operatorname{Re}\big(S(\gamma u)\big) = S\big(\operatorname{Re}(\gamma u)\big) \leqslant 1.$$

It is not difficult to show that this implies that $\|Su\|_\infty \leqslant 1$.

Conversely, if S is positive and L_∞-contractive, then $u \leqslant 1$ implies $Su \leqslant Su^+ \leqslant 1$.

We are looking for another characterisation involving more directly the generator A and not just its resolvent. This is possible in Hilbert spaces. Let H be a Hilbert space over $\mathbb{K}$ and let $\varnothing \neq C \subseteq H$ be closed and convex. We denote by $P_C \colon H \to C$ the **minimising projection** of H onto C, i.e., for $x \in H$ the element $P_C x \in C$ is the unique element satisfying

$$\|x - P_C x\| = \inf\{\|x - y\|\,;\ y \in C\}.$$

In other words, $P_C x$ is the best approximation to x in C. The mapping P_C is a contraction; in particular, P_C is continuous. It will be important for us that $P_C x$ can also be characterised as the unique element of C such that

$$\operatorname{Re}\,(y - P_C x \,|\, x - P_C x) \leqslant 0 \qquad (y \in C); \tag{9.1}$$

see [Bou07; V, § 1, N° 5, Théorème 1], [Bre83; Théorème V.2]. Geometrically, this means that $x - P_C x$ is 'orthogonal' to the boundary of C. Clearly, the mapping P_C satisfies $P_C \circ P_C = P_C$; so it deserves the name 'projection'. We could not find a commonly accepted name for this mapping in the literature. One should keep in mind that in general P_C is not a linear operator.

The following result has a geometric appeal. It tells that C is invariant under the motion if the 'driving term' $Au(t)$ in the equation $u'(t) = Au(t)$ always points 'sufficiently' from $u(t)$ towards C. (For $\omega \leqslant 0$ this is quite intuitive. If $\omega > 0$, then one can interpret that it is more and more true, the closer $u(t)$ is to C.)

9.4 Proposition. *Let H, T, A be as before, $\varnothing \neq C \subseteq H$ a closed convex set, and denote by $P := P_C$ the minimising projection. Assume that there exists $\omega \in \mathbb{R}$ such that*

$$\operatorname{Re}\,(Ax \,|\, x - Px) \leqslant \omega \|x - Px\|^2 \tag{9.2}$$

for all $x \in \operatorname{dom}(A)$.

Then C is invariant under T.

Proof. Because of Proposition 9.1 we only have to show that $(I - rA)^{-1}(C) \subseteq C$ for small $r > 0$. (Observe that $(I - rA)^{-1} \in \mathcal{L}(H)$ for small $r > 0$.) Without loss of generality we assume $\omega > 0$. Let $0 < r < 1/\omega$ and let $x \in \operatorname{dom}(A)$ such that $(I - rA)x \in C$. We have to show that $x \in C$. Applying (9.1) with $y = (I - rA)x \in C$ we obtain

$$\operatorname{Re}\big((I - rA)x - Px \,\big|\, x - Px\big) \leqslant 0.$$

Thus

$$\begin{aligned}\|x - Px\|^2 &= \operatorname{Re}\big(rAx + (I - rA)x - Px \,\big|\, x - Px\big)\\ &\leqslant r \operatorname{Re}\,(Ax \,|\, x - Px) \leqslant r\omega \|x - Px\|^2.\end{aligned}$$

Using $r\omega < 1$ we conclude that $\|x - Px\| = 0$, $x = Px \in C$. □

The converse of Proposition 9.4 holds for quasi-contractive semigroups.

9.5 Proposition. *Let H, T, A be as before, and assume that T is quasi-contractive, i.e., there exists $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant \mathrm{e}^{\omega t}$ for all $t \geqslant 0$. Let $\varnothing \neq C \subseteq H$ be a closed and convex set, and assume that C is invariant under T.*

Then (9.2) *holds for all $x \in \operatorname{dom}(A)$, with the minimising projection $P := P_C$.*

Proof. Let $x \in \operatorname{dom}(A)$. Then (9.1) implies $\operatorname{Re}\,(T(t)Px - Px \,|\, x - Px) \leqslant 0$, and this inequality can be rewritten as $0 \leqslant \operatorname{Re}\,(-T(t)Px + Px \,|\, x - Px)$. One then obtains

$$\begin{aligned}\operatorname{Re}\,(T(t)x - x \,|\, x - Px) &\leqslant \operatorname{Re}\big(T(t)(x - Px) - (x - Px) \,\big|\, x - Px\big)\\ &\leqslant (e^{\omega t} - 1)\|x - Px\|^2.\end{aligned}$$

Dividing by t and taking the limit $t \to 0+$ we conclude that

$$\operatorname{Re}\,(Ax \,|\, x - Px) \leqslant \omega \|x - Px\|^2.$$

□

For the case of contractive C_0-semigroups we summarise the results of Propositions 9.4 and 9.5 as an equivalence.

9.6 Corollary. *Let T be a contractive C_0-semigroup on a Hilbert space H, let A be the generator of T, and let $C \neq \varnothing$ be a closed convex subset of H. Then C is invariant under T if and only if*

$$\operatorname{Re}\,(Ax \,|\, x - Px) \leqslant 0 \qquad (x \in \operatorname{dom}(A)), \tag{9.3}$$

where $P = P_C$ is the minimising projection.

For an illustration we expand the example where $H = L_2(\mu;\mathbb{R})$ and $C = L_2(\mu)_+$ is the positive cone; see Remark 9.3(a).

9.7 Corollary. *Let T be a contractive C_0-semigroup on $L_2(\mu;\mathbb{R})$. Then T is* ***positive****, i.e., $T(t)$ is positive for all $t \geqslant 0$, if and only if*

$$\big(Au \,\big|\, u^+\big) \leqslant 0 \qquad (u \in \operatorname{dom}(A)). \tag{9.4}$$

Here, $u^+ := u \vee 0$ is the ***positive part of*** *u. If A satisfies the condition* (9.4)*, then A is sometimes called* ***dispersive****.*

Proof. Clearly, the projection P_C is given by $P_C u = u^+$. With $u^- := (-u)^+ = u^+ - u$, the condition (9.3) translates to $(Au \,|\, -u^-) \leqslant 0$ $(u \in \operatorname{dom}(A))$. Replacing u by $-u$ one obtains

$$0 \geqslant \big(A(-u) \,\big|\, -(-u)^-\big) = \big(Au \,\big|\, u^+\big) \qquad (u \in \operatorname{dom}(A)). \qquad \square$$

9.2 Application to Laplacians

We recall that the Dirichlet Laplacian Δ_{D} is associated with the classical Dirichlet form on $V = H_0^1(\Omega)$, embedded into $L_2(\Omega)$. We also recall that the Neumann Laplacian Δ_{N} is associated with the classical Dirichlet form on $V = H^1(\Omega)$, embedded into $L_2(\Omega)$. (See Example 5.13 and Theorem 7.13.)

9.8 Example. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then the C_0-semigroup generated by the Dirichlet Laplacian Δ_{D} in $L_2(\Omega;\mathbb{C})$ leaves $L_2(\Omega;\mathbb{R})$ invariant. Moreover, $\mathrm{e}^{t\Delta_{\mathrm{D}}}$ is sub-Markovian for all $t \geqslant 0$, i.e., $\mathrm{e}^{t\Delta_{\mathrm{D}}}$ is positivity preserving and $\big\|\mathrm{e}^{t\Delta_{\mathrm{D}}}u\big\|_\infty \leqslant \|u\|_\infty$ for all $0 \leqslant u \in L_2 \cap L_\infty(\Omega)$.

Proof. (i) For the proof of the first property we use Corollary 9.6. As $-\Delta_{\mathrm{D}}$ is accretive, the semigroup $(\mathrm{e}^{t\Delta_{\mathrm{D}}})_{t\geqslant 0}$ is contractive. The minimising projection $P\colon L_2(\Omega;\mathbb{C}) \to C := L_2(\Omega;\mathbb{R})$ is given by $Pu := \operatorname{Re} u$. Let $u \in \operatorname{dom}(\Delta_{\mathrm{D}})$. Then

$$\operatorname{Re}\,(\Delta_{\mathrm{D}}u \,|\, u - Pu) = \operatorname{Re}\,(\Delta_{\mathrm{D}}(\operatorname{Re} u) + \mathrm{i}\Delta_{\mathrm{D}}(\operatorname{Im} u) \,|\, \mathrm{i}\operatorname{Im} u) = (\Delta_{\mathrm{D}}(\operatorname{Im} u) \,|\, \operatorname{Im} u) \leqslant 0.$$

Therefore Corollary 9.6 implies that $L_2(\Omega;\mathbb{R})$ is invariant under $(\mathrm{e}^{t\Delta_{\mathrm{D}}})_{t\geqslant 0}$.

(ii) We now restrict $\mathrm{e}^{t\Delta_{\mathrm{D}}}$ a priori to $L_2(\Omega;\mathbb{R})$, and we show that the (closed convex) set $C := \{u \in L_2(\Omega;\mathbb{R});\ u \leqslant 1\}$ is invariant under $(\mathrm{e}^{t\Delta_{\mathrm{D}}})_{t\geqslant 0}$. The minimising projection onto C is given by $Pu = u \wedge 1$. We have to show that

$$\operatorname{Re}\,(\Delta_{\mathrm{D}}u \,|\, u - u \wedge 1) \leqslant 0$$

for all $u \in \operatorname{dom}(\Delta_{\mathrm{D}})$.

We are going to use that $u \wedge 1 \in H_0^1(\Omega)$, and that $\nabla(u \wedge 1) = \mathbf{1}_{[u<1]}\nabla u$. These properties will be shown in the following section (and the present example should serve as a motivation for this treatment); see Theorem 9.14. Accepting these properties we obtain

$$(\Delta_{\mathrm{D}} u \,|\, u - u \wedge 1) = -\big(\nabla u \,\big|\, \mathbf{1}_{[u\geqslant 1]}\nabla u\big) = -\int_{[u\geqslant 1]} |\nabla u|^2 \leqslant 0.$$

Now the application of Corollary 9.6 yields the invariance of C. It was shown in Remark 9.3(d) that then $\mathrm{e}^{t\Delta_{\mathrm{D}}}$ is sub-Markovian for all $t \geqslant 0$. □

We note that identically the same arguments show the same properties for the Neumann Laplacian. We refer to Exercise 9.3 for the discussion of invariance properties for the Robin Laplacian.

9.3 Interlude: lattice properties of $H^1(\Omega)$

We start this section with a warm-up.

9.9 Lemma. *Let $-\infty \leqslant a < b \leqslant \infty$, and let $u\colon (a,b) \to \mathbb{R}$ be continuously differentiable. Then $\partial|u| = (\operatorname{sgn} u)u'$ in the distributional sense, where* $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ *is the signum function,* $\operatorname{sgn}\alpha := \frac{\alpha}{|\alpha|}$ *if $\alpha \neq 0$, and* $\operatorname{sgn} 0 := 0$.

For the proof we use a sequence (F_k) of functions $F_k \in C^\infty(\mathbb{R};\mathbb{R})$ with $F_k(t) = |t| - \frac{1}{k}$ for $|t| \geqslant \frac{2}{k}$, $F_k(t) = 0$ for $|t| \leqslant \frac{1}{2k}$ and $|F_k'(t)| \leqslant 1$ for all $t \in \mathbb{R}$. (F_1 can be obtained as a convolution of $t \mapsto (|t|-1)^+$ with a suitable C_{c}^∞-function; and then $F_k(t) := \frac{1}{k}F_1(kt)$ ($t \in \mathbb{R}$).)

Proof of Lemma 9.9. By the chain rule we have $(F_k \circ u)' = (F_k' \circ u)u'$. Then $F_k \circ u(x) \to |u(x)|$, uniformly for x in compact subsets of (a,b), and $(F_k \circ u)'(x) \to (\operatorname{sgn} u(x))u'(x)$ for all $x \in (a,b)$, with $|(F_k \circ u)'(x)| \leqslant |u'(x)|$ for all $x \in (a,b)$, $k \in \mathbb{N}$. Therefore $(F_k \circ u)' \to (\operatorname{sgn} u)u'$ locally in L_1 on (a,b). This implies $\partial|u| = (\operatorname{sgn} u)u'$. □

9.10 Remark. It is an interesting observation that, in the situation of Lemma 9.9, one also obtains that $u' = 0$ on $[u=0]$ $(= \{x \in (a,b)\,;\, u(x) = 0\})$.

Indeed, using a sequence (F_k) of functions in $C^\infty(\mathbb{R})$ satisfying $F_k(0) = 0$, $F_k'(0) = 1$, $0 \leqslant F_k' \leqslant 1$, $F_k(t) = 1/k$ $(t \geqslant 2/k)$, $F(t) = -1/k$ $(t \leqslant 2/k)$, we obtain $F_k \circ u \to 0$ uniformly, $(F_k \circ u)' = (F_k' \circ u)u' \to \mathbf{1}_{[u=0]}u'$ locally in L_1 in (a,b). This shows that $\mathbf{1}_{[u=0]}u'$ is the distributional derivative of the zero-function; hence $u' = 0$ a.e. on $[u=0]$.

Our aim is to show similar properties in more general situations. The first point is that the chain rule holds also for distributional derivatives.

In the following let $\Omega \subseteq \mathbb{R}^n$ be an open set. Until further notice all the function spaces will consist of real-valued functions.

9.11 Proposition. *Let $F \in C^1(\mathbb{R};\mathbb{R})$, $|F'(t)| \leqslant 1$ for all $t \in \mathbb{R}$, $u \in L_{1,\mathrm{loc}}(\Omega)$, $j \in \{1,\ldots,n\}$, $\partial_j u \in L_{1,\mathrm{loc}}(\Omega)$.*

Then $\partial_j(F \circ u) = (F' \circ u)\partial_j u$.

Proof. Without loss of generality we assume that $F(0) = 0$; then $|F(t)| \leqslant |t|$ for all $t \in \mathbb{R}$. Being the distributional derivative of a function is a local property, and therefore (after suitable multiplication by a C_{c}^∞-function) it is sufficient to treat the case that $\Omega = \mathbb{R}^n$ and $u, \partial_j u \in L_1(\mathbb{R}^n)$.

Let $(\rho_k)_{k\in\mathbb{N}}$ be a δ-sequence in $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$. Then $u_k := \rho_k * u \to u$, $\partial_j u_k = \rho_k * \partial_j u \to \partial_j u$ in $L_1(\mathbb{R}^n)$, and for a suitable subsequence (ρ_{k_i}) these convergences also hold a.e. as well as "boundedly", in the sense that there exists $h \in L_1(\mathbb{R}^n)$ such that $|u_{k_i}|, |\partial_j u_{k_i}| \leqslant h$. As $u_k \in C^\infty(\mathbb{R}^n)$, one has $F \circ u_{k_i} \in C^1(\mathbb{R}^n)$, and

$$F \circ u_{k_i} \to F \circ u, \quad \partial_j(F \circ u_{k_i}) = (F' \circ u_{k_i})\partial_j u_{k_i} \to (F' \circ u)\partial_j u \quad \text{a.e.},$$

since F, F' are continuous. Furthermore, $|F \circ u_{k_i}| \leqslant |u_{k_i}| \leqslant h$, $|\partial_j(F \circ u_{k_i})| \leqslant h$, by the hypotheses on F and the subsequence, and therefore $F \circ u_{k_i} \to F \circ u$, $\partial_j(F \circ u_{k_i}) \to (F' \circ u)\partial_j u$ in $L_1(\mathbb{R}^n)$, and this implies that $\partial_j(F \circ u) = (F' \circ u)\partial_j u$. □

Next, we extend the chain rule of Proposition 9.11 to more general composition functions F.

9.12 Proposition. *Let $F\colon \mathbb{R} \to \mathbb{R}$ be continuous, and assume that there exist a function $G\colon \mathbb{R} \to \mathbb{R}$ and a sequence (F_k) in $C^1(\mathbb{R};\mathbb{R})$ with $\|F_k'\|_\infty \leqslant 1$ ($k \in \mathbb{N}$), $F_k \to F$ pointwise, and $F_k' \to G$ pointwise ($k \to \infty$).*

Let $u \in L_{1,\mathrm{loc}}(\Omega)$, $j \in \{1, \dots, n\}$, $\partial_j u \in L_{1,\mathrm{loc}}(\Omega)$. Then $F \circ u \in L_{1,\mathrm{loc}}(\Omega)$, $\partial_j(F \circ u) = (G \circ u)\partial_j u$.

Proof. From Proposition 9.11 we know that $F_k \circ u \in L_{1,\mathrm{loc}}(\Omega)$, $\partial_j(F_k \circ u) = (F_k' \circ u)\partial_j u$. Applying the dominated convergence theorem on relatively compact subsets of Ω one obtains the assertions. □

9.13 Corollary. *Let $u \in L_{1,\mathrm{loc}}(\Omega)$, $j \in \{1, \dots, n\}$, $\partial_j u \in L_{1,\mathrm{loc}}(\Omega)$. Then $u^+, u \wedge 1 \in L_{1,\mathrm{loc}}(\Omega)$, $\partial_j(u^+) = \mathbf{1}_{[u>0]}\partial_j u$, $\partial_j(u \wedge 1) = \mathbf{1}_{[u<1]}\partial_j u$.*

Proof. Similarly to the construction of the sequence (F_k) at the beginning of the section one can construct a sequence (F_k) converging to $F(t) := t^+$, with the properties mentioned in Proposition 9.12 and such that $F_k' \to \mathbf{1}_{(0,\infty)}$ pointwise. Then Proposition 9.12 implies the assertion for u^+. The reasoning for $u \wedge 1$ is analogous. □

The following result is the conclusion for the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. It implies that they are Stonian sublattices of $L_2(\mu)$.

9.14 Theorem. *Let $u \in H^1(\Omega)$. Then $u^+, u \wedge 1 \in H^1(\Omega)$, $\nabla u^+ = \mathbf{1}_{[u>0]}\nabla u$, $\nabla(u \wedge 1) = \mathbf{1}_{[u<1]}\nabla u$.*

If $u \in H_0^1(\Omega)$, then $u^+, u \wedge 1 \in H_0^1(\Omega)$.

Proof. It was shown in Corollary 9.13 that the indicated derivatives for u^+ and $u \wedge 1$ are the distributional derivatives. As they belong to $L_2(\Omega;\mathbb{R}^n)$, the first part of the theorem is proved.

Now let $u \in H_0^1(\Omega)$. There exists a sequence (u_k) in $H_{\mathrm{c}}^1(\Omega)$ such that $u_k \to u$ in $H^1(\Omega)$. Then (u_k^+) is a bounded sequence in $H_{\mathrm{c}}^1(\Omega)$, and $u_k^+ \to u^+$ in $L_2(\Omega)$. The fact formulated in Remark 9.15 below shows that u belongs to the closure of $H_{\mathrm{c}}^1(\Omega)$ in $H^1(\Omega)$, i.e., to $H_0^1(\Omega)$. The argument for $u \wedge 1$ is similar. □

9.15 Remark. Let V, H be Hilbert spaces, $V \subseteq H$ with continuous embedding. Let (v_n) be a bounded sequence in V that is weakly convergent in H to $u \in H$. Then $u \in V$, and $v_n \to u$ weakly in V.

Indeed, there exist $v \in V$ and a subsequence (v_{n_k}), $v_{n_k} \to v$ weakly in V. Then also $v_{n_k} \to v$ weakly in H; hence $v = u$. A standard sub-sub-sequence argument shows $v_n \to u$ weakly in V.

9.4 Invariance described by forms

In this section we transform the invariance criteria obtained in Propositions 9.4 and 9.5 to conditions on forms instead of operators. This means that we only treat C_0-semigroups associated with forms. In this case the C_0-semigroup is quasi-contractive, and the condition (9.2) is equivalent to the invariance of C under the semigroup.

We restrict our treatment to the case of embedded forms, i.e., we assume that V is a Hilbert space that is densely embedded into H and that $a\colon V \times V \to \mathbb{K}$ is a bounded H-elliptic form. We recall that this means that there exist $\omega \in \mathbb{R}$, $\alpha > 0$ such that

$$\operatorname{Re} a(u) + \omega \|u\|_H^2 \geqslant \alpha \|u\|_V^2 \qquad (u \in V). \tag{9.5}$$

In the following the quantity

$$\omega_0(a) := \inf\bigl\{\omega \in \mathbb{R};\ \operatorname{Re} a(u) + \omega \|u\|_H^2 \geqslant 0 \ (u \in V)\bigr\}$$

will be needed. Then $-\omega_0(a)$ is the 'lower bound' of a, in particular $\operatorname{Re} a(u) \geqslant -\omega_0(a)\|u\|_H^2$ for all $u \in V$. It follows from Proposition 5.5(b) that $\|T(t)\| \leqslant \mathrm{e}^{\omega_0(a)t}$ for all $t \geqslant 0$, where T is the C_0-semigroup associated with a.

The notation used above will be fixed troughout this section. Coming back to invariance, let $\varnothing \neq C \subseteq H$ be convex and closed, and let $P := P_C$ be the minimising projection.

9.16 Proposition. *Let C be invariant under T. Then $P(V) \subseteq V$.*

9.17 Remark. At the first glance, this property might look rather unexpected, because the elements of V have some quality (or 'regularity'), and it is surprising that this quality is preserved under P. To make the point, the elements of the domain of the generator will not be mapped to the domain of the generator, in general.

For the proof we single out a technical detail, which will be useful in several of the subsequent proofs.

9.18 Lemma. *Let $(u_n), (v_n)$ be sequences in V, $u_n \to u$ in H, (v_n) bounded in V, and*

$$\operatorname{Re} a(u_n, u_n - v_n) \leqslant 0 \qquad (n \in \mathbb{N}).$$

Then $u \in V$, and $u_n \to u$ weakly in V.

Proof. Using (9.5) we estimate

$$\begin{aligned}\alpha\|u_n\|_V^2 &\leqslant \operatorname{Re} a(u_n,u_n)+\omega\|u_n\|_H^2 \leqslant \operatorname{Re} a(u_n,v_n)+\omega\|u_n\|_H^2\\ &\leqslant M\|u_n\|_V\|v_n\|_V+\omega\|u_n\|_H\|u_n\|_V\end{aligned}$$

(where M denotes the bound of a, and without loss of generality we have assumed $\|\cdot\|_H \leqslant \|\cdot\|_V$). This implies that (u_n) is bounded in V, and therefore Remark 9.15 implies that $u_n \to u$ weakly in V. □

Proof of Proposition 9.16. Adding the inequality $(y-Px \,|\, Px-y) \leqslant 0$ to (9.1) one obtains

$$\operatorname{Re}(Px-y \,|\, y-x) = \operatorname{Re}(y-Px \,|\, x-y) \leqslant 0 \qquad (x \in H,\ y \in C). \tag{9.6}$$

For $r > 0$ small enough (say, $0 < r < r_0 < \infty$) we define $R_r := (I+rA)^{-1}$. Then $AR_r = \frac{1}{r}(I+rA-I)R_r = \frac{1}{r}(I-R_r)$, so

$$a(R_r u, v) = (AR_r u \,|\, v) = \frac{1}{r}(u-R_r u \,|\, v) \qquad (u,v \in V). \tag{9.7}$$

We recall from Proposition 9.1 that the invariance of C under T implies that $R_r(C) \subseteq C$. Let $u \in V$. Using (9.7) and applying (9.6) with $y = R_r Pu \in C$, we obtain

$$\operatorname{Re} a(R_r Pu, R_r Pu - u) = \frac{1}{r}\operatorname{Re}(Pu - R_r Pu \,|\, R_r Pu - u) \leqslant 0.$$

Since $R_r Pu \to Pu$ in H as $r \to 0$ (by Lemma 2.10(a)), Lemma 9.18 implies that $Pu \in V$. □

We insert an auxiliary result that will be used in the proof of the next theorem.

9.19 Lemma. (a) *As in the proof of Proposition 9.16 we define* $R_r := (I+rA)^{-1}$ *for* $0 < r < r_0$ *(suitable). Then* $R_r u \to u$ $(r \to 0)$ *weakly in* V *for all* $u \in V$.

(b) *Assume that* (u_n) *is a sequence converging weakly in* V *to* u. *Then* $a(u_n, v) \to a(u,v)$ *for all* $v \in V$.

Proof. (a) Let $u \in V$. By (9.7) we obtain

$$a(R_r u, R_r u - u) = \frac{1}{r}(u - R_r u \,|\, R_r u - u) \leqslant 0.$$

Since $R_r u \to u$ $(r \to 0)$ in H, Lemma 9.18 implies that $R_r u \to u$ weakly in V as $r \to 0$.

(b) For $v \in V$, the functional $V \ni u \mapsto a(u,v) \in \mathbb{K}$ is continuous, by the boundedness of a. This implies the assertion. □

Now we come to the fundamental result concerning invariance characterised by conditions on the form. The inequality (9.9) appearing below has already been commented upon before Proposition 9.4. In order to give a geometrical interpretation to (9.8) we note that, loosely speaking, $a(Pu, u-Pu)$ can be understood as $(A(Pu) \,|\, u-Pu)$ (only Pu is not necessarily in $\operatorname{dom}(A)$). So, the condition gives information on the driving term $-Au(t)$, whenever $u(t)$ is the image Pu of some $u \in H \setminus C$: in these points, the driving term 'points towards C'.

9.20 Theorem. *Under the previous assumptions the following properties are equivalent.*

(i) *C is invariant under T;*

(ii) *$P(V) \subseteq V$, and*

$$\operatorname{Re} a(Pu, u - Pu) \geqslant 0 \tag{9.8}$$

for all $u \in V$;

(iii) *there exists a dense subset D of V such that $P(D) \subseteq V$, and* (9.8) *holds for all $u \in D$;*

(iv) *$P(V) \subseteq V$, and*

$$\operatorname{Re} a(u, u - Pu) \geqslant -\omega \|u - Pu\|^2 \qquad (u \in V), \tag{9.9}$$

for some $\omega \in \mathbb{R}$ / for $\omega = \omega_0(a)$.

Proof. (i)⇒(ii). $P(V) \subseteq V$ was shown in Proposition 9.16. Let $u \in V$. Then for $0 < r < r_0$ we have

$$\operatorname{Re} a(R_r Pu, u - Pu) = \frac{1}{r} \operatorname{Re}\,(Pu - R_r Pu \,|\, u - Pu) \geqslant 0$$

by (9.7) and (9.1), and from Lemma 9.19 we obtain

$$\operatorname{Re} a(Pu, u - Pu) \geqslant 0.$$

(ii)⇔(iii). '(ii) ⇒ (iii)' is trivial. For the proof of '(iii) ⇒ (ii)' let $u \in V$. There exists a sequence (u_n) in D such that $u_n \to u$ in V as $n \to \infty$. By the hypothesis we have

$$\operatorname{Re} a(Pu_n, Pu_n - u_n) \leqslant 0 \qquad (n \in \mathbb{N}). \tag{9.10}$$

From the continuity of P we obtain $Pu_n \to Pu$ in H, and therefore Lemma 9.18 implies that $Pu \in V$, and $Pu_n \to Pu$ weakly in V.

In order to show (9.8) we use that on V an equivalent norm is given by

$$\|v\|_a := \big(\operatorname{Re} a(v) + \omega \|v\|_H^2\big)^{1/2} \qquad (v \in V),$$

associated with the scalar product $\frac{1}{2}(a + a^*) + \omega\,(\cdot\,|\,\cdot)_H$ (with ω from (9.5)). It follows that for the weakly convergent sequence (Pu_n) in V one has $\|Pu\|_a \leqslant \liminf_{n\to\infty} \|Pu_n\|_a$. Using (9.10) we obtain

$$\begin{aligned}\operatorname{Re} a(Pu) + \omega \|Pu\|_H^2 &\leqslant \liminf_{n\to\infty} \big(\operatorname{Re} a(Pu_n) + \omega \|Pu_n\|_H^2\big)\\ &\leqslant \liminf_{n\to\infty} \operatorname{Re} a(Pu_n, u_n) + \omega \|Pu\|_H^2.\end{aligned}$$

Since $a(Pu_n, u_n) = a(Pu_n, u_n - u) + a(Pu_n, u) \to 0 + a(Pu, u)$ as $n \to \infty$, we conclude that $\operatorname{Re} a(Pu, u - Pu) = \operatorname{Re} a(Pu, u) - \operatorname{Re} a(Pu) \geqslant 0$.

(ii)⇒(iv) with '$\omega = \omega_0(a)$' follows from the identity

$$a(u, u - Pu) = a(Pu, u - Pu) + a(u - Pu)$$

and the definition of $\omega_0(a)$.

(iv) with 'some $\omega \in \mathbb{R}$'⇒(i). Because of $a(u, u - Pu) = (Au \,|\, x - Pu)$ $(u \in \operatorname{dom}(A))$, condition (9.9) implies (9.2) for the generator $-A$ of T. Then the assertion follows from Proposition 9.4. □

9.21 Example. We come back to Example 9.8 and show again part (ii), i.e., that the C_0-semigroup generated by the Dirichlet or Neumann Laplacian is sub-Markovian. Using $C = \{u \in L_2(\Omega;\mathbb{R});\ u \leqslant 1\}$ and the minimising projection $Pu = u \wedge 1$, we check property (ii) of Theorem 9.20: Theorem 9.14 implies that P leaves $V = H_0^1(\Omega)$ (and also $V = H^1(\Omega)$) invariant, and

$$a(Pu, u - Pu) = \int \nabla(u \wedge 1) \cdot (\nabla u - \nabla(u \wedge 1)) = \int \mathbf{1}_{[u<1]} \nabla u \cdot \mathbf{1}_{[u \geqslant 1]} \nabla u = 0$$

shows that property (ii) is satisfied. Hence C is invariant by Theorem 9.20.

9.22 Remark. We emphasise that in Theorem 9.20 the associated semigroup is not assumed to be contractive. If the convex set C is not a cone, then the quasi-contractive case cannot be reduced to the contractive case by scaling.

An example for an application to a non-contractive semigroup can be found in Exercise 9.5.

Notes

Clearly, it is of fundamental interest to ask for criteria describing when certain sets are invariant under the time evolution of a system, and questions of this kind have a long history, in particular in the finite-dimensional case, for linear and non-linear problems.

The seminal papers for investigating such questions in infinite-dimensional spaces are probably the papers by Beurling and Deny, [BD58], [BD59]. Another fundamental paper on positive contraction semigroups is by Phillips [Phi62]. Skipping a lot of history we refer to Kunita [Kun70] for the idea to include non-contractive non-symmetric semigroups in the treatment, and we mention the more recent papers by Ouhabaz, [Ouh96] and [MVV05] as well as Ouhabaz' book [Ouh05] and refer to the literature mentioned in these sources. The invariance criterion in Theorem 9.20 is taken from [MVV05; Theorem 2.1]. An investigation on invariance for non-linear evolution equations can be found in [Bar96].

The lattice properties of the Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ have been developed in the 70's (at the latest). We refer to a paper of Marcus and Mizel [MM79] where also earlier references can be found. Meanwhile, these properties can be found in several books on Sobolev spaces or partial differential equations. We refer to [EE87; p. VI.2] for a more general chain rule than presented in the lecture.

Exercises

9.1 Let (Ω, μ) be a measure space, $1 \leqslant p \leqslant \infty$, and let $A \in \mathcal{L}(L_p(\mu))$ be positive, i.e., $Au \geqslant 0$ for all $u \in L_p(\mu)$ with $u \geqslant 0$.

(a) Show that $|Au| \leqslant A|u|$ for all $u \in L_p(\mu)$.

(b) Show that

$$\|A\| = \sup\big\{\|Au\|_p;\ u \in L_p(\mu),\ u \geqslant 0,\ \|u\|_p \leqslant 1\big\}.$$

9.2 Let (Ω, μ) be a measure space, $\check{C} \subseteq \mathbb{K}$ convex and closed, $0 \in \check{C}$; let $\check{P}\colon \mathbb{K} \to \check{C}$ be the minimising projection. Then clearly

$$C := \bigl\{u \in L_2(\mu)\,;\; u(x) \in \check{C} \text{ for } \mu\text{-a.e. } x\bigr\} \neq \varnothing$$

is convex and closed.

Show that the minimising projection $P\colon L_2(\mu) \to C$ is given by $(Pu)(x) = \check{P}(u(x))$ $(x \in \Omega)$.

9.3 Let Δ_β be the Robin Laplacian from Section 7.5.

(a) Let β be real-valued. Show that the C_0-semigroup generated by Δ_β is positive.

(b) Let $\beta \geqslant 0$. Show that the C_0-semigroup generated by Δ_β is sub-Markovian.

9.4 (a) Assume that H, V, a, j are as in Proposition 5.5, and such that minus the operator associated with (a, j) is a generator. Let $\varnothing \neq C \subseteq H$ be convex and closed, P the minimising projection onto C. Let $\hat{P}\colon V \to V$ be a mapping satisfying $Pj = j\hat{P}$. Further assume

$$a(u, u - \hat{P}u) \geqslant 0 \qquad (v \in V). \tag{9.11}$$

Show that C is invariant under the C_0-semigroup associated with (a, j). (Hint: Use Proposition 9.4.)

(For the more ambitious: Show the assertion if (9.11) is replaced by

$$a(\hat{P}u, u - \hat{P}u) \geqslant 0 \qquad (v \in V),$$

additionally assuming that a is j-elliptic.)

(b) Show that the C_0-semigroup generated by the Dirichlet-to-Neumann operator of Section 8.1 is sub-Markovian. (Hint: Show that the operation $u \mapsto u \wedge 1$ for an H^1-function is consistent with the trace operator.)

9.5 Let $\Omega \subseteq \mathbb{R}^n$ be open, let $b \in L_\infty(\Omega; \mathbb{R}^n)$, and define the operator A in $L_2(\Omega)$ by

$$\begin{aligned} \operatorname{dom}(A) &:= \bigl\{u \in H_0^1(\Omega)\,;\; -\Delta u + b \cdot \nabla u \in L_2(\Omega)\bigr\}, \\ Au &:= -\Delta u + b \cdot \nabla u \qquad (u \in \operatorname{dom}(A)). \end{aligned}$$

(a) Show that A is associated with an H-elliptic form on $V \times V$, with $V := H_0^1(\Omega) \subseteq H := L_2(\Omega)$ (and therefore $-A$ generates a quasi-contractive C_0-semigroup). Show that the semigroup generated by $-A$ is holomorphic of angle $\pi/2$, if $\mathbb{K} = \mathbb{C}$ (cf. Exercises 7.3 and 7.4).

(b) Show that the C_0-semigroup generated by $-A$ is sub-Markovian.

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