

Lecture 8

The Dirichlet-to-Neumann operator

The Dirichlet-to-Neumann operator plays an important role in the theory of inverse problems. In fact, from measurements of electrical currents at the surface of the human body one wishes to determine conductivity inside the body. But the Dirichlet-to-Neumann operator also plays a big role in many parts of analysis. Here we prove by form methods that it is a self-adjoint operator in $L_2(\partial\Omega)$. Our form methods come to fruition; in particular, to allow general mappings j is very useful here: throughout this lecture j will be the trace operator.

8.1 The Dirichlet-to-Neumann operator for the Laplacian

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1 -boundary. We use the classical Dirichlet form

$$a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} \quad (u, v \in H^1(\Omega)). \quad (8.1)$$

If we choose the canonical injection of $H^1(\Omega)$ into $L_2(\Omega)$, the associated operator is the Neumann Laplacian. Here we will choose as j the trace operator from $H^1(\Omega)$ to $L_2(\partial\Omega)$, introduced in Theorem 7.9. We will show that the form a is j -elliptic. Thus we obtain as associated operator a self-adjoint operator in $L_2(\partial\Omega)$. It turns out that this is the Dirichlet-to-Neumann operator D_0 which maps $g \in \text{dom}(D_0)$ to $\partial_\nu u \in L_2(\partial\Omega)$ where $u \in H^1(\Omega)$ is the harmonic function with $u|_{\partial\Omega} = g$. This will be made more precise in the following. (We recall that a function u defined on an open set is called **harmonic** if it is twice continuously differentiable, and $\Delta u = 0$.)

Observe that the trace operator $\text{tr}: H^1(\Omega) \rightarrow L_2(\partial\Omega)$ has dense image. In fact, by the Stone-Weierstrass theorem ([Yos68; Section 0.2]) the set $\{\varphi|_{\partial\Omega}; \varphi \in C_c^\infty(\mathbb{R}^n)\}$ is dense in $C(\partial\Omega)$, and $C(\partial\Omega)$ is dense in $L_2(\partial\Omega)$. Next we prove j -ellipticity.

8.1 Proposition. *With $j = \text{tr}$, the classical Dirichlet form is j -elliptic.*

For the proof we need an auxiliary result. We state it for the general case of Banach spaces; in our context it will only be needed for Hilbert spaces.

8.2 Lemma. *Let X, Y, Z be Banach spaces, X reflexive, $K \in \mathcal{L}(X, Y)$ compact and $S \in \mathcal{L}(X, Z)$ injective. Then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that*

$$\|Kx\|_Y \leq \varepsilon \|x\|_X + c_\varepsilon \|Sx\|_Z.$$

Proof. If not, there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$\|Kx_n\|_Y > \varepsilon \|x_n\|_X + n \|Sx_n\|_Z$$

and $\|x_n\|_X = 1$. Passing to a subsequence we may assume that $x_n \rightarrow x$ weakly. Hence $Sx_n \rightarrow Sx$ weakly. The inequality implies that $\|Sx_n\|_Z \rightarrow 0$, hence $Sx = 0$. Since S is injective, it follows that $x = 0$. Since K is compact it follows that $Kx_n \rightarrow 0$ in norm. But $\|Kx_n\|_Y \geq \varepsilon$ for all $n \in \mathbb{N}$, a contradiction. \square

Proof of Proposition 8.1. (i) Define $S: H^1(\Omega) \rightarrow L_2(\Omega; \mathbb{K}^n) \oplus L_2(\partial\Omega)$ by

$$Su := (\nabla u, u|_{\partial\Omega}).$$

We will show below that S is injective. Because the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, by Theorem 7.11, the application of Lemma 8.2 yields a constant $c \geq 0$ such that

$$\begin{aligned} \int_{\Omega} |u|^2 dx &\leq \frac{1}{2} \|u\|_{H^1(\Omega)}^2 + c \|Su\|^2 \\ &= \frac{1}{2} \int_{\Omega} |u|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + c \int_{\Omega} |\nabla u|^2 dx + c \int_{\partial\Omega} |u|^2 d\sigma. \end{aligned}$$

Adding $\frac{1}{2}(\int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |u|^2 dx)$ to this inequality we obtain

$$\frac{1}{2} \|u\|_{H^1(\Omega)}^2 \leq c \|u\|_{L_2(\partial\Omega)}^2 + (c+1)a(u) \quad (u \in H^1(\Omega)),$$

and this shows that a is j -elliptic.

(ii) For the proof that S is injective let $u \in H^1(\Omega)$ be such that $Su = 0$. Then $\nabla u = 0$, and therefore u is constant on each connected component ω of Ω . As $\text{tr } u = 0$, this constant is zero. Therefore $u = 0$. \square

8.3 Remark. In the proof given above (as well as in Exercise 7.1) it was used that a function u on a connected open set Ω with $\nabla u = 0$ (distributionally) is constant.

This is shown as follows. If one takes the convolution of u with a function $\rho \in C_c^\infty(\mathbb{R}^n)$ with support in a ‘small’ neighbourhood of 0, then $\rho * u$ is infinitely differentiable and satisfies $\nabla(\rho * u) = 0$ far enough away from $\partial\Omega$, and therefore is locally constant far enough away from $\partial\Omega$. Convoluting u with a δ -sequence in $C_c^\infty(\mathbb{R}^n)$ one therefore infers that, locally on Ω , u is the limit of constant functions. Thus u has a locally constant representative, which is constant since Ω is connected.

8.4 Theorem. Let j be the trace operator, and let a be the classical Dirichlet form (8.1). Then the operator D_0 in $L_2(\partial\Omega)$ associated with (a, j) is described by

$$D_0 = \{(g, h) \in L_2(\partial\Omega) \times L_2(\partial\Omega); \exists u \in H^1(\Omega): \Delta u = 0, u|_{\partial\Omega} = g, \partial_\nu u = h\}.$$

The operator D_0 is self-adjoint and positive and has compact resolvent. We call D_0 the **Dirichlet-to-Neumann operator** (with respect to Δ).

Proof. Let $(g, h) \in D_0$. Then there exists $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = g$ and

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} = a(u, v) = \int_{\partial\Omega} h \bar{v} \quad (8.2)$$

for all $v \in H^1(\Omega)$. Employing this equality with $v \in C_c^\infty(\Omega)$ we obtain $-\Delta u = 0$. Thus adding $\Delta u = 0$ to (8.2) we have

$$\int_{\Omega} (\Delta u) \bar{v} + \int_{\Omega} \nabla u \cdot \overline{\nabla v} = \int_{\partial\Omega} h \bar{v} \quad (v \in H^1(\Omega)).$$

Hence $\partial_\nu u = h$ by our definition of the normal derivative. This shows “ \subseteq ” in the asserted equality for D_0 .

In order to get the converse inclusion let $u \in H^1(\Omega)$ such that $\Delta u = 0$, $g := u|_{\partial\Omega}$, $h := \partial_\nu u \in L_2(\partial\Omega)$. Then

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Omega} (\Delta u) \bar{v} = \int_{\partial\Omega} h \bar{v} \quad (v \in H^1(\Omega)).$$

Thus $a(u, v) = (h | j(v))_{L_2(\partial\Omega)}$ for all $v \in H^1(\Omega)$. Consequently $(g, h) \in D_0$.

The symmetry of a implies that D_0 is self-adjoint. Finally, it was shown in Remark 7.12 that $\text{tr}: H^1(\Omega) \rightarrow L_2(\partial\Omega)$ is compact; hence D_0 has compact resolvent, by Proposition 6.15. \square

Our next aim is to define Dirichlet-to-Neumann operators with respect to more general Dirichlet problems. The following interlude is a preparation for this treatment.

8.2 Interlude: the Fredholm alternative in Hilbert space

We need a detail from the spectral theory of compact operators which we formulate and prove only for operators in Hilbert spaces. It will be used in the proof of Proposition 8.10.

8.5 Proposition. *Let H be a Hilbert space, $K \in \mathcal{L}(H)$ compact. Then $I + K$ is injective if and only if $\text{ran}(I + K) = H$, and in this case $I + K$ is invertible in $\mathcal{L}(H)$.*

Proof. (i) First we note that $K(B_H[0, 1])$ is a relatively compact set in a metric space, and as such is separable; therefore $\text{ran}(K)$ is separable.

(ii) In this step we treat the case that $\dim \text{ran}(K) < \infty$. We define $H_1 := \ker(K)$, $H_2 := \ker(K)^\perp$ and denote by P_1, P_2 the orthogonal projections onto H_1, H_2 , respectively. Note that $\dim H_2 < \infty$. (Indeed, from $K = \sum_{j=1}^n (\cdot | x_j) y_j$ one obtains $\ker(K) \supseteq \{x_1, \dots, x_n\}^\perp$, and therefore $\ker(K)^\perp \subseteq \text{lin}\{x_1, \dots, x_n\}$.) On the orthogonal sum $H_1 \oplus H_2$ the operator $I + K$ can be written as the operator matrix

$$I + K = \begin{pmatrix} I_1 & P_1 K|_{H_2} \\ 0 & I_2 + P_2 K|_{H_2} \end{pmatrix},$$

where I_1, I_2 are the identity operators in H_1, H_2 , respectively. In the matrix representation it is easy to see that $I + K$ is injective if and only if $I_2 + P_2 K|_{H_2}$ is injective. From

(finite-dimensional) linear algebra it is known that the operator $I_2 + P_2K|_{H_2}$ is injective if and only if it is surjective. Again, using the matrix representation one easily can see that $I_2 + P_2K|_{H_2}$ is surjective if and only if $I + K$ is surjective.

If $I + K$ is injective, then in the (upper diagonal) block matrix version of $I + K$ the diagonal entries are invertible, and it is straightforward to show that then $I + K$ is invertible in $\mathcal{L}(H)$.

(iii) If $\text{ran}(K)$ is not finite-dimensional, let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $\overline{\text{ran}(K)}$, and let P_n be the orthogonal projection onto $\text{lin}\{e_1, \dots, e_n\}$. Then $P_n x \rightarrow x$ ($n \rightarrow \infty$) for all $x \in \overline{\text{ran}(K)}$, and the compactness of K implies that $P_n K \rightarrow K$ in the operator norm.

Therefore K can be written as a sum $K = K_1 + K_2$, where $\|K_1\| < 1$ and $\text{ran}(K_2)$ is finite-dimensional. Then $I + K_1$ is invertible in $\mathcal{L}(H)$ (recall Remark 2.3(a)), and therefore $(I + K_1)^{-1}(I + K) = I + F$, with the finite rank operator $F = (I + K_1)^{-1}K_2$. Hence $I + K$ is injective if and only if $I + F$ is injective if and only if $I + F$ is surjective (by step (ii)) if and only if $I + K$ is surjective.

If $I + K$ is injective, then the continuity of the inverse follows from part (ii) and $(I + K)^{-1} = (I + F)^{-1}(I + K_1)^{-1}$. \square

8.6 Remark. If, in the situation of Proposition 8.5, $B \in \mathcal{L}(H)$ is invertible in $\mathcal{L}(H)$, then $B + K$ is injective if and only if $\text{ran}(B + K) = H$. This is immediate from $B + K = B(I + B^{-1}K)$ and Proposition 8.5, applied to $I + B^{-1}K$.

Proposition 8.5 carries the name ‘alternative’ because the only *alternative* to $I + K$ being invertible is that $I + K$ is not injective *and* $I + K$ is not surjective.

8.3 Quasi-m-accretive and self-adjoint operators via compactly elliptic forms

Let V, H be Hilbert spaces, and let $a: V \times V \rightarrow \mathbb{K}$ be a continuous sesquilinear form. Let $j \in \mathcal{L}(V, H)$ have dense range. We assume that

$$u \in \ker(j), a(u, v) = 0 \text{ for all } v \in \ker(j) \text{ implies } u = 0. \quad (8.3)$$

This is slightly more general than (5.4).

As in Section 5.3, let

$$A := \{(x, y) \in H \times H; \exists u \in V: j(u) = x, a(u, v) = (y | j(v)) \ (v \in V)\}.$$

8.7 Proposition. *Assume (8.3). Then the relation A defined above is an operator. As before, we call A the operator associated with (a, j) . If (8.3) is also satisfied for a^* , and B is the operator associated with (a^*, j) , then $B \subseteq A^*$ and $A \subseteq B^*$. In particular, if a is symmetric and $\text{dom}(A)$ is dense, then A is symmetric.*

The proof is delegated to Exercise 8.1. For the following we define

$$V_j(a) := \{u \in V; a(u, v) = 0 \ (v \in \ker(j))\}.$$

8.8 Remarks. (a) Condition (8.3) is equivalent to $V_j(a) \cap \ker(j) = \{0\}$.

(b) $V_j(a)$ is a closed subspace of V . One can think of $V_j(a)$ as the ‘orthogonal complement’ of $\ker(j)$ with respect to a .

We call the form a **compactly elliptic** if there exist a Hilbert space \tilde{H} and a compact operator $\tilde{j}: V \rightarrow \tilde{H}$ such that a is ‘ \tilde{j} -elliptic’, i.e.,

$$\operatorname{Re} a(u) + \|\tilde{j}(u)\|_{\tilde{H}}^2 \geq \tilde{\alpha} \|u\|_V^2 \quad (8.4)$$

for all $u \in V$ and some $\tilde{\alpha} > 0$. We are going to show that the operator associated with (a, j) is quasi- m -accretive if in addition (8.3) is satisfied.

8.9 Lemma. *If (8.3) is satisfied and a is compactly elliptic, then there exist $\omega \geq 0$, $\alpha > 0$ such that*

$$\operatorname{Re} a(u) + \omega \|j(u)\|_H^2 \geq \alpha \|u\|_V^2 \quad (u \in V_j(a)). \quad (8.5)$$

Proof. According to Remark 8.8(a), j is injective on $V_j(a)$. Therefore Lemma 8.2 implies

$$\|\tilde{j}(u)\|_{\tilde{H}}^2 \leq \frac{\tilde{\alpha}}{2} \|u\|_V^2 + c \|j(u)\|_H^2 \quad (u \in V_j(a)),$$

with $\tilde{\alpha} > 0$ from (8.4) and some $c \geq 0$. From (8.4) we then obtain

$$\operatorname{Re} a(u) + c \|j(u)\|_H^2 \geq \frac{\tilde{\alpha}}{2} \|u\|_V^2. \quad \square$$

8.10 Proposition. *Assume that (8.3) is satisfied and that a is compactly elliptic. Then $V = V_j(a) \oplus \ker(j)$ is a (not necessarily orthogonal) topological direct sum.*

Proof. There exists $R \in \mathcal{L}(V)$ such that

$$(Ru | v)_V = a(u, v) + \omega (j(u) | j(v))_H \quad (u, v \in V),$$

with ω from (8.5). (For fixed u , the right hand side, as a function of v , belongs to V^* , and Ru results from the Riesz-Fréchet theorem.) Then R is injective: If $Ru = 0$, then $a(u, v) = 0$ for all $v \in \ker(j)$, i.e., $u \in V_j(a)$, and then $\alpha \|u\|_V^2 \leq \operatorname{Re} (Ru | u) = 0$ by (8.5).

Define $B \in \mathcal{L}(V)$ by

$$(Bu | v)_V = a(u, v) + \omega (j(u) | j(v))_H + (\tilde{j}(u) | \tilde{j}(v))_{\tilde{H}} \quad (u, v \in V).$$

Then the ‘operator version’ of the Lax-Milgram lemma (Remark 5.3) implies that B is invertible in $\mathcal{L}(V)$, by (8.4). The operator $K := \tilde{j}^* \tilde{j} \in \mathcal{L}(V)$ is compact, and from the definitions we see that $R = B - K$. As R is injective, the ‘Fredholm alternative’, Proposition 8.5 in the guise of Remark 8.6, shows that R is invertible in $\mathcal{L}(V)$. Note that then also R^* is invertible in $\mathcal{L}(V)$; see Exercise 8.2(c).

Let $J: V_j(a) \hookrightarrow V$ be the embedding and $S := J^* R^* J$. Then $S \in \mathcal{L}(V_j(a))$, and

$$\operatorname{Re} (Su | u)_{V_j(a)} = \operatorname{Re} (R^* u | u)_V = \operatorname{Re} (u | Ru)_V \geq \alpha \|u\|_V^2 \quad (u \in V_j(a)).$$

This shows that S is coercive, hence invertible in $\mathcal{L}(V_j(a))$, by the operator version of the Lax-Milgram lemma, Remark 5.3. Note that J^* is the orthogonal projection onto $V_j(a)$; see Exercise 8.2(d).

Now, let $P := JS^{-1}J^*R^* \in \mathcal{L}(V)$. Then $P^2 = JS^{-1}SS^{-1}J^*R^* = P$, and clearly $\text{ran}(P) = V_j(a)$. This shows that P is a projection onto $V_j(a)$. We note that

$$\text{ran}(RJ) = R(V_j(a)) = \ker(j)^\perp.$$

Indeed, $u \in V_j(a)$ if and only if $(Ru | v) = a(u, v) = 0$ for all $v \in \ker(j)$, i.e., $Ru \in \ker(j)^\perp$. As R is invertible, this shows the equality. This implies

$$\ker P = \ker(J^*R^*) = \text{ran}(RJ)^\perp = \ker(j).$$

(For the second of these equalities we refer to Exercise 8.2(b) and Lemma 6.7.) Therefore P is the (continuous) projection onto $V_j(a)$ along $\ker(j)$, and this shows the assertion. \square

We insert the definition that a self-adjoint operator A in a Hilbert space H is called **bounded below** or **bounded above** if the set $\{(Ax | x); x \in \text{dom}(A), \|x\| = 1\}$ is bounded below or bounded above, respectively. Now we – finally – can show that the operator associated with (a, j) is as one should hope for.

8.11 Theorem. *Let $a: V \times V \rightarrow \mathbb{K}$ be a bounded form, and let $j \in \mathcal{L}(V, H)$ have dense range. Assume that (8.3) is satisfied and that a is compactly elliptic.*

Let A be the operator associated with (a, j) . Then A is quasi- m -accretive. If a is symmetric, then A is self-adjoint and bounded below. If j is compact, then A has compact resolvent.

Moreover, let $\check{a} := a|_{V_j(a) \times V_j(a)}$ and $\check{j} := j|_{V_j(a)}$. Then $\text{ran}(\check{j})$ is dense, and A is also associated with (\check{a}, \check{j}) .

Proof. From Proposition 8.10 one obtains $\check{j}(V_j(a)) = j(V)$, in particular $\text{ran}(\check{j})$ is dense. By Lemma 8.9, \check{a} is \check{j} -elliptic. Hence the operator \check{A} associated with (\check{a}, \check{j}) is quasi- m -accretive. If a is symmetric, then \check{a} is symmetric, and \check{A} is self-adjoint. If j is compact, then \check{j} is compact, and \check{A} has compact resolvent, by Proposition 6.15. It remains to show that $A = \check{A}$.

Let $(x, y) \in A$. Then there exists $u \in V$ such that $j(u) = x$ and $a(u, v) = (y | j(v))$ for all $v \in V$. In particular, for $v \in \ker(j)$ one has $a(u, v) = (y | j(v)) = 0$. Hence $u \in V_j(a)$ and $a(u, v) = (y | j(v))$ for all $v \in V_j(a)$. This implies that $(x, y) \in \check{A}$.

Conversely, let $(x, y) \in \check{A}$. Then there exists $u \in V_j(a)$ such that $j(u) = x$ and $a(u, v) = (y | j(v))$ for all $v \in V_j(a)$. For $v \in \ker(j)$ one has $a(u, v) = 0 = (y | j(v))$, because $u \in V_j(a)$. Now Proposition 8.10 implies that $a(u, v) = (y | j(v))$ for all $v \in V$, and therefore $(x, y) \in A$. \square

8.12 Remark. The last statement of Theorem 8.11 says that it is possible to specialise to the case of *embedded forms*, i.e., to the case that $j: V \rightarrow H$ is injective. This also applies to the situation that a is j -elliptic, even if j is not compact, which can be seen as follows.

There exists $\omega \in \mathbb{R}$ such that the form b defined by $b(u, v) := a(u, v) + \omega(j(u) | j(v))$ is coercive. Recall from Remark 5.6 that (b, j) is associated with $\omega + A$. One easily sees that $V_j(b) = V_j(a)$. Moreover, b is compactly elliptic, with $\tilde{j} = 0$, so Proposition 8.10 yields $V = V_j(b) \oplus \ker(j)$, and Theorem 8.11 applies to b . Let $\check{a}, \check{b}, \check{j}$ be the restrictions as in the last assertion of the theorem. Then $\omega + A$ is associated with (\check{b}, \check{j}) , and from Remark 5.6 we conclude that A is associated with (\check{a}, \check{j}) .

8.4 The Dirichlet-to-Neumann operator with respect to

$$\Delta + m$$

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1 -boundary. Let $m \in L_\infty(\Omega)$ be real-valued. Our aim is to consider the Dirichlet-to-Neumann operator D_m with respect to $(\Delta + m)$ -harmonic functions. This means that we define D_m in $L_2(\partial\Omega)$ by requiring that for $g, h \in L_2(\partial\Omega)$ one has $g \in \text{dom}(D_m)$ and $D_m g = h$ if there is a solution $u \in H^1(\Omega)$ of $\Delta u + mu = 0$, $u|_{\partial\Omega} = g$ such that $\partial_\nu u = h$. We will show that D_m is a self-adjoint operator if

$$0 \notin \sigma(\Delta_D + m) \quad (8.6)$$

which we want to suppose throughout. Here $\Delta_D + m$ is the Dirichlet Laplacian perturbed by the bounded multiplication operator by the function m . We observe that this operator is self-adjoint, bounded above and has compact resolvent; see Exercise 8.3.

As in Section 8.1 we consider $H = L_2(\partial\Omega)$, $V = H^1(\Omega)$ and as $j \in \mathcal{L}(H^1(\Omega), L_2(\partial\Omega))$ the trace operator. According to the intended setup we now define the form $a: V \times V \rightarrow \mathbb{C}$ by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} - \int_{\Omega} mu\bar{v} \quad (u, v \in H^1(\Omega)).$$

8.13 Remark. The form a is in general not j -elliptic. In fact, let $\lambda > \lambda_1$, where λ_1 is the first Dirichlet eigenvalue, and $m := \lambda$. Choose $u \in H_0^1(\Omega)$ as an eigenfunction of $-\Delta_D$ belonging to λ_1 . Then $\int_{\Omega} |\nabla u|^2 = \lambda_1 \int_{\Omega} |u|^2$, and

$$\int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} |u|^2 + \omega \int_{\partial\Omega} |u|^2 = (\lambda_1 - \lambda) \int_{\Omega} |u|^2 < 0$$

for all $\omega \geq 0$. Thus the form is not j -elliptic.

Since the form a is not j -elliptic the theory developed in the previous lectures is not applicable. However, we can apply the results of the previous section.

8.14 Theorem. *Suppose that (8.6) holds. Then*

$$D_m := \{(g, h) \in L_2(\partial\Omega) \times L_2(\partial\Omega); \exists u \in H^1(\Omega): \Delta u + mu = 0, u|_{\partial\Omega} = g, \partial_\nu u = h\}$$

defines a self-adjoint operator with compact resolvent, and D_m is bounded below.

Proof. Let a be the form defined above, and let $j \in \mathcal{L}(H^1(\Omega), L_2(\partial\Omega))$ be the trace. We first show that condition (8.3) is satisfied. We recall from Theorem 7.10 that $\ker(j) = H_0^1(\Omega)$. Let $u \in \ker(j)$ such that $a(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} - \int_{\Omega} mu\bar{v} = 0$ for all $v \in \ker(j) = H_0^1(\Omega)$. Then $u \in \text{dom}(\Delta_D + m)$ and $\Delta_D u + mu = 0$, by the definition of Δ_D . This implies $u = 0$ since $0 \notin \sigma(\Delta_D + m)$ by our assumption.

In order to show that a is compactly elliptic we choose $\tilde{H} := L_2(\Omega)$ and as \tilde{j} the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$, multiplied by $c := (\|m\|_\infty + 1)^{1/2}$. Then

$$a(u) + \|\tilde{j}(u)\|_2^2 = \int_{\Omega} |\nabla u|^2 - \int_{\Omega} m|u|^2 + (\|m\|_\infty + 1)\|u\|_2^2 \geq \int_{\Omega} |\nabla u|^2 + \|u\|_2^2 = \|u\|_{H^1}^2$$

for all $u \in H^1(\Omega)$, and from Theorem 7.11 we know that \tilde{j} is compact.

Let A be the operator associated with (a, j) . By Theorem 8.11, A is self-adjoint and bounded below. We show that $A = D_m$. In fact, for $g, h \in L_2(\partial\Omega)$ we have $(g, h) \in A$ if and only if there exists $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = g$ and

$$\int_{\Omega} \nabla u \cdot \overline{\nabla v} - \int_{\Omega} mu\bar{v} = \int_{\partial\Omega} h\bar{v} \quad (v \in H^1(\Omega)). \quad (8.7)$$

Inserting test functions $v \in C_c^\infty(\Omega)$ one concludes that $-\Delta u - mu = 0$. Plugging $mu = -\Delta u$ into (8.7) we deduce that $\partial_\nu u = h$. Thus $(g, h) \in D_m$. Conversely, if $(g, h) \in D_m$, then there exists $u \in H^1(\Omega)$ such that $\Delta u + mu = 0$, $u|_{\partial\Omega} = g$ and $\partial_\nu u = h$. Thus, by the definition of the normal derivative,

$$\int_{\partial\Omega} h\bar{v} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + \int_{\Omega} (\Delta u)\bar{v} = \int_{\Omega} \nabla u \cdot \overline{\nabla v} - \int_{\Omega} mu\bar{v} = a(u, v) \quad (v \in H^1(\Omega)),$$

hence $(g, h) \in A$.

As j is compact, by Remark 7.12, Theorem 8.11 implies that A has compact resolvent. \square

Notes

A large part of the material of this lecture is an extract from [AEKS14]. The main result of Section 8.3, Theorem 8.11, goes beyond this paper and is due to H. Vogt. Our proof of Theorem 8.11 for the general non-symmetric case is based on the decomposition in Proposition 8.10. A different proof can be given based on results in [Sau13].

We may add that assumption (8.6) is not really needed. However, if $0 \in \sigma(\Delta_D + m)$, then D_m is a self-adjoint relation and no longer an operator. We did not want to introduce the notion of self-adjoint relations here. However, it is well motivated. In the complex case, the resolvent $(is - D_m)^{-1}$ is a bounded operator on $L_2(\partial\Omega)$, and the mapping $L_\infty(\Omega) \ni m \mapsto (is - D_m)^{-1} \in \mathcal{L}(L_2(\partial\Omega))$ is continuous. This gives valuable information on the stability of the inverse problem which interests engineers and doctors at the same time.

Exercises

8.1 Prove Proposition 8.7.

8.2 Let F, G, H be Hilbert spaces.

(a) Let A be a densely defined operator from G to H , $B \in \mathcal{L}(G, H)$. Show that $(A + B)^* = A^* + B^*$.

(b) Let $A \in \mathcal{L}(F, G)$, $B \in \mathcal{L}(G, H)$. Show that $(BA)^* = A^*B^*$.

(c) Let $A \in \mathcal{L}(H)$ be invertible in $\mathcal{L}(H)$. Show that A^* is invertible in $\mathcal{L}(H)$.

(d) Let $H_0 \subseteq H$ be a closed subspace, $J: H_0 \rightarrow H$ the embedding. Show that J^* is the orthogonal projection from H onto H_0 .

8.3 Let $\Omega \subseteq \mathbb{R}^n$ be open, $m \in L_\infty(\Omega)$ real-valued. Show that $\Delta_D + m$ is self-adjoint, bounded above and has compact resolvent. (Hint: Use Exercise 8.2(a).)

8.4 Let $-\infty < a < b < \infty$. (a) Compute the Dirichlet-to-Neumann operator D_0 for $\Omega = (a, b)$, and compute the C_0 -semigroup generated by $-D_0$.

(b) For $a = -1, b = 1$, interpret the result in the light of Exercise 8.5.

8.5 Let $U_n := B_{\mathbb{R}^n}(0, 1)$ be the open unit ball in \mathbb{R}^n , $S_{n-1} := \partial U_n$ the unit sphere. The following facts can be used for the solution of this exercise: For each $\varphi \in C(S_{n-1})$ there exists a unique solution $u \in C(\overline{U_n})$ of the Dirichlet problem,

$$u|_{S_{n-1}} = \varphi, \quad u|_{U_n} \text{ harmonic.}$$

(We mention that the solution can be written down explicitly with the aid of the Poisson kernel, but this will not be needed for solving the exercise.) The solution satisfies $u|_{U_n} \in C^\infty(U_n)$ and $\|u\|_\infty \leq \|\varphi\|_\infty$. We will use the notation $G\varphi := u$; thus $G \in \mathcal{L}(C(S_{n-1}), C(\overline{U_n}))$.

Define $T(t) \in \mathcal{L}(C(S_{n-1}))$ by

$$T(t)\varphi(z) := u(e^{-t}z) \quad (z \in S_{n-1}, t \geq 0)$$

(with φ and $u = G\varphi$ as above).

(a) Show that T is a C_0 -semigroup of contractions on $C(S_{n-1})$.

(b) Let A be the generator of T . Show that $D := \bigcup_{t>0} \text{ran}(T(t))$ is a core for A , and that $A\varphi = -\partial_\nu(G\varphi)$ for all $\varphi \in D$.

(c) Define $A_{\min} := A|_D$. Show that $-A_{\min}$ is a restriction of the Dirichlet-to-Neumann operator in $L_2(S_{n-1})$, and that $D_0 = -A_{\min}$ (where A_{\min} is interpreted as an operator in $L_2(S_{n-1})$). Conclude that T extends to a C_0 -semigroup T_2 of contractions on $L_2(S_{n-1})$, and that $-D_0$ is the generator of T_2 .

(We refer to [Lax02; Section 36.2] for this exercise.)

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