Lecture 7

Robin boundary conditions

So far we have only studied the Laplacian with Dirichlet boundary conditions. Our aim of this lecture is to investigate Neumann boundary conditions

$$\partial_\nu u = 0 \quad \text{on } \partial\Omega$$

and more generally Robin boundary conditions

$$\partial_\nu u + \beta u = 0 \quad \text{on } \partial\Omega.$$

If we think of heat conduction in a body Ω, then the Neumann boundary condition describes an isolated body, whereas Robin boundary conditions describe when part of the heat is absorbed at the boundary.

We start with the description of properties of the boundary for an open subset of $\mathbb{R}^n$. The main issue of Section 7.1 will be the statement of Gauss' theorem and some discussion of its consequences. In particular we point out that it can be considered as an n-dimensional version of the fundamental theorem of calculus.

In an interlude in Section 7.2 we present properties of $H^1(\Omega)$ that will be needed in order to formulate Neumann and Robin boundary conditions and to derive properties of the associated operators.

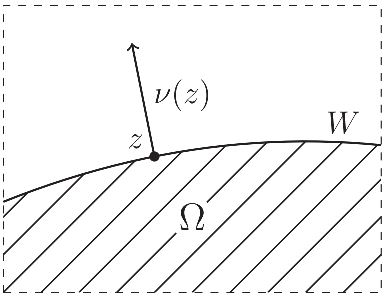
7.1 Gauss' theorem

The aim of this section is to generalise the fundamental theorem of calculus to higher dimensions. For this we need to define the outer normal. We will mainly consider open sets with C^1-boundary, but in order to be complete we occasionally mention domains with more general boundaries. For $x, y \in \mathbb{K}^n$ we use the notation $x \cdot y := \sum_{j=1}^n x_j y_j$ (which for $\mathbb{R}^n$ is the natural scalar product) and $|x| := \sqrt{x \cdot \overline{x}}$ (which for $\mathbb{R}^n$ is the Euclidean norm). All over this section, Ω is a bounded open subset of $\mathbb{R}^n$.

Let $W \subseteq \partial\Omega$ be an open subset (of the metric space $\partial\Omega$). Then we say that W is a **normal C^1-graph** (with respect to Ω) if $W' := \{(z_1, \ldots, z_{n-1});\ z = (z_1, \ldots, z_n) \in W\} \subseteq \mathbb{R}^{n-1}$ is open and there exist an open interval $(a, b) \subseteq \mathbb{R}$ and a C^1-function $g\colon W' \to (a, b)$ such that $W = \{(y, g(y));\ y \in W'\}$, i.e., W is the graph of g, and for a point $(y, t) \in W' \times (a, b)$ one has

$$(y, t) \in \Omega \quad \text{if and only if} \quad t < g(y).$$

It is easy to see that then $(y,t) \notin \overline{\Omega}$ if and only if $t > g(y)$. The set W is a C^1**-graph** (with respect to Ω), if there exists an orthogonal matrix $B \in \mathbb{R}^{n\times n}$ such that $\Phi(W)$ is a normal C^1-graph with respect to $\Phi(\Omega)$, where $\Phi(x) = Bx$ $(x \in \mathbb{R}^n)$. This means of course that W is a normal C^1-graph with respect to another cartesian coordinate system. We say that Ω has C^1**-boundary** if for each $z \in \partial\Omega$ there exists an open neighbourhood $W \subseteq \partial\Omega$ of z such that W is a C^1-graph.



We define right away other possible regularity properties of the boundary.

7.1 Remarks. (C^k-, Lipschitz, continuous boundary)

(a) For $k \in \mathbb{N} \cup \{\infty\}$ we call W a C^k**-graph** if the function g in the definition given above is a C^k-function.

We talk of a **Lipschitz graph** if $g\colon W' \to (a,b)$ satisfies $|g(x) - g(y)| \leqslant L|x-y|$ for all $x, y \in W'$ and some $L > 0$. We call V a **continuous graph** if we merely require that $g\colon W' \to (a,b)$ is continuous.

(b) We say that Ω has C^k**-boundary** (**Lipschitz boundary**, **continuous boundary**), if for each $z \in \partial\Omega$ there exists an open neighbourhood $W \subseteq \partial\Omega$ of z such that W is a C^k-graph (or Lipschitz graph, or continuous graph).

In this way we have defined a hierarchy of regularity properties. If Ω has C^1-boundary, then it also has Lipschitz boundary. Continuous boundary is the weakest property we consider and C^∞-boundary is the strongest. Each polygon in $\mathbb{R}^2$ and each convex polyhedron in $\mathbb{R}^3$ has Lipschitz boundary, but not C^1-boundary. So there are good reasons to consider Lipschitz boundary. But things become much easier in C^1-domains, on which we will focus.

At first we introduce the outer (or exterior) normal of a C^1-domain. This can be done intrinsically without mentioning the graph.

7.2 Proposition. *Assume that Ω has C^1-boundary. Then for each $z \in \partial\Omega$ there is a unique vector $\nu(z) \in \mathbb{R}^n$ satisfying*

(i) $|\nu(z)| = 1$*;*

(ii) *if $\psi \in C^1(-1,1;\mathbb{R}^n)$ is such that $\psi(0) = z$ and $\psi(t) \in \partial\Omega$ for all $t \in (-1,1)$, then $\nu(z) \cdot \psi'(0) = 0$;*

(iii) *there exists $\varepsilon > 0$ such that $z + t\nu(z) \notin \overline{\Omega}$ (and $z - t\nu(z) \in \Omega$) for all $0 < t < \varepsilon$.*

We call $\nu(z)$ the ***outer normal at*** *z. It is a continuous function on $\partial\Omega$ with values in $\mathbb{R}^n$.*

Condition (ii) says that $\nu(z)$ is orthogonal to the boundary and (iii) that $\nu(z)$ points out of Ω.

7.3 Remark. We do not give a proof of Proposition 7.2 but we mention that

$$\nu(z) = \frac{(-\nabla g(z'), 1)}{\sqrt{|\nabla g(z')|^2 + 1}}$$

if $z = (z', z_n) \in W$, with W as in the description of a normal C^1-graph.

Now we can formulate Gauss' theorem. By $C^1(\overline{\Omega})$ we denote the space of all functions $u \in C^1(\Omega) \cap C(\overline{\Omega})$ for which $\partial_j u$ has a continuous extension to $\overline{\Omega}$ for each $j \in \{1, \dots, n\}$. We keep the notation $\partial_j u$ for this extension.

7.4 Theorem. *(Gauss) There exists a unique Borel measure σ on $\partial\Omega$, the **surface measure** on $\partial\Omega$, such that*

$$\int_\Omega \partial_j u(x)\,\mathrm{d}x = \int_{\partial\Omega} u(z)\nu_j(z)\,\mathrm{d}\sigma(z)$$

for each $j \in \{1, \dots, n\}$ and all $u \in C^1(\overline{\Omega})$. Here $\nu \in C(\partial\Omega; \mathbb{R}^n)$ is the outer normal with coordinates $\nu(z) = (\nu_1(z), \dots, \nu_n(z))$.

7.5 Remarks. (a) If $W \subseteq \partial\Omega$ is a normal graph and $u \in C(\partial\Omega)$ has support in W, then

$$\int_{\partial\Omega} u(z)\,\mathrm{d}\sigma(z) = \int_{W'} u(z', g(z'))\sqrt{|\nabla g(z')|^2 + 1}\,\mathrm{d}z',$$

where g is as in the definition for the normal graph. This formula is one of the ingredients for the construction of the measure σ. The weight factor in the integral on the right hand side is such that the $(n-1)$-dimensional Lebesgue measure on W' is transferred to the appropriate $(n-1)$-dimensional measure on the $(n-1)$-dimensional manifold $\partial\Omega$.

(b) For the proof of Theorem 7.4 and Proposition 7.2 we refer to [AU10; Sections 7.1, 7.2] or to calculus books.

Gauss' theorem can be considered as the n-dimensional version of the fundamental theorem of calculus. In fact, for $\Omega = (a, b)$ we have $\partial\Omega = \{a, b\}$, $\nu(a) = -1$, $\nu(b) = 1$. Then for $u \in C^1[a, b]$ we can write

$$\int_a^b u'(x)\,\mathrm{d}x = u(b) - u(a) = \int_{\{a,b\}} u(z)\nu(z)\,\mathrm{d}\sigma(z),$$

with the counting measure σ.

Next we derive an important consequence of Gauss' theorem. We define $C^2(\overline{\Omega}) := \big\{u \in C^1(\overline{\Omega});\ \partial_j u \in C^1(\overline{\Omega})\ (j = 1, \dots, n)\big\}$. Then for $u \in C^2(\overline{\Omega})$ the functions $\partial_i\partial_j u$ are in $C(\overline{\Omega})$ for all $i, j = 1, \dots, n$. For $u \in C^1(\overline{\Omega})$ the function $\partial_\nu u \colon \partial\Omega \to \mathbb{K}$, given by

$$\partial_\nu u(z) := \nabla u(z) \cdot \nu(z) = \sum_{j=1}^n \partial_j u(z)\nu_j(z),$$

is called the **normal derivative** of u. Note that $\partial_\nu u \in C(\partial\Omega)$.

7.6 Corollary. *(Green's formulas) Let $u \in C^2(\overline{\Omega})$. Then*

$$\int_\Omega (\Delta u)v\,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x = \int_{\partial\Omega} (\partial_\nu u)v\,\mathrm{d}\sigma \qquad (v \in C^1(\overline{\Omega})), \tag{7.1}$$

$$\int_\Omega (v\Delta u - u\Delta v)\,\mathrm{d}x = \int_{\partial\Omega} (v\partial_\nu u - u\partial_\nu v)\,\mathrm{d}\sigma \qquad (v \in C^2(\overline{\Omega})). \tag{7.2}$$

Proof. By Gauss' theorem one has

$$\int_\Omega \partial_j u\,\partial_j v = -\int_\Omega (\partial_j^2 u)v + \int_\Omega \partial_j((\partial_j u)v) = -\int_\Omega (\partial_j^2 u)v + \int_{\partial\Omega} (\partial_j u)v\nu_j\,\mathrm{d}\sigma.$$

Summation over $j = 1, \ldots, n$ yields (7.1).

Exchanging u and v in (7.1) and substracting the result from (7.1) one obtains (7.2). ☐

7.2 Interlude: more on $H^1(\Omega)$; denseness, trace and compactness

In this section we give some information on $H^1(\Omega)$ that will be needed in the following. We could have simply quoted these results, because our main interest is to present form methods. However, we felt that just quoting the results would have the effect that the reader will not be aware of the analysis facts behind the treated situations. So we give the proofs, but we will be somewhat sketchy in the presentation. In a first reading there should be no problem if you just take notice of the results and first look at the further development.

The first issue is a denseness theorem for $H^1(\Omega)$. To recall, denseness theorems are needed to transfer properties one can show classically to more general functions if suitable estimates are provided. An example for this procedure is the proof of Poincaré's inequality in Lecture 5; further examples follow in this section.

7.7 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with continuous boundary. Then the set*

$$\check{C}^\infty(\Omega) := \{\varphi|_\Omega\,;\ \varphi \in C_\mathrm{c}^\infty(\mathbb{R}^n)\}$$

is dense in $H^1(\Omega)$. As a consequence, $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$.

Proof. (i) Let $W \subseteq \partial\Omega$ be a normal continuous graph, with (a, b) and g as described at the beginning of Section 7.1. Let $u \in H^1(\Omega)$ be such that $\operatorname{spt} u$ is a relatively compact subset of $W' \times (a, b)$, and extend u by 0 to $\mathbb{R}^n$. For $\tau > 0$ we define

$$u_\tau(x) := u(x', x_n - \tau) \qquad (x = (x', x_n) \in \mathbb{R}^n).$$

Then $u_\tau|_\Omega \in H^1(\Omega)$ for small τ, and $u_\tau|_\Omega \to u$ in $H^1(\Omega)$ as $\tau \to 0$. Moreover, for $\tau > 0$ one has that $u_\tau \in H^1(\Omega + \tau e_n)$ (with the n-th unit vector $e_n \in \mathbb{R}^n$). Let $(\rho_k)_{k\in\mathbb{N}}$ be a δ-sequence in $C_\mathrm{c}^\infty(\mathbb{R}^n)$. Then $\rho_k * u_\tau \in C_\mathrm{c}^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, and it is not too difficult to see that $(\rho_k * u_\tau)|_\Omega \to u_\tau|_\Omega$ in $H^1(\Omega)$ as $k \to \infty$. (In the last step one has to use a 'local version' of Lemma 4.13.)

From the above we conclude: for each continuous graph $W \subseteq \partial\Omega$ there exists an open set $U \subseteq \mathbb{R}^n$ such that $W = U \cap \partial\Omega$, and such that each $u \in H^1(\Omega)$ with relatively compact support in U can be approximated by functions in $\check{C}^\infty(\Omega)$.

(ii) A compactness argument shows that $\partial\Omega$ can be covered by open sets $W_1, \ldots, W_m$ such that each W_k is a continuous graph, with a corresponding open set $U_k \subseteq \mathbb{R}^n$ as indicated at the end of part (i) above. Then, defining $U_0 := \Omega$, the family $(U_k)_{k\in\{0,\ldots,m\}}$

is an open covering of $\overline{\Omega}$, and there exists a partition of unity $(\varphi_k)_{k\in\{0,\dots,m\}}$ in $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ on $\overline{\Omega}$, subordinate to $(U_k)_{k\in\{0,\dots,m\}}$. ('On $\overline{\Omega}$' means that $\sum_{k=0}^m \varphi_k|_{\overline{\Omega}} = 1$, and 'subordinate' means that $\operatorname{spt}\varphi_k \subseteq U_k$ $(k = 0,\dots,m)$. We refer to [AU10; Satz 7.12] for the existence of a partition of unity as above.)

Let $u \in H^1(\Omega)$. Then $\varphi_0 u \in H_0^1(\Omega)$ can be approximated by $C_{\mathrm{c}}^\infty(\Omega)$-functions, by Theorem 4.12(b), and $\varphi_1 u,\dots,\varphi_m u$ can be approximated by $\check{C}^\infty(\Omega)$-functions, by part (i). In consequence, u can be approximated by $\check{C}^\infty(\Omega)$-functions. □

7.8 Remarks. (a) The proof we have given uses that Ω having continuous boundary implies that Ω satisfies the 'segment property'. We refer to [Ada75; 3.17] for this property and to [Ada75; Theorem 3.18] for Theorem 7.7. It is not difficult to show that the segment property is in fact equivalent to continuous boundary.

(b) The following important observation will be used in the proofs of Theorems 7.9 and 7.10. The procedure used in the proof of Theorem 7.7 can be adapted to yield simultaneous approximation with respect to other properties. For instance, if $u \in H^1(\Omega) \cap C(\overline{\Omega})$, then the approximations can be chosen to additionally approximate u in the sup-norm. And also: if $u \in H^1(\Omega)$ is such that $\Delta u \in L_2(\Omega)$, then the approximations u_k can be chosen such that additionally $\Delta u_k \to \Delta u$ in $L_2(\Omega)$.

In order to explain this in somewhat more detail we first mention that multiplying u by a function $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ does not effect the additional properties mentioned in the previous paragraph. This is clear for the case that u is continuous. But also, if $\Delta u \in L_2(\Omega)$, then $\Delta(\varphi u) = (\Delta\varphi)u + 2\nabla\varphi\cdot\nabla u + \varphi\Delta u \in L_2(\Omega)$. In view of part (ii) of the proof of Theorem 7.7, this means that for the approximation we only have to treat the 'local' case considered in part (i).

Concerning part (i) of the proof of Theorem 7.7, the case of continuous u is done by invoking a local version of Proposition 4.3(a). The case $\Delta u \in L_2(\Omega)$ ist slightly more involved: one has to use a local version of Lemma 4.13(a), with ∂^α replaced by Δ, and for the convergence a local version of Proposition 4.3(b).

Next we show that for Ω with C^1-boundary one can define a trace mapping $\operatorname{tr}\colon H^1(\Omega) \to L_2(\Omega)$ such that for $u \in C^1(\overline{\Omega})$ one has $\operatorname{tr} u = u|_{\partial\Omega}$.

7.9 Theorem. *Assume that $\Omega \subseteq \mathbb{R}^n$ is open, bounded, and with C^1-boundary. Then there exists $c \geqslant 0$ such that*

$$\|u|_{\partial\Omega}\|_{L_2(\partial\Omega)}^2 \leqslant c\|u\|_{L_2(\Omega)}\|u\|_{H^1(\Omega)} \tag{7.3}$$

for all $u \in C^1(\overline{\Omega})$.

There is a unique bounded operator $\operatorname{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$, *called the* ***trace operator***, *such that* $\operatorname{tr} u = u|_{\partial\Omega}$ *for all $u \in C(\overline{\Omega}) \cap H^1(\Omega)$, and then* (7.3) *holds for all $u \in H^1(\Omega)$ (with $u|_{\partial\Omega}$ replaced by* $\operatorname{tr} u$ *on the left hand side).*

Proof. (i) Let $W \subseteq \partial\Omega$ be a normal C^1-graph, with (a,b) and g as in the definition. Let $x \in W$, and let $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ satisfy $\operatorname{spt}\varphi \subseteq W' \times (a,b)$, $\varphi \geqslant 0$, and $\varphi = 1$ on a neighbourhood of x. Then there exists an open neighbourhood $W_x \subseteq W$ of x such that $\varphi = 1$ on W_x. As $W \ni z \mapsto \nu(z) \in \mathbb{R}^n$ is continuous and $\nu_n(z) > 0$ for all $z \in W$, one concludes that $\delta := \inf_{z\in W_x} \nu_n(z) > 0$.

Let $u \in C^1(\overline{\Omega})$. Then, by Gauss' theorem,

$$\begin{aligned}\delta \int_{W_x} |u|^2 \,\mathrm{d}\sigma \leqslant \int_{\partial\Omega} (\varphi u)\overline{u}\,\nu_n \,\mathrm{d}\sigma &= \int_\Omega \bigl(\partial_n(\varphi u)\overline{u} + (\varphi u)\partial_n \overline{u}\bigr) \\ &\leqslant \|\varphi u\|_{H^1}\|u\|_{L_2} + \|\varphi u\|_{L_2}\|u\|_{H^1} \leqslant c_\varphi \|u\|_{L_2}\|u\|_{H^1},\end{aligned}$$

with c_φ only depending on φ.

We have shown that for each $x \in \partial\Omega$ there exist an open neighbourhood $W_x \subseteq \partial\Omega$ and a constant $c_x \geqslant 0$ such that

$$\int_{W_x} |u|^2 \,\mathrm{d}\sigma \leqslant c_x \|u\|_{L_2}\|u\|_{H^1}$$

for all $u \in C^1(\overline{\Omega})$. A standard compactness argument finishes the proof of (7.3) for $u \in C^1(\overline{\Omega})$.

(ii) The inequality (7.3) together with the denseness of $C^1(\overline{\Omega})$ in $H^1(\Omega)$ implies that the mapping $u \mapsto u|_{\partial\Omega}$ has a continuous extension $\mathrm{tr}\colon H^1(\Omega) \to L_2(\partial\Omega)$.

(iii) So far, we only have shown that $\mathrm{tr}\, u = u|_{\partial\Omega}$ holds for $u \in C^1(\overline{\Omega})$. In order to show it for $u \in C(\overline{\Omega}) \cap H^1(\Omega)$, we use the remarkable feature of the proof of Theorem 7.7 mentioned in Remark 7.8(b). As explained there, for $u \in C(\overline{\Omega}) \cap H^1(\Omega)$ an approximating sequence (u_k) in $C^1(\overline{\Omega})$ can be chosen converging to u in $C(\overline{\Omega})$ as well as in $H^1(\Omega)$. For this sequence, $\bigl(u_k|_{\partial\Omega}\bigr)$ converges to $\mathrm{tr}\, u$ in $L_2(\partial\Omega)$ and uniformly to $u|_{\partial\Omega}$, and this implies $\mathrm{tr}\, u = u|_{\partial\Omega}$. □

By abuse of notation, we still write $u|_{\partial\Omega} := \mathrm{tr}\, u$ for $u \in H^1(\Omega)$. In integrals we will frequently omit the trace notation to make things more readable. Here and in the following we always understand $L_2(\partial\Omega)$ with respect to the surface measure.

The trace is compatible with our definition of $H^1_0(\Omega)$ as the following result shows.

7.10 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with C^1-boundary. Then*

$$H^1_0(\Omega) = \bigl\{u \in H^1(\Omega);\ \mathrm{tr}\, u = 0\bigr\}.$$

Proof. The inclusion '$\subseteq$' follows from the continuity of the trace operator since $C^\infty_{\mathrm{c}}(\Omega)$ is dense in $H^1_0(\Omega)$.

Let $u \in H^1(\Omega)$ be such that $\mathrm{tr}\, u = 0$. Our aim is to show that $u \perp H^1_0(\Omega)^\perp$ (where throughout this proof the orthogonality symbol '$\perp$' refers to the scalar product $(\cdot\,|\,\cdot)_1$ in $H^1(\Omega)$). If this is proved we are done, because $H^1_0(\Omega)^{\perp\perp} = H^1_0(\Omega)$.

As a first step we note that one easily shows – using the definition of distributional derivatives – that $H^1_0(\Omega)^\perp = \{v \in H^1(\Omega);\ \Delta v = v\}$.

Now, let $v \in H^1_0(\Omega)^\perp$; then $\Delta v = v$. Thus, as explained in Remark 7.8(b), there exists a sequence (v_k) in $C^2(\overline{\Omega})$ such that $v_k \to v$ in $H^1(\Omega)$ and $\Delta v_k \to \Delta v = v$ in $L_2(\Omega)$. Then (7.1) in combination with Theorem 7.9 implies

$$\int_\Omega (\Delta v_k)\overline{u} + \int_\Omega \nabla v_k \cdot \overline{\nabla u} = \int_{\partial\Omega} (\partial_\nu v_k)\,\mathrm{tr}\,\overline{u}\,\mathrm{d}\sigma = 0$$

for all $k \in \mathbb{N}$, and for $k \to \infty$ one obtains

$$(v\,|\,u)_1 = \int_\Omega v\overline{u} + \int_\Omega \nabla v \cdot \overline{\nabla u} = 0.$$

□

For another proof of Theorem 7.10 we refer to [Eva10; Section 5.5, Theorem 2].

We close the section by transferring the compactness of the embedding $H_0^1(\Omega) \subseteq L_2(\Omega)$ to $H^1(\Omega)$.

7.11 Theorem. *(Rellich-Kondrachov) Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with continuous boundary. Then the embedding $j\colon H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact.*

Proof. The hypothesis implies that there exist a covering of $\partial\Omega$ by continuous graphs $W_1, \dots, W_m$ and vectors $y^1, \dots, y^m \in \mathbb{R}^n$ with the following properties: $\Omega \setminus \bigcup_{k=1}^m U_{k,\varepsilon}$ is a compact subset of Ω for all $\varepsilon > 0$, where

$$U_{k,\varepsilon} := \bigcup_{0<s<\varepsilon} \bigl(W_k + sy^k\bigr) \qquad (k = 1, \dots, m,\ \varepsilon > 0),$$

and $U_{k,2} \subseteq \Omega$ for $k = 1, \dots, m$.

Let $u \in C^1(\overline{\Omega})$. Let $k \in \{1, \dots, m\}$, $0 < \varepsilon < 1$. Then for $x \in U_{k,\varepsilon}$ we have

$$u(x) = -\int_0^1 \frac{d}{\mathrm{d}t}\bigl((1-t)u(x+ty^k)\bigr)\,\mathrm{d}t.$$

It follows that

$$\int_0^\varepsilon |u(z+sy^k)|^2\,\mathrm{d}s \leqslant \varepsilon \int_0^2 2\bigl(|u(z+ty^k)|^2 + |\nabla u(z+ty^k)|^2|y^k|^2\bigr)\,\mathrm{d}t$$

for all $z \in W_k$. Therefore, $\int_{U_{k,\varepsilon}} |u|^2\,\mathrm{d}x \leqslant 2\varepsilon\bigl(\|u\|^2_{L_2(U_{k,2})} + |y^k|^2\|\nabla u\|^2_{L_2(U_{k,2};\mathbb{K}^n)}\bigr)$. Summing over $k = 1, \dots, m$ we obtain

$$\int_{U_\varepsilon} |u|^2\,\mathrm{d}x \leqslant \varepsilon C\|u\|^2_{H^1(\Omega)}, \tag{7.4}$$

with the 'boundary layer' $U_\varepsilon := \bigcup_{k=1}^m U_{k,\varepsilon}$ and $C := \sum_{k=1}^m 2\bigl(1+|y^k|^2\bigr)$. From Theorem 7.7 we conclude that (7.4) holds for all $u \in H^1(\Omega)$.

There exists a function $\psi_\varepsilon \in C_c^\infty(\Omega)$ with $\mathbf{1}_{\Omega\setminus U_\varepsilon} \leqslant \psi_\varepsilon \leqslant 1$. Define $j_\varepsilon\colon H^1(\Omega) \to L_2(\Omega)$ by $j_\varepsilon(u) := \psi_\varepsilon u$. Then Theorem 6.21 implies that j_ε is compact. Further, (7.4) implies that $\|j - j_\varepsilon\|_{\mathcal{L}(H^1(\Omega),L_2(\Omega))} \leqslant (\varepsilon C)^{1/2}$.

Summing up, we have shown that the embedding j can be approximated in $\mathcal{L}(H^1(\Omega), L_2(\Omega))$ by compact operators, and this shows that j is compact. □

We also refer to [EE87; Theorem 4.17] for Theorem 7.11.

7.12 Remark. We note that (7.3) in combination with Theorem 7.11 also shows that the trace mapping in Theorem 7.9 is compact. Indeed, if (u_k) is a bounded sequence in $H^1(\Omega)$, then Theorem 7.11 implies that there exists a subsequence $\bigl(u_{k_i}\bigr)$ converging in $L_2(\Omega)$, and then (7.3) implies that $\bigl(\operatorname{tr} u_{k_i}\bigr)$ is a Cauchy sequence, hence convergent.

7.3 Weak normal derivative

Now we want to define the normal derivative for certain functions in $H^1(\Omega)$. At first we recall the weak definition of the Laplace operator.

Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $u, f \in L_2(\Omega)$. Then $\Delta u = f$ if

$$\int_\Omega u\Delta\varphi = \int_\Omega f\varphi \qquad (\varphi \in C_{\mathrm{c}}^\infty(\Omega)).$$

For $u \in L_2(\Omega)$ we say that $\Delta u \in L_2(\Omega)$ if there exists $f \in L_2(\Omega)$ such that $\Delta u = f$.

Now we define the normal derivative in a weak sense by asking Green's formula (7.1) to be valid.

Let Ω be bounded, with C^1-boundary, and let $u \in H^1(\Omega)$ such that $\Delta u \in L_2(\Omega)$. We say that $\partial_\nu u \in L_2(\partial\Omega)$ if there exists $h \in L_2(\partial\Omega)$ such that

$$\int_\Omega (\Delta u)v + \int_\Omega \nabla u \cdot \nabla v = \int_{\partial\Omega} hv \qquad (v \in H^1(\Omega)).$$

In that case we let $\partial_\nu u := h$. Here, in the integral over $\partial\Omega$ we did omit the trace sign. The integral over Ω is always with respect to the Lebesgue measure and that over $\partial\Omega$ always with respect to the surface measure σ.

7.4 The Neumann Laplacian

Now we consider Neumann boundary conditions. Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with C^1-boundary, and let $u \in H^1(\Omega)$ be such that $\Delta u \in L_2(\Omega)$. Then, by the definition given in the previous section, $\partial_\nu u = 0$ if and only if

$$\int_\Omega (\Delta u)v + \int_\Omega \nabla u \cdot \nabla v = 0 \qquad (v \in H^1(\Omega)). \tag{7.5}$$

It is remarkable that (7.5) makes sense for arbitrary open sets. Therefore, for open sets $\Omega \subseteq \mathbb{R}^n$ and $u \in H^1(\Omega)$ we will write '$\partial_\nu u = 0$' (including the quotes!) if (7.5) holds. This leads us to the following definition.

Let $\Omega \subseteq \mathbb{R}^n$ be open (not necessarily bounded). We define the operator Δ_{N} in $L_2(\Omega)$ by

$$\begin{aligned} \operatorname{dom}(\Delta_{\mathrm{N}}) &:= \bigl\{u \in H^1(\Omega);\ \Delta u \in L_2(\Omega),\ \text{`}\partial_\nu u = 0\text{'}\bigr\}, \\ \Delta_{\mathrm{N}} u &:= \Delta u \qquad (u \in \operatorname{dom}(\Delta_{\mathrm{N}})). \end{aligned}$$

We call Δ_{N} the **Laplacian with Neumann boundary condition** or simply the **Neumann Laplacian**.

7.13 Theorem. *The negative Neumann Laplacian $-\Delta_{\mathrm{N}}$ is self-adjoint and positive. It is associated with the classical Dirichlet form on $H^1(\Omega) \times H^1(\Omega)$.*

Proof. Define $a\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{K}$ by $a(u,v)=\int_\Omega \nabla u\cdot\overline{\nabla v}\,\mathrm{d}x$. Then a is continuous. We consider $H^1(\Omega)$ as a subspace of $H:=L_2(\Omega)$. Since $a(u)+\|u\|^2_{L_2(\Omega)}=\|u\|^2_{H^1(\Omega)}$, the form a is H-elliptic. Moreover, a is accretive. Let $A\sim a$. We show that $A=-\Delta_{\mathrm{N}}$. Let $u\in\operatorname{dom}(A)$, $Au=f$. Then by definition $\int_\Omega\nabla u\cdot\overline{\nabla v}=\int_\Omega f\bar v$ for all $v\in H^1(\Omega)$. Inserting test functions $v\in C_{\mathrm{c}}^\infty(\Omega)$ one obtains $-\Delta u=f$. Thus $\int_\Omega\nabla u\cdot\overline{\nabla v}+\int_\Omega(\Delta u)\bar v=0$ for all $v\in H_1(\Omega)$, i.e., u satisfies (7.5). We have shown that $A\subseteq-\Delta_{\mathrm{N}}$. Conversely, if $u\in\operatorname{dom}(\Delta_{\mathrm{N}})$ and $-\Delta_{\mathrm{N}}u=f$, then

$$\int_\Omega\nabla u\cdot\overline{\nabla v}=\int_\Omega\nabla u\cdot\overline{\nabla v}+\int_\Omega(\Delta u)\bar v+\int_\Omega f\bar v=\int_\Omega f\bar v\qquad(v\in H^1(\Omega)).$$

Thus $u\in\operatorname{dom}(A)$ and $Au=f$. □

Applying Theorem 7.11 one concludes that A has compact resolvent if Ω satisfies our weakest regularity property.

7.14 Theorem. *(Spectral decomposition of the Neumann Laplacian) If $\Omega\subseteq\mathbb{R}^n$ is open, bounded and has continuous boundary, then Δ_{N} has compact resolvent. There exist an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of $L_2(\Omega)$ and an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $[0,\infty)$, with $\lambda_1=0$ and $\lim_{k\to\infty}\lambda_k=\infty$, such that $-\Delta_{\mathrm{N}}$ is the associated diagonal operator. In particular, $\varphi_k\in\operatorname{dom}(\Delta_{\mathrm{N}})$ and*

$$-\Delta_{\mathrm{N}}\varphi_k=\lambda_k\varphi_k$$

for all $k\in\mathbb{N}$.

Proof. From Theorem 7.11 we know that the embedding $j\colon H^1(\Omega)\hookrightarrow L_2(\Omega)$ is compact. Therefore Proposition 6.15 – in combination with Theorem 7.13 – implies that Δ_{N} has compact resolvent.

The statement concerning the eigenfunctions and eigenvalues now follows from Theorem 6.17, except for the property that $\lambda_1=0$. However, it is immediate that $\varphi_1=\operatorname{vol}(\Omega)^{-1/2}\mathbf{1}_\Omega$ is an eigenfunction of $-\Delta_{\mathrm{N}}$ with eigenvalue 0. □

As a consequence, we deduce the following:

$$\begin{aligned}\operatorname{dom}(\Delta_{\mathrm{N}})&=\Big\{u\in L_2(\Omega);\ \sum_{j=1}^\infty\lambda_j^2\big|(u\,|\,\varphi_j)_{L_2(\Omega)}\big|^2<\infty\Big\},\\ -\Delta_{\mathrm{N}}u&=\sum_{j=1}^\infty\lambda_j\,(u\,|\,\varphi_j)_{L_2(\Omega)}\,\varphi_j,\\ T(t)u&=\sum_{j=1}^\infty\mathrm{e}^{-\lambda_jt}\,(u\,|\,\varphi_j)_{L_2(\Omega)}\,\varphi_j,\end{aligned}$$

where T denotes the C_0-semigroup generated by Δ_{N}.

We will see later that $\lambda_2>0$ if Ω is connected. This gives important information concerning the asymptotic behaviour of $T(t)$ as $t\to\infty$. We have to wait until we have discussed positivity.

7.5 The Robin Laplacian

Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with C^1-boundary. Here we consider Robin boundary conditions. Given $\beta \in L_\infty(\partial\Omega)$ we define the operator Δ_β in $L_2(\Omega)$ by

$$\begin{aligned}\operatorname{dom}(\Delta_\beta) &:= \bigl\{u \in H^1(\Omega);\, \Delta u \in L_2(\Omega),\ \partial_\nu u + \beta u|_{\partial\Omega} = 0\bigr\},\\ \Delta_\beta u &:= \Delta u.\end{aligned}$$

Note that the condition '$\partial_\nu u + \beta u|_{\partial\Omega} = 0$' should be read as '$\partial_\nu u = -\beta u|_{\partial\Omega}$', in the sense of the definition in Section 7.3. We call Δ_β the **Laplacian with Robin boundary conditions** or briefly **Robin Laplacian**. We now state and prove properties about the Robin Laplacian, announced in the title of the lecture.

7.15 Theorem. *Let β be real-valued. Then the operator $-\Delta_\beta$ is self-adjoint and quasi-accretive, with compact resolvent. In particular, Δ_β generates a quasi-contractive C_0-semigroup T_β on $L_2(\Omega)$. If $\beta \geqslant 0$, then $-\Delta_\beta$ is accretive and $\|T_\beta(t)\| \leqslant 1$ for all $t \geqslant 0$.*

Proof of Theorem 7.15. Consider the form $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \beta uv$. Then $|a(u,v)| \leqslant \|\nabla u\|_2 \|\nabla v\|_2 + \|\beta\|_{L_\infty(\partial\Omega)} \|\operatorname{tr} u\|_{L_2(\partial\Omega)} \|\operatorname{tr} v\|_{L_2(\partial\Omega)}$. Since the trace is continuous, it follows that a is continuous.

We consider $H^1(\Omega)$ as a subspace of $H := L_2(\Omega)$ and claim that a is H-elliptic, i.e., that

$$\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \beta |u|^2 + \omega \int_\Omega |u|^2 \geqslant \alpha \|u\|^2_{H^1(\Omega)}$$

for some $\omega \geqslant 0$, $\alpha > 0$ and all $u \in H^1(\Omega)$. Let $0 < \varepsilon < 1$. By Theorem 7.9 and Euclid's inequality (i.e., $ab \leqslant \frac{1}{2}\bigl(\gamma a^2 + \frac{1}{\gamma} b^2\bigr)$ for all $a, b, \gamma > 0$) there exists $c > 0$ such that

$$\|\beta\|_{L_\infty(\partial\Omega)} \int_{\partial\Omega} |u|^2 \leqslant \frac{1}{2} \|u\|^2_{H^1(\Omega)} + c\|u\|^2_{L_2(\Omega)}$$

for all $u \in H^1(\Omega)$. Hence

$$\int_{\partial\Omega} \beta |u|^2 \geqslant -\frac{1}{2} \int_\Omega |\nabla u|^2 - \Bigl(\frac{1}{2} + c\Bigr) \int_\Omega |u|^2,$$

and therefore

$$\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \beta |u|^2 \geqslant \frac{1}{2} \int_\Omega |\nabla u|^2 - \Bigl(\frac{1}{2} + c\Bigr) \int_\Omega |u|^2$$

for all $u \in H^1(\Omega)$. This proves the claim.

Let A be the operator associated with a. We show that $A = -\Delta_\beta$. Let $(u,f) \in A$. Then

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_{\partial\Omega} \beta u \bar{v} = \int_\Omega f \bar{v} \qquad (v \in H^1(\Omega)). \tag{7.6}$$

Taking $v \in C_\mathrm{c}^\infty(\Omega)$ we see that $-\Delta u = f$. Replacing f by $-\Delta u$ in (7.6) we find

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u) \bar{v} = -\int_{\partial\Omega} \beta u \bar{v} \qquad (v \in H^1(\Omega)). \tag{7.7}$$

This is equivalent to $\partial_\nu u = -\beta u|_{\partial\Omega}$. Thus $(u,f) \in -\Delta_\beta$. Conversely, if $u \in \operatorname{dom}(\Delta_\beta)$, then (7.7) holds. Letting $f = -\Delta u$ we obtain (7.6) and thus $(u,f) \in A$.

Finally, since by Theorem 7.11 the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, the operator A has compact resolvent. Since a is symmetric, A is self-adjoint, and since a is H-elliptic, A is quasi-accretive. $\square$

Notes

In Section 7.1 we partially follow [AU10]. It is possible to extend Gauss' theorem to Lipschitz domains, for which a suitable surface measure on $\partial\Omega$ is defined analogously. Most of the properties of $H^1(\Omega)$ presented in Section 7.2 can be found in the standard literature on Sobolev spaces. The proofs of (7.3) and Theorem 7.10 have been contributed by H. Vogt.

The theorem of Gauss is due to Lagrange in 1792 but has been rediscovered by Carl Friedrich Gauss in 1813, by George Green in 1825, and by Mikhail V. Ostrogradsky in 1831. For this reason one finds it in the literature under these different names. We could call it 'fundamental theorem of calculus' but this is not usual. Obviously, it can also be written as

$$\int_\Omega \operatorname{div} u = \int_{\partial\Omega} u \cdot \nu \, \mathrm{d}\sigma,$$

for each vector field $u \in C^1(\overline{\Omega}; \mathbb{R}^n)$. In this form it is frequently called the **divergence theorem**. Physicists and engineers love this theorem because of immediate interpretation.

Victor Gustave Robin (1855–1897) was a French mathematician and the reader should correctly pronounce the nasal. He was teaching mathematical physics at the Sorbonne in Paris. Not much is known about him since he burnt his manuscripts. But he worked on thermodynamics, and the Russian school introduced the name Robin boundary conditions. These boundary conditions had already been introduced by Isaac Newton (1643–1727). Neumann boundary conditions carry their name to honour Carl G. Neumann (1832–1925) who was professor at Halle, Basel, Tübingen and Leipzig. He introduced the Neumann series for matrices.

None of these 'forefathers' used forms it seems. Time was not yet ripe and Hilbert had to come into play first.

Exercises

7.1 Let $\Omega \subseteq \mathbb{R}^n$ be open, connected, bounded, and with C^1-boundary. Let $0 \leqslant \beta \in L_\infty(\partial\Omega)$ such that $\int_{\partial\Omega} \beta \, \mathrm{d}\sigma > 0$ (i.e., β is not 0 in $L_\infty(\partial\Omega)$). Denote by T_β the C_0-semigroup generated by the Robin Laplacian Δ_β. Show that

$$\|T_\beta(t)\| \leqslant \mathrm{e}^{-\varepsilon t} \qquad (t \geqslant 0)$$

for some $\varepsilon > 0$. (Hint: If $u \in H^1(\Omega)$ is such that $\nabla u = 0$, then u is constant. This can be used without proof; it holds because Ω is connected.)

7.2 Let $\Omega \subseteq \mathbb{R}^n$ be a C^1-domain with a hole, i.e., we assume that there exist bounded open sets $\tilde{\Omega}$ and ω with C^1-boundary such that $\overline{\omega} \subseteq \tilde{\Omega}$ and $\Omega = \tilde{\Omega} \setminus \overline{\omega}$. Let $\Gamma_1 = \partial\tilde{\Omega}$,

$\Gamma_2 = \partial\omega$ so that $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Let $\beta \in L_\infty(\Gamma_2)$ be real-valued. Define the Laplacian with Robin boundary condition $\partial_\nu u + \beta u|_{\Gamma_2} = 0$ on Γ_2 and Dirichlet boundary condition zero on Γ_1. Show that it is a self-adjoint operator.

7.3 Let $a\colon V \times V \to \mathbb{C}$ be a symmetric continuous form that is j-elliptic, where V, H are complex Hilbert spaces and $j \in \mathcal{L}(V, H)$ has dense range. Let $b\colon V \times V \to \mathbb{C}$ be sesquilinear and assume that there exists $c \geqslant 0$ such that

$$|b(u,v)| \leqslant c\|u\|_V \|j(v)\|_H \qquad (u, v \in V).$$

(a) Show that $a + b\colon V \times V \to \mathbb{C}$ is continuous and j-elliptic.

(b) Denote by A the operator associated with $(a + b, j)$. Show that the numerical range $\operatorname{num}(A)$ lies in a parabola with vertex on the real axis and opened in the direction of the positive real axis.

(c) Assume that $a = 0$ and that b is j-elliptic. Denote by B the operator associated with (b, j). Show that j is an isomorphism and that B is bounded.

7.4 In this problem we let $\mathbb{K} = \mathbb{C}$. Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with C^1-boundary. Let $\beta \in L_\infty(\partial\Omega)$ (not necessarily real-valued), and let Δ_β be the Robin Laplacian.

(a) Show that $-\Delta_\beta$ is H-elliptic (where $H := L_2(\Omega)$), and that $-\Delta_\beta$ is m-accretive if $\operatorname{Re}\beta \geqslant 0$.

(b) Show that $\operatorname{num}(-\Delta_\beta)$ is contained in the region 'surrounded' by a parabola with vertex on the real axis and opened in the direction of the positive real axis.

(c) Show that $-\Delta_\beta$ is quasi-m-sectorial of any angle $\varphi < \pi/2$ and that Δ_β generates a holomorphic semigroup of angle $\pi/2$.

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