Lecture 6

Adjoint operators, and compactness

The main objective of this lecture is to show that, for a bounded open set $\Omega \subseteq \mathbb{R}^n$, the Dirichlet Laplacian $\Delta_{\rm D}$ on Ω has an orthonormal basis consisting of eigenfunctions. The main point is that one can show that $(I + \Delta_{\rm D})^{-1}$ is a compact self-adjoint operator, and one shows that for such operators there exist sufficiently many eigenfunctions. One of the aims of this lecture is to explain all the notions occuring in the previous sentence.

6.1 Adjoints of operators, and self-adjoint operators

The principal objective of this section is to prove the following result.

6.1 Theorem. *Let H be a Hilbert space, A an operator in H. Then the following properties are equivalent.*

(a) *A is a positive self-adjoint operator.*

(b) *A is a symmetric m-accretive operator.*

If H is a complex Hilbert space, then there is the following additional equivalent property.

(c) *A is m-sectorial of angle* 0.

Evidently, before proceeding to the proof, we first have to explain the notions of self-adjointness and symmetry. The proof will be given at the very end of this section.

Let G, H be Hilbert spaces over $\mathbb{K}$. Before defining the adjoint of an operator we want to explain the idea behind this notion. If A is an operator from G to H, then the adjoint A^* should be the maximal operator from H to G such that

$$(Ax \,|\, y)_H = (x \,|\, A^*y)_G \qquad \bigl(x \in \operatorname{dom}(A),\ y \in \operatorname{dom}(A^*)\bigr).$$

In the following, the product space $G \times H$ will be provided with the scalar product

$$\bigl((x,y) \,\big|\, (x_1,y_1)\bigr) := (x \,|\, x_1)_G + (y \,|\, y_1)_H \qquad \bigl((x,y),(x_1,y_1) \in G \times H\bigr),$$

which makes $G \times H$ a Hilbert space; we will use the notation $G \oplus H$ for this space.

For an operator A from G to H we define the **adjoint**

$$\begin{aligned} A^* &:= \bigl\{(y,x) \in H \times G;\, \forall x_1 \in \operatorname{dom}(A)\colon\ (Ax_1 \,|\, y)_H = (x_1 \,|\, x)_G\bigr\} \\ &= \bigl\{(y,x) \in H \times G;\, \forall (x_1,y_1) \in A\colon\ \bigl((x_1,-y_1) \,\big|\, (x,y)\bigr) = 0\bigr\} \\ &= ((-A)^\perp)^{-1}. \end{aligned}$$

It is obvious from the last equality that A^* is a closed subspace of $H \times G$. We mention that $-A$, as a linear relation, is given by

$$-A = \{(x, -y)\,;\ (x, y) \in A\}.$$

(It is somewhat unpleasant that, in principle, A^* is already defined as the antidual space to the subspace A of $G \oplus H$. We will have to live with this ambiguity.)

6.2 Remarks. Let A be an operator from G to H.

(a) If B is an operator from H to G such that

$$(Ax \,|\, y) = (x \,|\, By) \qquad \big(x \in \operatorname{dom}(A),\ y \in \operatorname{dom}(B)\big),$$

then it is immediate that $B \subseteq A^*$ and $A \subseteq B^*$.

(b) If A^* is an operator, then the definition implies that $(Ax \,|\, y) = (x \,|\, A^*y)$ for all $x \in \operatorname{dom}(A)$, $y \in \operatorname{dom}(A^*)$.

6.3 Theorem. *Let A be an operator from G to H. Then:*

(a) *A^* is an operator if and only if $\operatorname{dom}(A)$ is dense in G.*

(b) *Assume that $\operatorname{dom}(A)$ is dense. Then $A^{**} := (A^*)^* = \overline{A}$, and $\operatorname{dom}(A^*)$ is dense if and only if A is closable.*

Proof. (a) From the definition of A^* it is easy to see that $\{x \in G;\ (0, x) \in A^*\} = \operatorname{dom}(A)^\perp$. This shows the assertion.

(b) It is easy to see that in the expression $((-A)^\perp)^{-1}$ for A^* the order of the operations $A \mapsto -A$, $A \mapsto A^\perp$ and $A \mapsto A^{-1}$ does not matter. It follows that $A^{**} = A^{\perp\perp} = \overline{A}$.

By part (a), $\operatorname{dom}(A^*)$ is dense if and only if $(A^*)^*$ is an operator, and by $A^{**} = \overline{A}$ this is equivalent to the closability of A. $\square$

6.4 Example. Define the operator A in $L_2(\mathbb{R})$ by

$$\operatorname{dom}(A) := C_c^\infty(\mathbb{R}), \qquad Af := f' \quad (f \in D(A)).$$

For $f, g \in L_2(\mathbb{R})$ we then obtain:

$$\begin{aligned}(g, f) \in A^* &\iff \forall \varphi \in C_c^\infty(\mathbb{R})\colon\ (A\varphi \,|\, g) = (\varphi \,|\, f)\\ &\iff \forall \varphi \in C_c^\infty(\mathbb{R})\colon\ \int g\varphi' = \int f\varphi\\ &\iff f = -g \text{ in the distributional sense.}\end{aligned}$$

This shows that $D(A^*) = H^1(\mathbb{R})$, $A^*g = -g'$ $(g \in \operatorname{dom}(A^*))$.

We claim that $\overline{A} = -A^*$.

The previous considerations show that $A \subseteq -A^*$, and since $-A^*$ is closed we obtain $\overline{A} \subseteq -A^*$. The subset A^* of $L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$ is isometrically isomorphic to $H^1(\mathbb{R})$, under the mapping $A^* \ni (f, -f') \mapsto f \in H^1(\mathbb{R})$. From Theorem 4.12 we know that $\operatorname{dom}(A) = C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$. This shows that $\overline{A} = -A^*$.

An operator A in H is called **symmetric** if $\operatorname{dom}(A)$ is dense, and $A \subseteq A^*$; A is called **self-adjoint** if $\operatorname{dom}(A)$ is dense, and $A = A^*$.

We collect some properties of symmetric operators.

6.5 Remarks. Let A be an operator in H, $\operatorname{dom}(A)$ dense.

(a) Then A is symmetric if and only if $(Ax \,|\, y) = (x \,|\, Ay)$ for all $x, y \in \operatorname{dom}(A)$. This follows immediately from Remark 6.2.

(b) If A is symmetric, then A is closable, and $\overline{A}$ is symmetric. Indeed from $A \subseteq A^*$ and the fact that A^* is a closed operator one concludes that A is closable and that $\overline{A} \subseteq A^* = \overline{A}^*$ (where the last equality is immediate from the definition).

(c) Let H be complex. Then A is symmetric if and only if $(Ax \,|\, x) \in \mathbb{R}$ for all $x \in \operatorname{dom}(A)$. This follows from Remark 5.1(a), applied to the form $\operatorname{dom}(A) \times \operatorname{dom}(A) \ni (x, y) \mapsto (Ax \,|\, y)$, and part (a) above.

An operator A in H is called **essentially self-adjoint** if it is symmetric and $\overline{A}$ is self-adjoint. A symmetric operator is called **positive**, $A \geqslant 0$, if $(Ax \,|\, x) \geqslant 0$ for all $x \in \operatorname{dom}(A)$ (in other words, if A is accretive).

Particularly simple examples of self-adjoint operators are those possessing ‘sufficiently many’ eigenelements, which we will present next. We note that an eigenvalue of a symmetric operator A is always real: If $0 \neq x \in H$, $Ax = \lambda x$, then $\lambda\,(x \,|\, x) = (Ax \,|\, x) \in \mathbb{R}$.

6.6 Example. Diagonal self-adjoint operators.

Let A be a self-adjoint operator in an infinite-dimensional separable Hilbert space, and assume that there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ consisting of eigenelements of A, with corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$.

Then

$$\begin{aligned} \operatorname{dom}(A) &= \Big\{x \in H\,;\ \sum_{n\in\mathbb{N}} |\lambda_n|^2 |(x \,|\, e_n)|^2 < \infty\Big\}, \\ Ax &= \sum_{n\in\mathbb{N}} \lambda_n\,(x \,|\, e_n)\, e_n \qquad (x \in \operatorname{dom}(A)). \end{aligned}$$

Define $A_{\mathrm{c}} \subseteq A$ by $\operatorname{dom}(A_{\mathrm{c}}) := \operatorname{lin}\{e_n\,;\ n \in \mathbb{N}\}$. Then A_{c} is essentially self-adjoint (and $\overline{A_{\mathrm{c}}} = A$).

We call A the **diagonal operator** associated with the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and the sequence $(\lambda_n)_{n\in\mathbb{N}}$.

Proof. Using the unitary operator $J\colon H \to \ell_2,\ x \mapsto \big((x \,|\, e_n)\big)_{n\in\mathbb{N}}$, we transform the situation to the case where A is a self-adjoint operator in the Hilbert space ℓ_2, possessing the canonical unit vectors as eigenelements.

Now we refer to Exercise 6.2, for the measure space $\mathbb{N}$ with counting measure, with the multiplying function m in Exercise 6.2 given by the sequence $(\lambda_n)_{n\in\mathbb{N}}$. Then $A_{\mathrm{c}} = A_0$, with A_0 defined in Exercise 6.2, and by Exercise 6.2(b) the operator A_c^* is the maximal multiplication operator M_λ by the ‘function’ $(\overline{\lambda_n})_{n\in\mathbb{N}} = (\lambda_n)_{n\in\mathbb{N}}$. From Exercise 6.2(c) we know that then $\overline{A_{\mathrm{c}}} = M_\lambda$; hence A_{c} is essentially self-adjoint. Because A is closed, one concludes that $M_\lambda \subseteq A$. Thus $A^* \subseteq M_\lambda \subseteq A = A^*$, and A is the maximal multiplication operator, i.e., A is the operator described above. $\square$

In order to connect self-adjointness with m-accretivity and m-sectoriality we need further preparations.

6.7 Lemma. *Let G, H be Hilbert spaces, A an operator from G to H.*
Then $\ker(A^*) = \operatorname{ran}(A)^\perp$.

Proof. By the definition of A^*, $y \in \ker(A^*)$ (which by definition means $(y,0) \in A^*$) is equivalent to $(Ax_1 \,|\, y)_H = 0$ for all $x_1 \in \operatorname{dom}(A)$, i.e., to $y \perp \operatorname{ran}(A)$. □

We point out that in the following result the operator A is not supposed to be densely defined.

6.8 Proposition. *Let A be an operator in H, $(Ax \,|\, y) = (x \,|\, Ay)$ for all $x, y \in \operatorname{dom}(A)$, and $\operatorname{ran}(A) = H$.*
Then A is self-adjoint.

Proof. Remark 6.2(a) shows that $A \subseteq A^*$, and Lemma 6.7 implies that $\ker(A^*) = \operatorname{ran}(A)^\perp = \{0\}$. These two facts, together with $\operatorname{ran}(A) = H$, imply $A^* = A$. (This comes under the heading that 'a surjective mapping cannot have a proper injective extension' – which also holds for relations.) From Theorem 6.3(a) one now infers that $\operatorname{dom}(A)$ is dense, because $A^* = A$ is an operator. □

6.9 Lemma. *Let A be an operator in H, $\operatorname{dom}(A)$ dense, and let $\lambda \in \mathbb{K}$. Then $(\lambda I + A)^* = \overline{\lambda} I + A^*$.*

Proof. Let $x_1 \in \operatorname{dom}(A)$, $y \in \operatorname{dom}(A^*)$. Then

$$\big((\lambda + A)x_1 \,\big|\, y\big) = \big(x_1 \,\big|\, \overline{\lambda} y\big) + (x_1 \,|\, A^* y) = \big(x_1 \,\big|\, (\overline{\lambda} + A^*)y\big)\,.$$

Therefore we obtain $\overline{\lambda} + A^* \subseteq (\lambda + A)^*$. This also implies that

$$A^* = (-\lambda + (\lambda + A))^* \supseteq -\overline{\lambda} + (\lambda + A)^*,$$

i.e., $\overline{\lambda} + A^* \supseteq (\lambda + A)^*$. The two inclusions prove the assertion. □

After these preparations we will now prove the main result of this section.

Proof of Theorem 6.1. (a) ⇒ (b), (c). Clearly, it is sufficient to show that $\operatorname{ran}(I + A) = H$. As A is closed and accretive, the latter is satisfied if $\operatorname{ran}(I + A)$ is dense: the inequality $\|(I + A)x\| \geqslant \|x\|$ $(x \in \operatorname{dom}(A))$ implies that $\operatorname{ran}(I + A)$ is closed.

By Lemma 6.9 the operator $I + A$ is self-adjoint, and $I + A$ is injective since A is accretive. Now Lemma 6.7 implies $\operatorname{ran}(I + A)^\perp = \ker((I + A)^*) = \ker(I + A) = \{0\}$.

(b) ⇒ (a). The hypothesis implies that $I + A$ satisfies the conditions in Proposition 6.8, and therefore $I + A$ is self-adjoint. Then Lemma 6.9 shows that A is self-adjoint.

(c) ⇒ (b) Being an m-sectorial operator, A is m-accretive. Sectoriality of angle 0 means that $(Ax \,|\, x) \geqslant 0$ for all $x \in \operatorname{dom}(A)$. Applying Remark 6.5(c) one concludes that A is symmetric. □

6.2 Adjoints of forms and operators

In this section we assume that V and H are Hilbert spaces over $\mathbb{K}$ and that $j \in \mathcal{L}(V, H)$ has dense range. Let $a \colon V \times V \to \mathbb{K}$ be a bounded form. For the reader's convenience we recall the definition of the operator A associated with a and j,

$$A = \big\{(x, y) \in H \times H\,;\, \exists u \in V : j(u) = x,\ a(u, v) = (y \,|\, j(v))\ \ (v \in V)\big\};$$

see Lecture 5.

The following result shows the close connection between adjoints of forms and operators.

6.10 Theorem. *Let V, H and j be as above, let $a \colon V \times V \to \mathbb{K}$ be bounded and coercive, and let $A \sim (a, j)$, $B \sim (a^*, j)$.*

Then $B = A^*$. If additionally a is symmetric, then A is self-adjoint and positive.

Proof. First we note that for $x \in \operatorname{dom}(A)$, $y \in \operatorname{dom}(B)$ there exist $u, v \in V$ such that $j(u) = x$, $j(v) = y$ and

$$(Ax \,|\, y) = (Ax \,|\, j(v)) = a(u, v) = \overline{a^*(v, u)} = \overline{(By \,|\, j(u))} = (x \,|\, By)\,.$$

Hence $B \subseteq A^*$, by Remark 6.2(a).

As in the proof of Theorem 5.7 we define the form b by

$$b(u, v) := a(u, v) + (j(u) \,|\, j(v)) \qquad (u, v \in V).$$

Then $(I+A) \sim (b, j)$, $(I+B) \sim (b^*, j)$, and as just noted this implies that $I+B \subseteq (I+A)^*$. From Theorem 5.7 we know that $\operatorname{ran}(I + B) = \operatorname{ran}(I + A) = H$, and therefore Lemma 6.7 implies that $(I + A)^*$ is injective. These properties imply that $I + B = (I + A)^*$ (recall that 'a surjective mapping cannot have a proper injective extension'). Using Lemma 6.9 we obtain $I + B = I + A^*$, and then $B = A^*$.

If a is symmetric, then $A = B = A^*$, and $a(u) \geqslant 0$ $(u \in V)$ implies $(Ax \,|\, x) \geqslant 0$ $(x \in \operatorname{dom}(A))$. $\square$

6.3 The spectral theorem for compact self-adjoint operators

For Banach spaces X, Y and an operator A from X to Y we recall that A is called **compact** if $A \in \mathcal{L}(X, Y)$ and $A(B_X(0, 1))$ is a relatively compact subset of Y, where $B_X(0, 1)$ is the open unit ball of X. We recall that the set $\mathcal{K}(X, Y)$ of compact operators is a closed subspace of $\mathcal{L}(X, Y)$, and that the composition of a compact operator with a bounded operator is compact. The latter is called the **ideal property** of compact operators.

In the following let H be a Hilbert space. The next theorem is the spectral theorem for compact self-adjoint operators.

6.11 Theorem. *(Hilbert) Let $A \in \mathcal{L}(H)$ be compact and self-adjoint. Then there exist $N \subseteq \mathbb{N}$ and an orthonormal system $(e_j)_{j\in N}$ of eigenelements of A with corresponding eigenvalues $(\lambda_j)_{j\in N}$ in $\mathbb{R}\setminus\{0\}$, $\lambda_j \to 0$ ($j\to\infty$), such that*

$$Ax = \sum_{j\in N} \lambda_j\,(x\,|\,e_j)\,e_j \qquad (x\in H).$$

For the proof we need two preparations.

6.12 Proposition. *Let $A \in \mathcal{L}(H)$ be self-adjoint. Then*

$$\|A\| = \sup\{|(Ax\,|\,x)|;\ \|x\|\leqslant 1\}.$$

Proof. "$\geqslant$" is obvious from $|(Ax\,|\,x)| \leqslant \|A\|\|x\|^2$.
"$\leqslant$". Let $a(x,y) := (Ax\,|\,y)$ $(x,y\in H)$, $c := \sup\{|a(x)|;\ \|x\|\leqslant 1\}$, $b(x,y) := c\,(x\,|\,y)$ $(x,y\in H)$. Then Schwarz's inequality (Proposition 5.2) implies $|(Ax\,|\,y)| \leqslant c\|x\|\|y\|$ $(x,y\in H)$; therefore $\|A\|\leqslant c$. □

6.13 Proposition. *Let $0\neq A\in\mathcal{L}(H)$ be compact and self-adjoint. Then $\|A\|$ or $-\|A\|$ is an eigenvalue of A.*

Proof. By Proposition 6.12 there exist a sequence (x_n), $\|x_n\| = 1$ $(n\in\mathbb{N})$, and $\lambda\in\mathbb{R}$ with $|\lambda| = \|A\|$ such that $(Ax_n\,|\,x_n)\to\lambda$. Therefore

$$0\leqslant \|Ax_n-\lambda x_n\|^2 = \|Ax_n\|^2 - 2\lambda\,(Ax_n\,|\,x_n)+\lambda^2 \leqslant 2\lambda^2 - 2\lambda\,(Ax_n\,|\,x_n)\to 0;$$

hence $\|Ax_n-\lambda x_n\|\to 0$ $(n\to\infty)$.
The compactness of A implies that there exists a subsequence (x_{n_j}) of (x_n) such that (Ax_{n_j}) is convergent, $Ax_{n_j}\to y$. This implies

$$x_{n_j} = \frac{1}{\lambda}(\lambda x_{n_j} - Ax_{n_j}) + \frac{1}{\lambda}Ax_{n_j} \to \frac{1}{\lambda}y =: x\neq 0,$$

$Ax = \lambda x$. □

Proof of Theorem 6.11. We show the assertion with $N=\mathbb{N}$ or $N=\{1,\dots,n\}$ for some $n\in\mathbb{N}_0$. By induction we construct an orthonormal sequence $(e_j)_{j\in N}$ with $\{e_j\,;\ j\in N\}^\perp = \ker(A)$ and a corresponding sequence of eigenvalues $(\lambda_j)_{j\in N}$, decreasing in absolute values.
Assume that $n\in\mathbb{N}_0$ and that $e_1,\dots,e_n$ and $\lambda_1,\dots,\lambda_n$ are constructed. Define $H_{n+1} := \{e_1,\dots,e_n\}^\perp$. Then $A(H_{n+1})\subseteq H_{n+1}$; indeed, $x\perp e_j$ implies $(Ax\,|\,e_j) = (x\,|\,Ae_j) = \lambda_j(x\,|\,e_j) = 0$ $(j=1,\dots,n)$. Then $A_{n+1} := A|_{H_{n+1}}$, considered as an operator in H_{n+1}, is compact. If $A_{n+1}=0$, then the construction is finished, with $N=\{1,\dots,n\}$. If $A_{n+1}\neq 0$, then Proposition 6.13 implies that there exist $\lambda_{n+1}\in\mathbb{R}$ with $|\lambda_{n+1}| = \|A_{n+1}\|$ and $e_{n+1}\in H_{n+1}$ with $\|e_{n+1}\|=1$ such that $A_{n+1}e_{n+1} = \lambda_{n+1}e_{n+1}$.
The construction yields $|\lambda_j| = \|A_j\|$ $(j\in N)$, and as $(\|A_j\|)_{j\in N}$ is decreasing, so is $(|\lambda_j|)_{j\in N}$.
If $N=\mathbb{N}$, then $e_j\to 0$ weakly $(j\to\infty)$. This implies that $|\lambda_j| = \|Ae_j\|\to 0$ $(j\to\infty)$, because A is compact; see Exercise 6.1.

Finally we show the representation of A. For $x \in H$ we have

$$\Big\| Ax - \sum_{j=1}^{n} \lambda_j \, (x \,|\, e_j) \, e_j \Big\| = \Big\| A_{n+1}\Big(x - \sum_{j=1}^{n} (x \,|\, e_j) \, e_j \Big) \Big\| \leqslant \|A_{n+1}\| \|x\| = |\lambda_{n+1}| \|x\|,$$

and this is 0 if $N = \{1, \ldots, n\}$ and converges to 0 as $n \to \infty$ if $N = \mathbb{N}$. The representation of A shows that $\{e_j \,;\, j \in N\}^{\perp} = \ker(A)$. □

We are going to draw consequences from Hilbert's theorem, for self-adjoint operators with compact resolvent. We will prepare this by some remarks and an example.

6.14 Remark. Let A be an operator in a Banach space X, and assume that there exists $\lambda \in \rho(A)$ such that $R(\lambda, A)$ is a compact operator. Then it follows from the resolvent equation and the ideal property of compact operators that $R(\mu, A)$ is compact for all $\mu \in \rho(A)$. In this case we will simply call A an **operator with compact resolvent**.

6.15 Proposition. *Let V, H be Hilbert spaces, let $j \in \mathcal{L}(V, H)$ have dense range, let $a \colon V \times V \to \mathbb{K}$ be a bounded coercive form, and let $A \sim (a, j)$. Assume additionally that j is compact.*

Then A has compact resolvent.

Proof. From Remark 5.8 we know that the inverse of $I + A$ can be expressed explicitly as $(I + A)^{-1} = j\mathcal{B}^{-1}k$. As $\mathcal{B}^{-1}$ and k are bounded operators, we see that $(I + A)^{-1}$ is compact. □

6.16 Example. Diagonal forms.

Let H be an infinite-dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0, \infty)$.

Define the space

$$V := \Big\{ u \in H \,;\, \sum_{n=1}^{\infty} \lambda_n |(u \,|\, e_n)_H|^2 < \infty \Big\}.$$

Then V is a Hilbert space for the scalar product

$$(u \,|\, v)_V := (u \,|\, v)_H + a(u, v),$$

where $a(u, v) := \sum_{n=1}^{\infty} \lambda_n \, (u \,|\, e_n)_H \, (e_n \,|\, v)_H$.

Then the injection j of V into H is continuous, and a is continuous and j-elliptic. The operator A associated with (a, j) is self-adjoint. It is easy to see that A is the diagonal operator associated with $(e_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ as defined in Example 6.6.

Assume additionally that $\lim_{n\to\infty} \lambda_n = \infty$. Then the injection j is compact. Indeed, it is easy to see that the finite rank operators $P_n j$, where P_n denotes the orthogonal projection onto $\mathrm{lin}\{e_1, \ldots, e_n\}$, approximate j in the operator norm. By Proposition 6.15 we see that A has compact resolvent (which is also easily obtained directly).

This example is generic for positive self-adjoint operators with compact resovent, as the following theorem shows.

6.17 Theorem. *Let A be a positive self-adjoint operator with compact resolvent in an infinite-dimensional Hilbert space H. Then there exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ with $\lim_{n\to\infty}\lambda_n = \infty$ such that A is the associated diagonal operator.*

Proof. From Theorem 6.1 we know that $(I+A)^{-1}$ exists in $\mathcal{L}(H)$, and it is easy to see that this operator is symmetric, hence self-adjoint. By hypothesis, $(I+A)^{-1}$ is compact. Applying Theorem 6.11 to $(I+A)^{-1}$ one obtains an orthonormal system $(e_n)_{n\in N}$, with a corresponding sequence $(\mu_n)_{n\in N}$ of eigenvalues. The representation $(I+A)^{-1}x = \sum_{n\in N}\mu_n(x\,|\,e_n)e_n$ together with the injectivity of $(I+A)^{-1}$ implies that $(e_n)_{n\in N}$ is an orthonormal basis. Hence N is countably infinite, without loss of generality $N=\mathbb{N}$.

It follows that e_n is an eigenelement of A with eigenvalue $\lambda_n := \mu_n^{-1}-1$, for all $n\in\mathbb{N}$. The positivity of A implies

$$\lambda_n = (\lambda_n e_n\,|\,e_n) = (Ae_n\,|\,e_n) \geqslant 0 \qquad (n\in\mathbb{N}).$$

Since $(\mu_n)_{n\in\mathbb{N}}$ is a null sequence, it follows that $\lim_{n\to\infty}\lambda_n=\infty$. □

A sequence (λ_n) in $\mathbb{R}$ with $\lim_{n\to\infty}\lambda_n=\infty$ can always be rearranged to an increasing sequence, and in applications of Theorem 6.17 one usually assumes the sequence of eigenvalues to be increasing.

Note that the hypotheses of Theorem 6.17 imply that H is separable. We formulate the consequence of this theorem for operators associated with forms.

6.18 Corollary. *Let V, H be infinite-dimensional Hilbert spaces, and let $a\colon V\times V\to\mathbb{K}$ be a bounded coercive symmetric form. Let $j\in\mathcal{L}(V,H)$ have dense range, and assume that j is compact. Let A be the operator associated with (a,j).*

Then A is a positive self-adjoint operator with compact resolvent, and therefore all the conclusions of Theorem 6.17 hold. Additionally, all eigenvalues of A are positive.

Proof. We know from Theorem 6.10 that A is a positive self-adjoint operator, and it was shown in Proposition 6.15 that A has compact resolvent. In order to show that all the eigenvalues are positive, we note that, if λ is an eigenvalue of A with eigenelement x, then $\lambda\|x\|^2 = (Ax\,|\,x) = a(u) > 0$, where $u\in V$ with $j(u)=x$. □

We finally come to the property announced in the introduction of this lecture. In order to draw the conclusion for the Dirichlet Laplacian Δ_{D} in $L_2(\Omega)$, for bounded open $\Omega\subseteq\mathbb{R}^n$ we will use the compactness of the embedding $H_0^1(\Omega)\hookrightarrow L_2(\Omega)$, which will be proved in the next section.

6.19 Example. Spectral decomposition of the Dirichlet Laplacian.

If $\Omega\subseteq\mathbb{R}^n$ is open and bounded, then Δ_{D} has compact resolvent. There exists an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of $L_2(\Omega)$ and an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $(0,\infty)$, with $\lim_{k\to\infty}\lambda_k=\infty$, such that $-\Delta_{\mathrm{D}}$ is the associated diagonal operator. In particular, $\varphi_k\in\operatorname{dom}(\Delta_{\mathrm{D}})$ and

$$-\Delta_{\mathrm{D}}\varphi_k = \lambda_k\varphi_k$$

for all $k\in\mathbb{N}$.

Proof. From Example 5.13 we know that $-\Delta_{\mathrm{D}}$ is associated with the classical Dirichlet form and $j\colon H_0^1(\Omega) \hookrightarrow L_2(\Omega)$. As the embedding j is compact, by Theorem 6.21, the assertions follow from Corollary 6.18. □

6.20 Remark. The existence of an orthonormal basis of eigenelements of $-\Delta_{\mathrm{D}}$ is a highlight and triumph of Hilbert space theory as applied to partial differential equations. There is no way to obtain this kind of result by computing eigenfunctions, even if the boundary is nice.

In the case of an interval in one dimension there is no problem (see Exercise 6.4), and this can be generalised to n-dimensional rectangles. For a ball one already has to use Bessel functions.

6.4 Compactness of the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$

The main result of this section is the following special case of the Rellich-Kondrachov theorem.

6.21 Theorem. *(Rellich-Kondrachov) Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then the inclusion map $j\colon H_0^1(\Omega) \to L_2(\Omega)$ is compact.*

The proof will be given at the end of the section. As a preparation we start with explaining the convolution of functions for a context different from Subsection 4.1.1.

6.22 Proposition. *Let $\rho \in L_1(\mathbb{R}^n)$, and let $1 \leqslant p \leqslant \infty$. Then, for all $u \in L_p(\mathbb{R}^n)$,*

$$\rho * u(x) := \int_{\mathbb{R}^n} \rho(x-y)u(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \rho(y)u(x-y)\,\mathrm{d}y$$

exists for a.e. $x \in \mathbb{R}^n$, and

$$\|\rho * u\|_p \leqslant \|\rho\|_1 \|u\|_p.$$

Proof. Let $u \in L_p(\mathbb{R}^n)$. Then the function $\mathbb{R}^{2n} \ni (x,y) \mapsto \rho(x-y)u(y)$ is measurable because the mapping $(x,y) \mapsto (x-y,y)$ is measurable. Let $1 \leqslant q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L_q(\mathbb{R}^n)$ we estimate, using Fubini-Tonelli and Hölder,

$$\begin{aligned} \iint |\rho(y)u(x-y)|\,\mathrm{d}y\,|g(x)|\,\mathrm{d}x &= \int \rho(y) \int |u(x-y)||g(x)|\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \int |\rho(y)| \|u(\cdot - y)\|_p \|g\|_q\,\mathrm{d}y \leqslant \|\rho\|_1 \|u\|_p \|g\|_q. \end{aligned}$$

This inequality shows the assertions. □

The fundamental result in this context is the following theorem.

6.23 Theorem. *Let $\rho \in L_1(\mathbb{R}^n)$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $1 \leqslant p \leqslant \infty$. Then the mapping*

$$L_p(\mathbb{R}^n) \ni u \mapsto (\rho * u)|_\Omega \in L_p(\Omega)$$

is compact.

Proof. If $\rho \in C_{\mathrm{c}}(\mathbb{R}^n)$, then we show that

$$F := \big\{(\rho * u)|_{\overline{\Omega}};\ u \in L_p(\mathbb{R}^n),\ \|u\|_p \leqslant 1\big\}$$

is relatively compact in $C(\overline{\Omega})$. Clearly, the set is bounded because $|\rho * u(x)| \leqslant \|\rho\|_q \|u\|_p$ (with $\frac{1}{q} + \frac{1}{p} = 1$, as usual). Also, the estimate

$$|\rho * u(x) - \rho * u(y)| \leqslant \int \big|\big(\rho(x-z) - \rho(y-z)\big)u(z)\big|\,\mathrm{d}z \leqslant \|\rho(x-\cdot) - \rho(y-\cdot)\|_q \|u\|_p$$

shows that F is equicontinuous. By the Arzelà-Ascoli theorem it follows that F is relatively compact in $C(\overline{\Omega})$. The embedding $C(\overline{\Omega}) \hookrightarrow L_p(\Omega)$ is continuous, and therefore F is also relatively compact in $L_p(\Omega)$.

Now let $\rho \in L_1(\mathbb{R}^n)$. Then there exists a sequence (ρ_k) in $C_{\mathrm{c}}(\mathbb{R}^n)$ such that $\rho_k \to \rho$ in $L_1(\mathbb{R}^n)$. For $k \in \mathbb{N}$ let $J_k \in \mathcal{L}(L_p(\mathbb{R}^n), L_p(\Omega))$ be defined by $J_k u = (\rho_k * u)|_\Omega$ $(u \in L_p(\mathbb{R}^n))$, and let J be associated with ρ in the same way. As shown above, J_k is compact for all $k \in \mathbb{N}$. Proposition 6.22 implies that $\|J - J_k\| \leqslant \|\rho - \rho_k\|_1 \to 0$ $(k \to \infty)$, and as $\mathcal{K}(L_p(\mathbb{R}^n), L_p(\Omega))$ is closed in $\mathcal{L}(L_p(\mathbb{R}^n), L_p(\Omega))$ we conclude that J is compact. □

In order to use Theorem 6.23 for the compactness of the embedding $H_0^1 \hookrightarrow L_2$, we derive a representation of functions in $C_{\mathrm{c}}^1(\mathbb{R}^n)$ in terms of their derivatives.

6.24 Lemma. *Let $\varphi \in C_{\mathrm{c}}^1(\mathbb{R}^n)$. Then*

$$\varphi(0) = -\frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla\varphi(y) \cdot \frac{y}{|y|^n}\,\mathrm{d}y,$$

where σ_{n-1} is the $(n-1)$-dimensional volume of the unit sphere S_{n-1} of $\mathbb{R}^n$.

Proof. Let $z \in S_{n-1}$. Then

$$\varphi(0) = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\varphi(tz)\,\mathrm{d}t = -\int_0^\infty \nabla\varphi(tz) \cdot z\,\mathrm{d}t.$$

Integrating this formula over S_{n-1}, provided with the surface measure σ, we obtain

$$\begin{aligned}
\sigma_{n-1}\varphi(0) &= -\int_{z \in S_{n-1}} \int_{t=0}^\infty \nabla\varphi(tz) \cdot z\,\mathrm{d}t\,\mathrm{d}\sigma(z) \\
&= -\int_{t=0}^\infty \int_{z \in S_{n-1}} \nabla\varphi(tz) \cdot \frac{tz}{t^n}\,\mathrm{d}\sigma(z)\,t^{n-1}\,\mathrm{d}t = -\int_{\mathbb{R}^n} \nabla\varphi(y) \cdot \frac{y}{|y|^n}\,\mathrm{d}y,
\end{aligned}$$

where in the last equality we have used generalised polar coordinates. □

Proof of Theorem 6.21. In this proof we extend all functions defined on Ω to $\mathbb{R}^n$, by setting them 0 on $\mathbb{R}^n \setminus \Omega$.

There exists $R > 0$ such that $\Omega \subseteq B(0, R)$. For $\varphi \in C_c^1(\Omega)$, $x \in \Omega$ we obtain, applying Lemma 6.24 to $\varphi(x - \cdot)$,

$$\begin{aligned}\varphi(x) &= \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla\varphi(x-y) \cdot \frac{y}{|y|^n} \,\mathrm{d}y \\ &= \frac{1}{\sigma_{n-1}} \int_{B(0,2R)} \nabla\varphi(x-y) \cdot \frac{y}{|y|^n} \,\mathrm{d}y = \int_{\mathbb{R}^n} \nabla\varphi(x-y) \cdot \rho(y) \,\mathrm{d}y\end{aligned}$$

with $\rho(y) := \frac{1}{\sigma_{n-1}} \mathbf{1}_{B(0,2R)}(y) \frac{y}{|y|^n}$.

Due to the cut-off at $2R$, the function ρ belongs to the vector valued $L_1(\mathbb{R}^n; \mathbb{R}^n)$. Therefore Theorem 6.23 implies that the mapping

$$J_2 \colon L_2(\mathbb{R}^n; \mathbb{K}^n) \to L_2(\Omega), u \mapsto (\rho * u)|_\Omega$$

is compact. Obviously, the mapping

$$J_1 \colon H_0^1(\Omega) \to L_2(\Omega; \mathbb{K}^n), \; f \mapsto \nabla f$$

is continuous. This implies that $J_2 J_1 \colon H_0^1(\Omega) \to L_2(\Omega)$ is compact.

The formula presented above shows that for $\varphi \in C_c^1(\Omega)$ one has $j(\varphi) = J_2 J_1 \varphi$. As both operators j and $J_2 J_1$ are continuous and $C_c^1(\Omega)$ is dense in $H_0^1(\Omega)$ one concludes that $j = J_2 J_1$ is compact. □

Notes

It is difficult to attribute the development of adjoint operators to a source. Maybe one of the first more systematic treatments is given in [Neu32]. The idea to include also linear relations is contained in [Are61], where the adjoint is also defined for linear relations (whereas we define the adjoint only for operators).

The first version of the spectral theorem of compact self-adjoint operators is contained in [Hil06]. (This paper is also contained in the collection [Hil12].) The diagonal structure of the Dirichlet Laplacian seems to be classical and difficult to attribute; however, the use of compactness methods for this purpose can be attributed to Rellich [Rel30]. In this paper the first version of the Rellich-Kondrachov theorem appeared as well as the application to the Dirichlet Laplacian. The proof we give for the Rellich-Kondrachov theorem is not the one usually found in text books. In particular, the virtual lecturer could not find a version of Theorem 6.23 in the literature.

Exercises

6.1 Let G, H be Hilbert spaces, $A \in \mathcal{L}(G, H)$. Show that A is a compact operator if and only if A maps weakly convergent sequences in G to convergent sequences in H.

Hints: 1. Recall that (x_n) in G is called weakly convergent to $x \in G$ if $(x_n \,|\, y) \to (x \,|\, y)$ $(y \in G)$, and that every bounded sequence in G contains a weakly convergent subsequence.

2. Recall that $A \in \mathcal{L}(G, H)$ is also continuous with respect to the weak topologies in G and H; in particular, if (x_n) in G is weakly convergent to $x \in G$, then (Ax_n) is weakly convergent to Ax in H.

6.2 Let (Ω, μ) be a σ-finite measure space, and let $m\colon \Omega \to \mathbb{C}$ be measurable. Let A_0 be defined by

$$\operatorname{dom}(A_0) := \operatorname{lin}\{\mathbf{1}_C\,;\, C \in \mathcal{C}\},$$

with

$$\mathcal{C} := \Big\{C \subseteq \Omega;\, C \text{ measurable},\ \mu(C) < \infty,\ \sup_{x \in C} |m(x)| < \infty\Big\},$$

$$A_0 f := mf \qquad (f \in \operatorname{dom}(A_0)).$$

(a) Show that $\operatorname{dom}(A_0)$ is dense in $L_2(\Omega, \mu)$, and that $A_0^* = M_{\overline{m}}$, where $M_{\overline{m}}$ denotes the maximal multiplication operator by $\overline{m}$,

$$\begin{aligned} \operatorname{dom}(M_{\overline{m}}) &:= \{f \in L_2(\Omega, \mu)\,;\, \overline{m}f \in L_2(\Omega, \mu)\},\\ M_{\overline{m}} f &:= \overline{m} f \qquad (f \in \operatorname{dom}(M_{\overline{m}})). \end{aligned}$$

(b) Show further that $M_{\overline{m}}^* = M_m$ (maximal multiplication operator by m).
(c) Assume that m is real-valued. Show that A_0 is essentially self-adjoint, and $\overline{A_0} = M_m$.

6.3 (a) Let X be a vector space, Y an n-dimensional subspace, Z an $(n-1)$-codimensional subspace (i.e., there exists an $(n-1)$-dimensional subspace Z_0 of X such that $Z \cap Z_0 = \{0\}$, $Z + Z_0 = X$). Show that $Y \cap Z \neq \{0\}$.
(b) Let H be an infinite-dimensional separable Hilbert space, and let A be a self-adjoint diagonal operator in H associated with the form a, described as in Examples 6.6 and 6.16. Assume that the sequence of eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ is increasing, with $\lim_{n \to \infty} \lambda_n = \infty$.
For $x_1, \ldots, x_{n-1} \in H$ define

$$I(x_1, \ldots, x_{n-1}) := \inf\big\{a(x)\,;\, x \in \{x_1, \ldots, x_{n-1}\}^\perp \cap V,\ \|x\| = 1\big\}.$$

Show the **min-max principle**:

$$\lambda_n = \max\{I(x_1, \ldots, .x_{n-1})\,;\, x_1, \ldots, x_{n-1} \in H\} \qquad (n \in \mathbb{N}).$$

(c) Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$ be bounded open sets, and let Δ_j be the Dirichlet-Laplacian on Ω_j and $(\lambda_k^j)_{k \in \mathbb{N}}$ the corresponding increasing sequence of eigenvalues, for $j = 1, 2$.
Show the **domain monotonicity of eigenvalues**: $\lambda_k^1 \geqslant \lambda_k^2$ for all $k \in \mathbb{N}$.

6.4 (a) Let $f \in C(-1, 1)$, and assume that $g_1 := \big(f|_{(-1,0)}\big)' \in L_1(-1, 0)$, $g_2 := \big(f|_{(0,1)}\big)' \in L_1(0, 1)$. Define $g \in L_1(-1, 1)$ by $g|_{(-1,0)} := g_1$, $g|_{(0,1)} := g_2$. Show that $f' = g$ (all derivatives in the distributional sense).
(b) Let $a, b \in \mathbb{R}$, $a < b$. Show that $H_0^1(a, b) = \{f \in H^1(a, b)\,;\, f(a) = f(b) = 0\}$.
Hints: 1. For $f \in H^1(a, b)$ with $f(a) = f(b) = 0$, show that the extension of f to $\mathbb{R}$ by 0 belongs to $H^1(\mathbb{R})$.

2. Let $c \in (a, b)$, and let $f \in H^1(a, b)$, $f(a) = 0$, $f|_{(c,b)} = 0$. Show that $f \in H^1_0(a, b)$ (by using suitable translates of f).

(c) Compute the orthonormal basis of eigenfunctions and the eigenvalues of $-\Delta_{\mathrm{D}}$ for $\Omega = (0, \pi)$.

(d) Determine the optimal value of the Poincaré constant for the open set $(0, \pi)$ (see Section 5.4).

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