Lecture 4

The Sobolev space H^1, and applications

In Section 4.1 we present the definition and some basic properties of the Sobolev space H^1. This treatment is prepared by several important tools from analysis. The main objective of this lecture is the Hilbert space treatment of the Laplace operator in Section 4.2. In particular, the Dirichlet Laplacian will be presented as our first (non-trivial) example of a generator of a contractive holomorphic C_0-semigroup.

4.1 The Sobolev space H^1

4.1.1 Convolution

We recall the definition of **locally integrable functions** on an open subset Ω of $\mathbb{R}^n$,

$$\begin{aligned} L_{1,\mathrm{loc}}(\Omega) := \bigl\{ f\colon \Omega \to \mathbb{K};\ &\text{for all } x \in \Omega \text{ there exists } r > 0 \text{ such that} \\ &B(x,r) \subseteq \Omega \text{ and } f|_{B(x,r)} \in L_1(B(x,r)) \bigr\}. \end{aligned}$$

Moreover, $C_\mathrm{c}^\infty(\Omega) := C^\infty(\Omega) \cap C_\mathrm{c}(\Omega)$ is the space of infinitely differentiable functions with compact support.

4.1 Lemma. *Let $u \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, $\rho \in C_\mathrm{c}^\infty(\mathbb{R}^n)$. We define the* ***convolution of*** ρ ***and*** u,

$$\rho * u(x) := \int_{\mathbb{R}^n} \rho(x-y)u(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \rho(y)u(x-y)\,\mathrm{d}y \qquad (x \in \mathbb{R}^n).$$

Then $\rho * u \in C^\infty(\mathbb{R}^n)$, and for all $\alpha \in \mathbb{N}_0^n$ one has

$$\partial^\alpha(\rho * u) = (\partial^\alpha \rho) * u.$$

Proof. (i) The integral exists because ρ is bounded and has compact support.

(ii) Continuity of $\rho * u$: There exists $R > 0$ such that $\operatorname{spt} \rho \subseteq B(0,R)$. Let $R' > 0$, $\delta > 0$. For $x, x' \in B(0,R')$, $|x - x'| < \delta$, one obtains

$$\begin{aligned} |\rho * u(x) - \rho * u(x')| &= \left| \int_{B(0,R+R')} \bigl(\rho(x-y) - \rho(x'-y)\bigr)u(y)\,\mathrm{d}y \right| \\ &\leqslant \sup\bigl\{|\rho(z) - \rho(z')|;\ |z - z'| < \delta\bigr\} \int_{B(0,R+R')} |u(y)|\,\mathrm{d}y. \end{aligned}$$

The second factor in the last expression is finite because u is locally integrable, and the first factor becomes small for small δ because ρ is uniformly continuous.

(iii) Let $1 \leqslant j \leqslant n$. The existence of the partial derivative of $\rho * u$ with respect to the j-th variable and the equality $\partial_j(\rho * u) = (\partial_j \rho) * u$ are a consequence of the differentiability of integrals with respect to a parameter. The function $(\partial_j \rho) * u$ is continuous, by step (ii) above, and therefore $\rho * u$ is continuously differentiable with respect to the j-th variable.

(iv) Induction shows the assertion for all $\alpha \in \mathbb{N}_0^n$. □

A sequence $(\rho_k)_{k\in\mathbb{N}}$ in $C_{\mathrm{c}}(\mathbb{R}^n)$ is called a **δ-sequence** if $\rho_k \geqslant 0$, $\int \rho_k(x)\,\mathrm{d}x = 1$ and $\operatorname{spt} \rho_k \subseteq B[0, 1/k]$ for all $k \in \mathbb{N}$. (The notation 'δ-sequence' is motivated by the fact that the sequence approximates the 'δ-distribution' $C_{\mathrm{c}}^\infty(\mathbb{R}^n) \ni \varphi \mapsto \varphi(0)$.)

4.2 Remarks. (a) We recall the standard example of a C_{c}^∞-function φ. The source of this function is the well-known function $\psi \in C^\infty(\mathbb{R})$,

$$\psi(t) := \begin{cases} 0 & \text{if } t \leqslant 0, \\ \mathrm{e}^{-1/t} & \text{if } t > 0. \end{cases}$$

Then $\varphi(x) := \psi(1 - |x|^2)$ $(x \in \mathbb{R}^n)$ defines a function $0 \leqslant \varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, with the property that $\varphi(x) \neq 0$ if and only if $|x| < 1$.

(b) If $0 \leqslant \rho \in C_{\mathrm{c}}(\mathbb{R}^n)$, $\int \rho(x)\,\mathrm{d}x = 1$, $\operatorname{spt} \rho \subseteq B[0, 1]$, and we define

$$\rho_k(x) := k^n \rho(kx) \qquad (x \in \mathbb{R}^n,\ k \in \mathbb{N}),$$

then (ρ_k) is a δ-sequence.

4.3 Proposition. *Let (ρ_k) be a δ-sequence in $C_{\mathrm{c}}(\mathbb{R}^n)$.*

(a) *Let $f \in C(\mathbb{R}^n)$. Then $\rho_k * f \to f$ uniformly on compact subsets of $\mathbb{R}^n$ as $k \to \infty$.*

(b) *Let $1 \leqslant p \leqslant \infty$, $f \in L_p(\mathbb{R}^n)$. Then $\rho_k * f \in L_p(\mathbb{R}^n)$,*

$$\|\rho_k * f\|_p \leqslant \|f\|_p \quad \textit{for all } k \in \mathbb{N}.$$

If $1 \leqslant p < \infty$, then

$$\|\rho_k * f - f\|_p \to 0 \quad (k \to \infty).$$

Proof. (a) The assertion is an easy consequence of the uniform continuity of f on compact subsets of $\mathbb{R}^n$.

(b) (i) For $p = \infty$ the estimate is straightforward.

If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we estimate, using Hölder's inequality in the second step,

$$\begin{aligned} |\rho_k * f(x)| &= \left| \int \rho_k(x-y)^{\frac{1}{q}+\frac{1}{p}} f(y)\,\mathrm{d}y \right| \\ &\leqslant \left(\int \rho_k(x-y)\,\mathrm{d}y \right)^{1/q} \left(\int \rho_k(x-y)|f(y)|^p\,\mathrm{d}y \right)^{1/p} \\ &= \left(\int \rho_k(x-y)|f(y)|^p\,\mathrm{d}y \right)^{1/p}. \end{aligned}$$

This estimate also holds (trivially) for $p = 1$. Then, with Fubini's theorem in the second step,

$$\int |\rho_k * f(x)|^p \,\mathrm{d}x \leqslant \int_x \int_y \rho_k(x-y)|f(y)|^p \,\mathrm{d}y \,\mathrm{d}x$$
$$= \int_y \int_x \rho_k(x-y) \,\mathrm{d}x \, |f(y)|^p \,\mathrm{d}y = \|f\|_p^p.$$

(ii) Now let $1 \leqslant p < \infty$. For $k \in \mathbb{N}$ we define $T_k \in \mathcal{L}(L_p(\mathbb{R}^n))$ by $T_k g := \rho_k * g$ for $g \in L_p(\mathbb{R}^n)$; then step (i) shows that $\|T_k\| \leqslant 1$.

If $g \in C_\mathrm{c}(\mathbb{R}^n)$, then $T_k g \to g$ in $L_p(\mathbb{R}^n)$ as $k \to \infty$. Indeed, there exists $R > 0$ such that $\operatorname{spt} g \subseteq B[0,R]$. It is easy to check that this implies that $\operatorname{spt}(\rho_k * g) \subseteq B[0,R+1]$ for all $k \in \mathbb{N}$. Also, $\rho_k * g \to g$ uniformly on $B[0,R+1]$, by part (a). This shows that $\rho_k * g \to g$ $(k \to \infty)$ in $L_p(\mathbb{R}^n)$.

Now, the denseness of $C_\mathrm{c}(\mathbb{R}^n)$ in $L_p(\mathbb{R}^n)$ (which we will accept as a fundamental fact from measure and integration theory) together with Proposition 1.6 implies that $T_k g \to g$ $(k \to \infty)$ in $L_p(\mathbb{R}^n)$, for all $g \in L_p(\mathbb{R}^n)$. □

4.4 Corollary. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \leqslant p < \infty$. Then $C_\mathrm{c}^\infty(\Omega)$ is dense in $L_p(\Omega)$.*

Proof. Let (ρ_k) be a δ-sequence in $C_\mathrm{c}^\infty(\mathbb{R}^n)$.

Let $g \in C_\mathrm{c}(\Omega)$, and extend g by zero to a function in $C_\mathrm{c}(\mathbb{R}^n)$. Then $\rho_k * g \in C^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, by Lemma 4.1. If $\frac{1}{k} < \operatorname{dist}(\operatorname{spt} g, \mathbb{R}^n \setminus \Omega)$, then $\operatorname{spt}(\rho_k * g) \subseteq \operatorname{spt} g + B[0,1/k] \subseteq \Omega$ (see Exercise 4.1(a)), and therefore $\rho_k * g \in C_\mathrm{c}^\infty(\Omega)$. From Lemma 4.3 we know that $\rho_k * g \to g$ $(k \to \infty)$ in $L_p(\mathbb{R}^n)$. So, we have shown that $C_\mathrm{c}^\infty(\Omega)$ is dense in $C_c(\Omega)$ with respect to the L_p-norm.

Now the denseness of $C_\mathrm{c}(\Omega)$ in $L_p(\Omega)$ yields the assertion. □

4.1.2 Distributional derivatives

Let $P(\partial) = \sum_{\alpha \in \mathbb{N}_0,\, |\alpha| \leqslant k} a_\alpha \partial^\alpha$ be a partial differential operator with constant coefficients $a_\alpha \in \mathbb{K}$ $(|\alpha| \leqslant k)$, with $k \in \mathbb{N}$. Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in C^k(\Omega)$. Then for all "test functions" $\varphi \in C_\mathrm{c}^\infty(\Omega)$ one has

$$\int_\Omega \big(P(\partial) f\big) \varphi \,\mathrm{d}x = \int_\Omega f \sum_{|\alpha| \leqslant k} (-1)^{|\alpha|} a_\alpha \partial^\alpha \varphi \,\mathrm{d}x$$

(integration by parts!).

Now, let $f \in L_{1,\mathrm{loc}}(\Omega)$. Then we say that $P(\partial) f \in L_{1,\mathrm{loc}}(\Omega)$ if there exists $g \in L_{1,\mathrm{loc}}(\Omega)$ such that

$$\int g \varphi \,\mathrm{d}x = \int f \sum_{|\alpha| \leqslant k} (-1)^{|\alpha|} a_\alpha \partial^\alpha \varphi \,\mathrm{d}x$$

for all $\varphi \in C_\mathrm{c}^\infty(\Omega)$, and we say that $P(\partial) f = g$ holds **in the distributional sense**. In particular, if $\partial^\alpha f \in L_{1,\mathrm{loc}}(\Omega)$, then we call $\partial^\alpha f$ the **distributional** (or **generalised** or **weak**) **derivative of** f.

In order to justify this definition we have to show that $g = P(\partial) f$ is unique. This is the content of the following "fundamental lemma of the calculus of variations".

4.5 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in L_{1,\mathrm{loc}}(\Omega)$,*

$$\int f\varphi \,\mathrm{d}x = 0 \qquad (\varphi \in C_c^\infty(\Omega)).$$

Then $f = 0$.

The statement '$f = 0$' means that f is the zero element of $L_{1,\mathrm{loc}}(\Omega)$, i.e., if f is a representative, then $f = 0$ a.e.

Proof of Lemma 4.5. Let $\varphi \in C_c^\infty(\Omega)$. We show that $\varphi f = 0$ a.e. Defining

$$g(x) := \begin{cases} \varphi(x)f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

we have $g \in L_1(\mathbb{R}^n)$.

Let (ρ_k) be a δ-sequence in $C_c^\infty(\mathbb{R}^n)$. From Lemma 4.3 we know hat $\rho_k * g \to g$ in $L_1(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ we have

$$\rho_k * g(x) = \int_\Omega \rho_k(x-y)\varphi(y)f(y)\,\mathrm{d}y = 0,$$

because $\rho_k(x - \cdot)\varphi \in C_c^\infty(\Omega)$. This shows that $\rho_k * g = 0$, and hence $g = 0$ (as an L_1-function).

From $\varphi f = 0$ a.e. for all $\varphi \in C_c^\infty(\Omega)$ we conclude that $f = 0$ a.e.; see Exercise 4.5. □

In the remainder of this subsection we give more information on the one-dimensional case. The aim is to show that an $L_{1,\mathrm{loc}}$-function with $L_{1,\mathrm{loc}}$-derivative can be written as the integral of its derivative.

4.6 Proposition. *Let $-\infty \leqslant a < x_0 < b \leqslant \infty$, $f, g \in L_{1,\mathrm{loc}}(a,b)$. Then $f' = g$ in the distributional sense if and only if there exists $c \in \mathbb{K}$ such that*

$$f(x) = c + \int_{x_0}^x g(y)\,\mathrm{d}y \qquad (a.e.\ x \in (a,b)).$$

Note that the right hand side of the previous equality is continuous as a function of x, and that therefore f has a continuous representative.

For the proof we need a preparation.

4.7 Lemma. *Let $-\infty \leqslant a < b \leqslant \infty$, $h \in L_{1,\mathrm{loc}}(a,b)$, and assume that $h' = 0$ in the distributional sense. Then there exists $c \in \mathbb{K}$ such that $h = c$.*

Proof. (i) We start with the observation that a function $\psi \in C_c^\infty(a,b)$ is the derivative of a function in $C_c^\infty(a,b)$ if and only if $\int \psi \,\mathrm{d}x = 0$.

(ii) Let $\rho \in C_c^\infty(a,b)$, $\int \rho(x)\,\mathrm{d}x = 1$, and let $c := \int \rho(x)h(x)\,\mathrm{d}x$. For all $\varphi \in C_c^\infty(a,b)$ one obtains

$$\begin{aligned} \int \varphi(x)\big(h(x) - c\big)\,\mathrm{d}x &= \int \varphi(x)h(x)\,\mathrm{d}x - \int \varphi(y)\,\mathrm{d}y \int \rho(x)h(x)\,\mathrm{d}x \\ &= \int \Big(\varphi(x) - \int \varphi(y)\,\mathrm{d}y\,\rho(x)\Big)h(x)\,\mathrm{d}x = 0 \end{aligned}$$

because of $\int \big(\varphi(x) - \int \varphi(y)\,\mathrm{d}y\,\rho(x)\big)\,\mathrm{d}x = 0$ and part (i). Now the assertion is a consequence of Lemma 4.5. □

Proof of Proposition 4.6. (i) We first show the sufficiency. Let $\varphi \in C_c^\infty(a,b)$, and choose $x_1 \in (a, \inf \operatorname{spt} \varphi)$. Then

$$f(x) = c_1 + \int_{x_1}^x g(y)\,\mathrm{d}y \qquad (\text{a.e. } x \in (a,b)),$$

with $c_1 := c + \int_{x_0}^{x_1} g(y)\,\mathrm{d}y$. We obtain

$$\begin{aligned}\int_a^b \varphi' f\,\mathrm{d}x &= \int_a^b \varphi'(x)\Big(c_1 + \int_{x_1}^x g(y)\,\mathrm{d}y\Big)\,\mathrm{d}x = c_1\int_a^b \varphi'\,\mathrm{d}x + \iint\limits_{x_1<y<x<b} \varphi'(x)g(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \iint\limits_{x_1<y<x<b} \varphi'(x)g(y)\,\mathrm{d}x\,\mathrm{d}y = -\int_a^b \varphi(y)g(y)\,\mathrm{d}y.\end{aligned}$$

Thus, $f' = g$.

(ii) For the proof of the necessity we define

$$h(x) := f(x) - \int_{x_0}^x g(y)\,\mathrm{d}y \qquad (a < x < b).$$

Then part (i) implies that $h' = f' - g = 0$ in the distributional sense, so by Lemma 4.7 there exists $c \in \mathbb{K}$ such that $h = c$. □

4.1.3 Definition of $H^1(\Omega)$

Let $\Omega \subseteq \mathbb{R}^n$ be open. We define the **Sobolev space**

$$H^1(\Omega) := \big\{ f \in L_2(\Omega);\ \partial_j f \in L_2(\Omega)\ (j \in \{1,\dots,n\})\big\},$$

with scalar product

$$(f\,|\,g)_1 := (f\,|\,g) + \sum_{j=1}^n (\partial_j f\,|\,\partial_j g)$$

(where

$$(f\,|\,g) := \int_\Omega f(x)\overline{g(x)}\,\mathrm{d}x$$

denotes the usual scalar product in $L_2(\Omega)$) and associated norm

$$\|f\|_{2,1} := \Big(\|f\|_2^2 + \sum_{j=1}^n \|\partial_j f\|_2^2\Big)^{1/2}.$$

4.8 Theorem. *The space $H^1(\Omega)$ is a separable Hilbert space.*

Proof. Clearly, $H^1(\Omega)$ is a pre-Hilbert space.

Let $J\colon H^1(\Omega) \to \bigoplus_{j=0}^n L_2(\Omega)$ be defined by $Jf := (f, \partial_1 f, \dots, \partial_n f)$, where $\bigoplus_{j=0}^n L_2(\Omega)$ denotes the orthogonal direct sum. Then evidently J is a norm preserving linear mapping.

Therefore $H^1(\Omega)$ is complete if and only if the range of J is a closed linear subspace of the Hilbert space $\bigoplus_{j=0}^n L_2(\Omega)$. Let (f_k) be a sequence in $H^1(\Omega)$ such that (Jf_k) is convergent in $\bigoplus_{j=0}^n L_2(\Omega)$ to an element $(f^0,\dots,f^n)$. This means that $f_k \to f^0$ and $\partial_j f_k \to f^j$ $(j=1,\dots,n)$ in $L_2(\Omega)$ as $k\to\infty$. We show that this implies $f^j = \partial_j f^0$ $(j=1,\dots,n)$. Indeed, for all $\varphi \in C_c^\infty(\Omega)$, $k\in\mathbb{N}$ we have

$$\int \partial_j f_k(x)\varphi(x)\,\mathrm{d}x = -\int f_k(x)\partial_j\varphi(x)\,\mathrm{d}x.$$

Taking $k\to\infty$ we obtain

$$\int f^j(x)\varphi(x)\,\mathrm{d}x = -\int f^0(x)\partial_j\varphi(x)\,\mathrm{d}x.$$

So we have shown that $f := f^0 \in H^1(\Omega)$ and that $Jf = (f^0,\dots,f^n)$.

The space $L_2(\Omega)$ is separable, therefore $\bigoplus_{j=0}^n L_2(\Omega)$ is separable, the closed subspace $J(H^1(\Omega))$ of $\bigoplus_{j=0}^n L_2(\Omega)$ is separable, and thus $H^1(\Omega)$ is separable because J is isometric. □

As in Subsection 4.1.2, we give additional information for the one-dimensional case.

4.9 Theorem. *Let $-\infty < a < b < \infty$. Then every $f\in H^1(a,b)$ possesses a representative in $C[a,b]$, and the inclusion $H^1(a,b)\subseteq C[a,b]$ thus defined is continuous.*

Proof. It is an immediate consequence of Proposition 4.6 that every function $f\in H^1(a,b)$ has a continuous representative, which also can be extended continuously to $[a,b]$.

Let $f\in H^1(a,b)$, with f chosen as the continuous representative. Then

$$\begin{aligned}\|f\|_{C[a,b]} &\leqslant \inf_{x\in(a,b)}|f(x)| + \int_a^b |f'(x)|\,\mathrm{d}x \leqslant \frac{1}{b-a}\int_a^b |f(x)|\,\mathrm{d}x + \int_a^b |f'(x)|\,\mathrm{d}x \\ &\leqslant (b-a)^{-1/2}\|f\|_2 + (b-a)^{1/2}\|f'\|_2 \leqslant \big((b-a)^{-1/2} + (b-a)^{1/2}\big)\|f\|_{2,1}.\end{aligned}$$

This inequality shows that the inclusion mapping is a bounded operator. □

4.10 Remarks. (a) It is not difficult to see that Theorem 4.9 implies that, for $c\in\mathbb{R}$, one also obtains continuous embeddings $H^1(-\infty,c)\subseteq C_0(-\infty,c]$, $H^1(c,\infty)\subseteq C_0[c,\infty)$ and $H^1(-\infty,\infty)\subseteq C_0(-\infty,\infty)$. (The important observation is that the norm of the inclusion mapping only depends on the length of the interval (a,b).)

In fact, the inequality derived in the proof of Theorem 4.9 shows that one obtains

$$\|f\|_\infty \leqslant \big(d^{-1/2}+d^{1/2}\big)\|f\|_{2,1} \qquad (f\in H^1(a,b))$$

if (a,b) contains an interval of length d.

(b) Theorem 4.9 is a simple instance of the Sobolev embedding theorems.

4.1.4 Denseness properties

Let $\Omega \subseteq \mathbb{R}^n$ be open. For $f \in L_{1,\mathrm{loc}}(\Omega)$ we define the **support of** f by

$$\operatorname{spt} f := \Omega \setminus \bigcup \bigl\{ U \subseteq \Omega;\, U \text{ open},\ f|_U = 0 \bigr\}.$$

(This definition is consistent with the already defined support for continuous functions.) Furthermore we define

$$\begin{aligned} H^1_{\mathrm{c}}(\Omega) &:= \bigl\{ f \in H^1(\Omega);\ \operatorname{spt} f \text{ compact in } \Omega \bigr\}, \\ H^1_0(\Omega) &:= \overline{H^1_{\mathrm{c}}(\Omega)}^{H^1(\Omega)}. \end{aligned}$$

4.11 Remarks. (a) In a generalised sense, functions in $H^1_0(\Omega)$ 'vanish at the boundary of Ω'. This will be made more precise in Lecture 7.

(b) For $f \in H^1_0(\Omega)$ the extension to $\mathbb{R}^n$ by zero belongs to $H^1(\mathbb{R}^n)$. This is easy to see if $f \in H^1_{\mathrm{c}}(\Omega)$ and then carries over to the closure; see Exercise 4.2.

4.12 Theorem. (a) $H^1_0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

(b) $H^1_0(\Omega) = \overline{C^\infty_{\mathrm{c}}(\Omega)}^{H^1(\Omega)}$.

We need the following auxiliary result.

4.13 Lemma. (a) *Let $\alpha \in \mathbb{N}_0^n$, $f, \partial^\alpha f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Then*

$$\partial^\alpha(\rho * f) = \rho * (\partial^\alpha f).$$

(b) *Let $f \in H^1(\mathbb{R}^n)$, and let (ρ_k) be a δ-sequence in $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Then $\rho_k * f \to f$ in $H^1(\mathbb{R}^n)$ as $k \to \infty$.*

Proof. (a) Using Lemma 4.1 in the first equality and the definition of the distributional derivative in the third, one obtains

$$\begin{aligned} \partial^\alpha(\rho * f)(x) &= \int \partial^\alpha \rho(x-y) f(y)\,\mathrm{d}y = (-1)^{|\alpha|} \int \partial^\alpha \bigl(y \mapsto \rho(x-y)\bigr) f(y)\,\mathrm{d}y \\ &= \int \rho(x-y) \partial^\alpha f(y)\,\mathrm{d}y = \rho * \partial^\alpha f(x). \end{aligned}$$

(b) From Proposition 4.3(b) we know that $\rho_k * f \to f$ in $L_2(\mathbb{R}^n)$ as $k \to \infty$. Also, using part (a), for $1 \leqslant j \leqslant n$ one obtains

$$\partial_j(\rho_k * f) = \rho_k * \partial_j f \to \partial_j f \qquad (k \to \infty)$$

in $L_2(\mathbb{R}^n)$, again by Proposition 4.3(b). □

Proof of Theorem 4.12. (a) Let $f \in H^1(\mathbb{R}^n)$, $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. A straightforward computation shows that $\partial_j(\varphi f) = \partial_j \varphi f + \varphi \partial_j f$, and therefore $\partial_j(\varphi f) \in L_2(\mathbb{R}^n)$ $(1 \leqslant j \leqslant n)$. Hence $\varphi f \in H^1_{\mathrm{c}}(\mathbb{R}^n)$.

Choose $\psi \in C_c^\infty(\mathbb{R}^n)$, $\psi|_{B(0,1)} = 1$, and define $\psi_k := \psi(\cdot/k)$ $(k \in \mathbb{N})$. Then, with the aid of the previous observation, it is not difficult to show that $\psi_k f \to f$ in $H^1(\mathbb{R}^n)$ as $k \to \infty$.

(b) The inclusion '$\supseteq$' is trivial. For the inclusion '$\subseteq$' it is sufficient to show that each $f \in H_c^1(\Omega)$ can be approximated by elements of $C_c^\infty(\Omega)$. This, however, is a consequence of Remark 4.11(b) and Lemma 4.13(b). (Observe that the support of $\rho_k * f$ is a compact subset of Ω, for large $k \in \mathbb{N}$; see Exercise 4.1(a).) □

4.14 Remark. In general, the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ do not coincide. We show this for the case that $\Omega \neq \varnothing$ is bounded.

As a preparation we note that for all $f \in H_0^1(\Omega)$ one has $\int \partial_1 f(x)\,\mathrm{d}x = 0$. This is clear if $f \in C_c^\infty(\Omega)$ and carries over to $H_0^1(\Omega)$ by continuity. Let $f \in H^1(\Omega)$ be defined by $f(x) := x_1$ $(x \in \Omega)$. Then $\int \partial_1 f(x)\,\mathrm{d}x \neq 0$ and therefore $f \in H^1(\Omega) \setminus H_0^1(\Omega)$.

4.15 Example. Right translation semigroups.

We come back to Examples 1.7, but only for the case $p = 2$. We describe the generator A.

(a) On $L_2(\mathbb{R})$: It was mentioned in Remark 1.15 that $D := C_c^1(\mathbb{R})$ is a core for A, i.e., $A = \overline{A|_D}$, and that $Af = -f'$ for all $f \in D$. Since $A|_D$, considered as a subspace of $L_2(\mathbb{R}) \times L_2(\mathbb{R})$, is isomorphic to $C_c^1(\mathbb{R})$, as a subspace of $H^1(\mathbb{R})$, one obtains

$$\operatorname{dom}(A) = \overline{C_c^1(\mathbb{R})} = H_0^1(\mathbb{R}) = H^1(\mathbb{R}), \quad Af = -f' \quad (f \in \operatorname{dom}(A)).$$

(b) On $L_2(0,\infty)$: Similarly to part (a) one obtains

$$\operatorname{dom}(A) = \overline{C_c^1(0,\infty)} = H_0^1(0,\infty), \quad Af = -f' \quad (f \in \operatorname{dom}(A)).$$

(c) We leave it as a homework to show that for $L_2(-\infty,0)$ the generator is given by

$$\operatorname{dom}(A) = H^1(-\infty,0), \quad Af = -f' \quad (f \in \operatorname{dom}(A)),$$

whereas for $L_2(0,1)$ one obtains

$$\operatorname{dom}(A) = \big\{f \in H^1(0,1);\, f(0) = 0\big\}, \quad Af = -f' \quad (f \in \operatorname{dom}(A)).$$

4.2 The Hilbert space method for the solution of inhomogeneous problems, and the Dirichlet Laplacian

The first aim of the this section is to present a variant of the solution of the Poisson equation

$$-\Delta u = f \text{ on } \Omega, \qquad u|_{\partial\Omega} = 0.$$

Here, f should be a given function on the open subset $\Omega \subseteq \mathbb{R}^n$, and (for the moment) u should be thought twice differentiable on Ω and continuous on the closure. We will not treat the problem in this form but rather weaken the requirements.

More explicitly, the requirement on the boundary behaviour will be modified to the requirement that u should belong to $H^1_0(\Omega)$, and the equation itself will only be required to hold in the distributional sense.

In the second part of the section we will establish the connection to m-accretivity and holomorphic semigroups.

4.2.1 The equation $u - \Delta u = f$

4.16 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $f \in L_2(\Omega)$. Then there exists a unique function $u \in H^1_0(\Omega)$ such that $u - \Delta u = f$ in the distributional sense.*

We insert a lemma expressing the distributional equality in another form.

4.17 Lemma. *Let $u \in H^1(\Omega)$, $g \in L_2(\Omega)$. Then $\Delta u = g$ in the distributional sense if and only if*

$$(v \,|\, g)_{L_2(\Omega)} = -\sum_{j=1}^{n} (\partial_j v \,|\, \partial_j u)_{L_2(\Omega)} \qquad (v \in H^1_0(\Omega)). \tag{4.1}$$

Proof. By the definition of distributional derivatives, the equation $\Delta u = g$ is equivalent to

$$(\varphi \,|\, g) = (\Delta\varphi \,|\, u) = -\sum_{j=1}^{n} (\partial_j \varphi \,|\, \partial_j u) \qquad (\varphi \in C_c^\infty(\Omega)),$$

i.e., to the validity of Equation (4.1) for all $v \in C_c^\infty(\Omega)$. As both mappings $H^1(\Omega) \ni v \mapsto (v \,|\, g) \in \mathbb{K}$ and $H^1(\Omega) \ni v \mapsto \sum_{j=1}^n (\partial_j v \,|\, \partial_j u) \in \mathbb{K}$ are continuous, the equality of the terms in (4.1) extends to the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$, i.e., to $H^1_0(\Omega)$ (recall Theorem 4.12(b). $\square$

Proof of Theorem 4.16. We define a linear functional $\eta\colon H^1_0(\Omega) \to \mathbb{K}$ by

$$\eta(v) := (v \,|\, f)_{L_2(\Omega)} \qquad (v \in H^1_0(\Omega)).$$

Then η is continuous; indeed $|\eta(v)| \leqslant \|f\|_2 \|v\|_2 \leqslant \|g\|_2 \|v\|_{2,1}$ $(v \in H^1_0(\Omega))$. Applying the representation theorem of Riesz-Fréchet (see, for instance, [Bre83; Théorème V.5]) we obtain $u \in H^1_0(\Omega)$ such that

$$\eta(v) = (v \,|\, u)_1 \qquad (v \in H^1_0(\Omega)).$$

Putting this equation and the definition of η together we obtain

$$(v \,|\, f) = (v \,|\, u)_1 = (v \,|\, u) + \sum_{j=1}^{n} (\partial_j v \,|\, \partial_j u) \qquad (v \in H^1_0(\Omega)).$$

Shifting the first term on the right hand side to the left and applying Lemma 4.17 we conlude that $-\Delta u = f - u$ in the distributional sense.

The uniqueness of u is a consequence of the uniqueness in the Riesz-Fréchet representation theorem. $\square$

4.2.2 The Dirichlet Laplacian

In this subsection we reformulate the result of Subsection 4.2.1 in operator language. As before, let $\Omega \subseteq \mathbb{R}^n$ be open. In what follows the space $L_2(\Omega)$ will be complex.

We define the **Dirichlet Laplacian** Δ_{D} in $L_2(\Omega)$,

$$\Delta_{\mathrm{D}} := \bigl\{(u,f) \in L_2(\Omega) \times L_2(\Omega);\, u \in H_0^1(\Omega),\ \Delta u = f\bigr\}.$$

In other words,

$$\begin{aligned} \mathrm{dom}(\Delta_{\mathrm{D}}) &:= \bigl\{u \in H_0^1(\Omega);\, \Delta u \in L_2(\Omega)\bigr\}, \\ \Delta_{\mathrm{D}} u &:= \Delta u \qquad (u \in \mathrm{dom}(\Delta_{\mathrm{D}})). \end{aligned}$$

We will show that Δ_{D} generates a contractive holomorphic C_0-semigroup of angle $\pi/2$. The name 'Dirichlet Laplacian' may be somewhat misleading; so we give a short explanation. In principal, 'Dirichlet boundary conditions' are of the form $u|_{\partial\Omega} = \varphi$ for some function φ defined on $\partial\Omega$. We have explained above that the membership of u in H_0^1 is a version of Dirichlet boundary condition zero. So, 'Dirichlet Laplacian' should be regarded as an abbreviation of 'Laplacian with Dirichlet boundary condition zero'.

4.18 Theorem. *The negative Dirichlet Laplacian $-\Delta_{\mathrm{D}}$ is m-sectorial of angle 0. The operator Δ_{D} is the generator of a contractive holomorphic C_0-semigroup of angle $\pi/2$ on $L_2(\Omega)$.*

Proof. For $u \in \mathrm{dom}(\Delta_{\mathrm{D}})$ an application of Lemma 4.17 yields

$$(-\Delta_{\mathrm{D}} u \,|\, u) = (-\Delta u \,|\, u) = \sum_{j=1}^n \int \partial_j u \,\overline{\partial_j u} = \sum_{j=1}^n \int |\partial_j u|^2 \in [0,\infty).$$

This means that $\mathrm{num}(-\Delta_{\mathrm{D}}) \subseteq [0,\infty)$. Also, Theorem 4.16 states that $\mathrm{ran}(I - \Delta_{\mathrm{D}}) = L_2(\Omega)$. As a consequence, $-\Delta_{\mathrm{D}}$ is m-sectorial of angle 0.

Now Theorem 3.22 implies that Δ_{D} generates a contractive holomorphic C_0-semigroup of angle $\pi/2$. □

The statement that '$-\Delta_{\mathrm{D}}$ is m-sectorial of angle 0' is equivalent to saying that $-\Delta_{\mathrm{D}}$ is a positive self-adjoint operator; this will be explained in Lecture 6.

Notes

In Section 4.1 we have collected some basics of Sobolev spaces as far as we will need and use them in the following. For the general definition of Sobolev spaces and more information we refer to [Ada75]. The reader may have noticed that we state (and prove) some properties in more generality than used for the case of the Sobolev space H^1.

The treatment of the Dirichlet Laplacian as given in the lecture is well-established and can be found in many books on partial differential equations.

Exercises

4.1 (a) Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, $\varphi \in C_\mathrm{c}(\mathbb{R}^n)$. Show that $\operatorname{spt}(\varphi * f) \subseteq \operatorname{spt} f + \operatorname{spt} \varphi$. (Hint: Show first that $\operatorname{spt} f + \operatorname{spt} \varphi$ is closed.)

(b) Let $K \subseteq U \subseteq \mathbb{R}^n$, K compact, U open. Show that there exists $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^n)$ with $\operatorname{spt} \psi \subseteq U$, $\psi|_K = \mathbf{1}$ and $0 \leqslant \psi \leqslant 1$. (Hint: Note that $\operatorname{dist}(K, \mathbb{R}^n \setminus U) > 0$. Find ψ as the convolution of a suitable function $\rho \in C_\mathrm{c}^\infty(\mathbb{R}^n)$ with a suitable indicator function.)

4.2 Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Let $f \in H_\mathrm{c}^1(\Omega)$, and define $\tilde{f}$ as the extension of f to $\mathbb{R}^n$ by zero. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$. (Hint: Using Exercise 4.1, choose a function $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^n)$ with $\operatorname{spt} \psi \subseteq \Omega$ and $\psi = 1$ in a neighbourhood of $\operatorname{spt} f$. With the aid of this function show that $\partial_j \tilde{f}$ is the extension of $\partial_j f$ to $\mathbb{R}^n$ by zero.)

(b) Let $f \in H_0^1(\Omega)$, and define $\tilde{f}$ as the extension of f to $\mathbb{R}^n$ by zero. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$.

4.3 Let $H \subseteq \mathbb{R}^2$ be the half-plane $H := \{(x_1, x_2);\ x_1 \geqslant 0\}$, and let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^2)$ be defined by $f := \mathbf{1}_H$.

(a) Show that $\int \partial_1 \varphi f = \int_{x_2 \in \mathbb{R}} \varphi(0, x_2)\, \mathrm{d}x_2$ for all $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^2)$ and that there is no $g \in L_{1,\mathrm{loc}}(\mathbb{R}^2)$ such that $\int \partial_1 \varphi f = \int \varphi g$ for all $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^2)$.

(b) Decide which of the partial derivatives $\partial_1 f, \partial_2 f, \partial_1 \partial_2 f$ belong to $L_{1,\mathrm{loc}}(\mathbb{R}^2)$.

4.4 Let $n \geqslant 3$. Show that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n \setminus \{0\})$. For more ambitious participants: Show this also for $n = 2$.

4.5 Let $\Omega \subseteq \mathbb{R}^n$ be open.

(a) Show that there exists a **standard exhaustion** $(\Omega_k)_{k \in \mathbb{N}}$ of Ω, i.e., Ω_k is open, relatively compact in Ω_{k+1} ($k \in \mathbb{N}$), and $\bigcup_{k \in \mathbb{N}} \Omega_k = \Omega$. (Hint: For $\Omega \neq \mathbb{R}^n$ use

$$\Omega_k := \big\{x \in \Omega;\ |x| < k,\ \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) > \tfrac{1}{k}\big\}.)$$

(b) Let $f \in L_{1,\mathrm{loc}}(\Omega)$, and assume that $f = 0$ locally, i.e., for all $x \in \Omega$ there exists $r > 0$ such that $f|_{B(0,r)} = 0$. Then $f = 0$. (All '$= 0$' should be interpreted as a.e.)

References

[Ada75] R. A. Adams: *Sobolev spaces.* Academic Press, New York, 1975.

[Bre83] H. Brezis: *Analyse fonctionelle – Théorie et applications.* Masson, Paris, 1983.