Lecture 3

Holomorphic semigroups

The objective of this lecture is to introduce semigroups for which the 'time parameter' t can also be chosen in a complex neighbourhood of the positive reals. The foundations of these 'holomorphic semigroups' will be given in Section 3.2. The generation of such semigroups will be studied in Section 3.3. And in Section 3.4 the special case of holomorphic semigroups on Hilbert spaces will be treated. We start with an interlude on Banach space valued holomorphy.

3.1 Interlude: vector-valued holomorphic functions

In this section let X, Y be complex Banach spaces.

The first issue of the present section is to show that for Banach space valued functions several notions of holomorphy coincide.

Let $\Omega \subseteq \mathbb{C}$ be open, $f\colon \Omega \to X$. The function f is called **holomorphic** if f is (complex) differentiable (at each point of Ω). f is called **scalarly holomorphic** if $x' \circ f$ is holomorphic for all $x' \in X'$ ($= \mathcal{L}(X, \mathbb{C})$, the dual space of X). f is called **analytic** if f can be represented as a power series in a neighbourhood of each point of Ω.

In the following, we will use the notation $B(z_0, r) := \{z \in \mathbb{C};\ |z - z_0| < r\}$, $B[z_0, r] := \{z \in \mathbb{C};\ |z - z_0| \leqslant r\}$.

3.1 Remarks. (a) It is evident that holomorphy of a function implies scalar holomorphy.

(b) Let $\Omega \subseteq \mathbb{C}$ be open, and let $f\colon \Omega \to X$ be holomorphic. The following facts can be shown in the same way as in the case of $\mathbb{C}$-valued functions.

If Ω is convex, and γ is a piecewise continuously differentiable closed path in Ω, then $\int_\gamma f(z)\,dz = 0$ (Cauchy's integral theorem). Note that the path integral is defined by a parametrisation of the path and therefore reduces to integrals over intervals. As a consequence, path integrals enter the context explained in Subsection 1.3.2.

For $z_0 \in \Omega$, $r > 0$ such that $B[z_0, r] \subseteq \Omega$ the function f satisfies Cauchy's integral formula

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \qquad (z \in B(z_0, r)).$$

The function f is analytic, and one has Cauchy's integral formulas for the derivatives,

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta \qquad (z \in B(z_0, r),\ n \in \mathbb{N}),$$

with z_0 and r as before.

A set $E \subseteq X'$ is **separating** (**for** X) if for all $0 \neq x \in X$ there exists $x' \in E$ such that $x'(x) \neq 0$. The set E is called **almost norming** (**for** X) if

$$\|x\|_E := \sup\{|x'(x)|;\, x' \in E,\ \|x'\| \leqslant 1\}$$

defines a norm that is equivalent to the norm on X. E is called **norming** if $\|\cdot\|_E = \|\cdot\|$ on X.

3.2 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, $f\colon \Omega \to X$. Then the following properties are equivalent.*

(i) *f is holomorphic.*

(ii) *f is scalarly holomorphic.*

(iii) *There exists an almost norming closed subspace $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.*

(iv) *f is locally bounded, and there exists an almost norming subset $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.*

(v) *f is continuous, and there exists a separating set $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x' \in E$.*

Proof. (i) $\Rightarrow$ (ii) is clear (and was already noted above).

(ii) $\Rightarrow$ (iii) is trivial, with $E = X'$.

(iii) $\Rightarrow$ (iv). With the canonical embedding $X \subseteq E' = \mathcal{L}(E, \mathbb{C})$ we can consider f as an $\mathcal{L}(E, \mathbb{C})$-valued function. Then the uniform boundedness theorem implies that f is locally bounded with respect to the norm $\|\cdot\|_E$.

(iv) $\Rightarrow$ (v). Since almost norming subsets are separating we only have to show that f is continuous.

Let $z_0 \in \Omega$, $r > 0$, $B[z_0, r] \subseteq \Omega$, $M := \sup\bigl\{\|f(\zeta)\|;\, |\zeta - z_0| = r\bigr\}$ $(< \infty)$. For $x' \in E$, $z \in B(z_0, r/2)$ we then obtain, using Cauchy's integral formula for the derivative,

$$\left|\frac{\mathrm{d}}{\mathrm{d}z} x'(f(z))\right| = \left|\frac{1}{2\pi i} \int\limits_{|\zeta - z_0| = r} \frac{x'(f(\zeta))}{(\zeta - z)^2}\, \mathrm{d}\zeta\right| \leqslant \frac{1}{2\pi} 2\pi r \|x'\| \frac{4M}{r^2} = \frac{4M}{r}\|x'\|.$$

For $z', z'' \in B[z_0, r/2]$ this implies

$$|x'(f(z') - f(z''))| \leqslant \frac{4M}{r}|z' - z''|\|x'\| \qquad (x' \in E),$$

and therefore

$$\|f(z') - f(z'')\|_E \leqslant \frac{4M}{r}|z' - z''|.$$

Since the norm $\|\cdot\|_E$ is equivalent to the norm on X the continuity follows on $B(z_0, r/2)$.

(v) $\Rightarrow$ (i). Let $z_0 \in \Omega$. We show that f can be expanded into a power series about z_0.

Without loss of generality we assume $z_0 = 0$. There exists $r > 0$ such that $B[0, r] \subseteq \Omega$. For $n \in \mathbb{N}_0$ we define

$$a_n := \frac{1}{2\pi i} \int\limits_{|\zeta| = r} \frac{f(\zeta)}{\zeta^{n+1}}\, \mathrm{d}\zeta.$$

With $M := \sup\{\|f(\zeta)\|;\ |\zeta| = r\}$ $(< \infty)$ one obtains $\|a_n\| \leqslant \frac{M}{r^n}$ for all $n \in \mathbb{N}_0$, so $\limsup \|a_n\|^{1/n} \leqslant 1/r$, and therefore the radius of convergence of the power series $g(z) := \sum_{n=0}^{\infty} z^n a_n$ is greater or equal r. For $x' \in E$, $|z| < r$ one has

$$x'(g(z)) = \sum_{n=0}^{\infty} z^n \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{x'(f(\zeta))}{\zeta^{n+1}}\,\mathrm{d}\zeta = \sum_{n=0}^{\infty} \frac{(x' \circ f)^{(n)}(0)}{n!} z^n = x'(f(z)),$$

where the last equality is just the power series expansion of the holomorphic function $x' \circ f$. Since E is separating we therefore obtain $f(z) = g(z)$. □

3.3 Remark. If X is a dual Banach space then the predual is a norming closed subspace of its bidual X'. (For instance, c_0 is norming for ℓ_1.) This illustrates a possible application of condition (iii) of Theorem 3.2.

As a first application of Theorem 3.2 we show the identity theorem for holomorphic functions.

3.4 Corollary. *Let $\Omega \subseteq \mathbb{C}$ be open and connected, let $f\colon \Omega \to X$ be holomorphic, and assume that $[f = 0] = \{z \in \Omega;\ f(z) = 0\}$ has a cluster point in Ω. Then $f = 0$.*

Proof. For each $x' \in X'$ the zeros of the function $x' \circ f$ have a cluster point in Ω, and therefore $x' \circ f = 0$ follows from the identity theorem for $\mathbb{C}$-valued holomorphic functions.
From $x' \circ f = 0$ for all $x' \in X'$ one obtains $f = 0$. □

For the experts this proof may not really seem convincing, because the identity theorem can also be concluded in the same way as for $\mathbb{C}$-valued functions (in the spirit of Remark 3.1(b)).

Finally we come to the characterisation of holomorphy for $\mathcal{L}(X,Y)$-valued functions. As $\mathcal{L}(X,Y)$ is a Banach space, all previous criteria apply. However, scalar holomorphy is not a useful concept in this case, because the dual of $\mathcal{L}(X,Y)$ mostly is not easily accessible.

3.5 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, $F\colon \Omega \to \mathcal{L}(X,Y)$. Then the following are equivalent.*
(i) *F is holomorphic (as an $\mathcal{L}(X,Y)$-valued function).*
(ii) *For all $x \in X$, the function $F(\cdot)x\colon \Omega \to Y$ is holomorphic.*

Proof. (i) ⇒ (ii) is clear.
(ii) ⇒ (i). The uniform boundedness theorem implies that F is locally bounded (as an $\mathcal{L}(X,Y)$-valued function). The set

$$E := \{A \mapsto y'(Ax);\ x \in X,\ y' \in Y'\} \subseteq \mathcal{L}(X,Y)'$$

is norming for $\mathcal{L}(X,Y)$, and the function $y'(F(\cdot)x)$ is holomorphic for all $x \in X$, $y' \in Y'$. Therefore Theorem 3.2, (iv) ⇒ (i), implies the assertion. □

3.6 Remarks. (a) Evidently, condition (ii) of Theorem 3.5 can be weakened according to the conditions in Theorem 3.2.
(b) Note that the set E in the proof of Theorem 3.5 is not a sub*space* of $\mathcal{L}(X,Y)$. For this application it is convenient to have condition (iv) of Theorem 3.2 at our disposal.

The last issue of this section is the convergence of sequences of holomorphic functions.

3.7 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be open, and let (f_n) be a sequence of holomorphic functions $f_n \colon \Omega \to X$. Assume that (f_n) is locally bounded (i.e., for each $z_0 \in \Omega$ there exists $r > 0$ such that $B(z_0, r) \subseteq \Omega$ and $\sup\{\|f_n(z)\|;\ z \in B(z_0,r),\ n \in \mathbb{N}\} < \infty$) and that $f(z) := \lim_{n\to\infty} f_n(z)$ exists for all $z \in \Omega$.*

Then (f_n) converges to f locally uniformly, and f is holomorphic.

Proof. In the first step we show that the sequence (f_n) is locally uniformly equicontinuous. Let $z_0 \in \Omega$, $r > 0$ be such that $B[z_0, r] \subseteq \Omega$. Then $C := \sup_{z\in B[z_0,r],\, n\in\mathbb{N}} \|f_n(z)\| < \infty$, by hypothesis. From Cauchy's integral formula for the derivative,

$$f_n'(z) = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f_n(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta \qquad (z \in B(z_0, r),\ n \in \mathbb{N})$$

we conclude that for all $z \in B[z_0, r/2]$ one has

$$\|f_n'(z)\| \leqslant C\frac{4}{r} \qquad (n \in \mathbb{N}).$$

This implies that the sequence (f_n) is uniformly equicontinuous on $B[z_0, r/2]$.

The local uniform equicontinuity of the sequence (f_n) together with the pointwise convergence implies that (f_n) converges to f locally uniformly. In order to show that this implies that f is holomorphic, let again $z_0 \in \Omega$ and $r > 0$ be such that $B[z_0, r] \subseteq \Omega$. Then Cauchy's integral formula

$$f_n(z) = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f_n(\zeta)}{\zeta - z}\,\mathrm{d}\zeta,$$

valid for all $z \in B(z_0, r)$, $n \in \mathbb{N}$, carries over to f,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \qquad (z \in B(z_0, r)),$$

and this implies that f is holomorphic on $B(z_0, r)$. □

3.8 Corollary. *Let $\Omega \subseteq \mathbb{C}$ be open, and let (F_n) be a sequence of holomorphic functions $F_n \colon \Omega \to \mathcal{L}(X, Y)$. Assume that (F_n) is locally bounded and that $F(z) := \text{s-}\lim_{n\to\infty} F_n(z)$ exists for all $z \in \Omega$.*

Then F is holomorphic.

Proof. The hypotheses in combination with Theorem 3.7 imply that $F(\cdot)x$ is holomorphic for all $x \in X$. Therefore Theorem 3.5 implies that F is holomorphic. □

3.2 Holomorphic semigroups

Let X be a complex Banach space. For $\theta \in (0, \pi]$ we define the (open) sector

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\};\ |\mathrm{Arg}\, z| < \theta\} = \{r\mathrm{e}^{\mathrm{i}\alpha};\ r > 0,\ |\alpha| < \theta\}.$$

We will also use the notation $\Sigma_{\theta,0} := \Sigma_\theta \cup \{0\}$. A **holomorphic semigroup** (**of angle** θ, if we want to make precise the angle) is a function $T \colon \Sigma_{\theta,0} \to \mathcal{L}(X)$, holomorphic on Σ_θ, satisfying

(i) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_{\theta,0}$.

If additionally

(ii) $\lim_{z\to 0,\, z\in\Sigma_{\theta'}} T(z)x = x$ for all $x \in X$ and all $\theta' \in (0, \theta)$,

then T will be called a **holomorphic C_0-semigroup** (**of angle** θ).

Saying that T is a holomorphic semigroup we will always mean that T brings along its domain of definition, in particular, *the angle of* T is defined.

3.9 Remark. We note that the definition of a holomorphic C_0-semigroup implies that for all $\theta' \in (0, \theta)$ there exist $M' \geqslant 1$, $\omega' \in \mathbb{R}$ such that

$$\|T(z)\| \leqslant M' \mathrm{e}^{\omega' \operatorname{Re} z} \qquad (z \in \Sigma_{\theta'});$$

see Exercise 3.3.

The following lemma states that it suffices to check the semigroup property for real times.

3.10 Lemma. *Let $\theta \in (0, \pi/2]$, and let $T \colon \Sigma_\theta \to \mathcal{L}(X)$ be holomorphic and such that $T(t + s) = T(t)T(s)$ for all $t, s > 0$.*

Then $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\theta$.

Proof. Fixing $t > 0$, we know that the functions $\Sigma_\theta \ni z \mapsto T(t + z)$, $\Sigma_\theta \ni z \mapsto T(t)T(z)$ and $\Sigma_\theta \ni z \mapsto T(z)T(t)$ are holomorphic and coincide on $(0, \infty)$. The identity theorem, Theorem 3.4, implies that they are equal on Σ_θ. Repeating the argument with $t \in \Sigma_\theta$ we obtain the assertion. □

3.11 Proposition. *Let T be a C_0-semigroup on X, and assume that there exist $\theta \in (0, \pi/2]$ and an extension of T to $\Sigma_{\theta,0}$, also called T, holomorphic on Σ_θ and satisfying*

$$\sup_{z\in\Sigma_\theta,\, |z|<1} \|T(z)\| < \infty.$$

Then $\lim_{z\to 0,\, z\in\Sigma_\theta} T(z)x = x$ for all $x \in X$, and T is a holomorphic C_0-semigroup.

Proof. First note that Lemma 3.10 implies property (i) from above.

Let $x \in D := \bigcup_{t>0} \operatorname{ran}(T(t))$, i.e., there exist $y \in X$, $t > 0$ such that $x = T(t)y$. Then the continuity of $z \mapsto T(z)x = T(z)T(t)y = T(z + t)y$ at 0 implies $\lim_{z\to 0,\, z\in\Sigma_\theta} T(z)x = x$.

Note that D is dense, because $T(t) \to I$ strongly as $t \to 0$. Therefore the boundedness assumption implies the assertion (recall Proposition 1.6). □

3.12 Remarks. Let T be a holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$.

(a) Then for each $\alpha \in (-\theta, \theta)$ the mapping $[0, \infty) \ni t \mapsto T_\alpha(t) := T(\mathrm{e}^{\mathrm{i}\alpha} t)$ is a C_0-semigroup.

(b) The set $D := \bigcup_{z\in\Sigma_\theta} \operatorname{ran}(T(z))$ is a dense subspace of X. The denseness follows from property (ii). In order to show that D is a vector space we first note that $\operatorname{ran}(T(z_1 + z_2)) \subseteq \operatorname{ran}(T(z_1))$ for all $z_1, z_2 \in \Sigma_\theta$. Further, it is not difficult to see that for all $z_1, z_2 \in \Sigma_\theta$ there exists $z \in \Sigma_\theta$ such that $z_1 - z, z_2 - z \in \Sigma_\theta$, and then it follows that $\operatorname{ran}(T(z_j)) \subseteq \operatorname{ran}(T(z))$ for $j = 1, 2$.

3.13 Theorem. *Let T be a holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$, and let A be the generator of T_0 $(= T|_{[0,\infty)})$. Then:*

(a) *For all $\alpha \in (-\theta, \theta)$ the generator of T_α is given by $\mathrm{e}^{\mathrm{i}\alpha}A$.*

(b) *For all $x \in \operatorname{dom}(A)$, $\theta' \in (0, \theta)$ one has*

$$\lim_{z \to 0,\, z \in \Sigma_{\theta'}} \frac{1}{z}(T(z)x - x) = Ax.$$

Proof. (a) Let $\alpha \in (-\theta, \theta)$, and denote the generator of the C_0-semigroup T_α (see Remark 3.12(a)) by A_α. Let $D := \bigcup_{z \in \Sigma_\theta} \operatorname{ran}(T(z))$. Then the differentiability of T on Σ_θ implies that $D \subseteq \operatorname{dom}(A_\alpha)$, and one easily computes that $A_\alpha x = \mathrm{e}^{\mathrm{i}\alpha}Ax$ for all $x \in D$. Also it follows from the definition of D that D is invariant under T_α. Therefore Proposition 1.14 (together with Remark 3.12(b)) implies that D is a core for A_α. In particular, this holds for $\alpha = 0$, and therefore one obtains $A_\alpha = \mathrm{e}^{\mathrm{i}\alpha}A$.

(b) Let $x \in \operatorname{dom}(A)$, $\theta' \in (0, \theta)$. For $z \in \Sigma_{\theta'}$ one then obtains

$$\frac{1}{z}(T(z)x - x) = \frac{1}{z}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}T(sz)x \,\mathrm{d}s = \int_0^1 T(sz)Ax \,\mathrm{d}s.$$

In view of the continuity at 0 of the restriction of $T(\cdot)Ax$ to $\Sigma_{\theta',0}$ one therefore obtains the assertion. □

Let T be a holomorphic C_0-semigroup. In the light of Theorem 3.13 it is justified to call the generator of the C_0-semigroup $T|_{[0,\infty)}$ also the **generator** of the holomorphic C_0-semigroup T. Clearly, the application of Theorem 2.7 yields estimates for the resolvents of the generator of T. We will not pursue this issue but restrict our attention to a special kind of holomorphic semigroups.

3.3 Generation of contractive holomorphic semigroups

As before, X will be a complex Banach space. We will call a holomorphic semigroup of angle θ **contractive** if $\|T(z)\| \leqslant 1$ for all $z \in \Sigma_{\theta,0}$.

The following theorem characterises the generation of contractive holomorphic C_0-semigroups.

3.14 Theorem. *An operator A is the generator of a contractive holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$ if and only if it is closed, densely defined, and $\Sigma_\theta \subseteq \rho(A)$, with*

$$\big\|(\lambda I - A)^{-1}\big\| \leqslant \frac{1}{|\lambda|} \qquad (\lambda \in \Sigma_\theta).$$

Proof. The necessity of the condition is an almost immediate consequence of Theorem 3.13 and Theorem 2.7. 'Closed' and 'densely defined' are clear. For $\alpha \in (-\theta, \theta)$, we know from Theorem 3.13(a) that $\mathrm{e}^{\mathrm{i}\alpha}A$ is the generator of a contractive C_0-semigroup; therefore $(0, \infty) \subseteq \rho(\mathrm{e}^{\mathrm{i}\alpha}A)$ and $\|(\lambda I - \mathrm{e}^{\mathrm{i}\alpha}A)^{-1}\| \leqslant 1/\lambda$ for all $\lambda > 0$, by Theorem 2.7. The equation $(\lambda I - \mathrm{e}^{\mathrm{i}\alpha}A) = \mathrm{e}^{\mathrm{i}\alpha}(\mathrm{e}^{-\mathrm{i}\alpha}\lambda I - A)$ then shows that $\big\{\mathrm{e}^{-\mathrm{i}\alpha}\lambda;\ \lambda > 0\big\} \subseteq \rho(A)$ and $\|(\mathrm{e}^{-\mathrm{i}\alpha}\lambda I - A)^{-1}\| \leqslant 1/\lambda$ for all $\lambda > 0$. This finishes the proof of the necessity.

In the proof of the sufficiency we will employ the exponential formula, Theorem 2.12. For $n \in \mathbb{N}$ we define the holomorphic function $F_n\colon \Sigma_\theta \to \mathcal{L}(X)$,

$$F_n(z) := (I - \tfrac{z}{n}A)^{-n}.$$

Then the hypotheses imply that $\|F_n(z)\| \leqslant 1$ for all $z \in \Sigma_\theta$, $n \in \mathbb{N}$. It also follows from the hypotheses and Theorem 2.9 that for each $\alpha \in (-\theta, \theta)$ the operator $e^{i\alpha}A$ generates a contractive C_0-semigroup; call it T_α. Let $z \in \Sigma_\theta$, $z = e^{i\alpha}t$ with suitable $t > 0$, $\alpha \in (-\theta, \theta)$. Then $F_n(z) = (I - \frac{t}{n}e^{i\alpha}A)^{-n} \to T_\alpha(t)$ strongly as $n \to \infty$, by Theorem 2.12, so

$$T(z) := \operatorname*{s-lim}_{n\to\infty} F_n(z)$$

exists for all $z \in \Sigma_{\theta,0}$. From Corollary 3.8 we derive that T is holomorphic on Σ_θ and from Theorem 2.12 that $T|_{[0,\infty)}$ is the C_0-semigroup generated by A. □

We add some comments on the generation of holomorphic semigroups which are not necessarily contractive.

3.15 Remarks. (a) The same proof as given for Theorem 3.14 can be used to show the following equivalence.

Let $\theta \in (0, \pi/2]$, $M \geqslant 1$. Then an operator A is the generator of a **bounded holomorphic C_0-semigroup** of angle θ and with bound M if and only if for each $\alpha \in (-\theta, \theta)$ the operator $e^{i\alpha}A$ is the generator of a bounded C_0-semigroup with bound M. (Note that we call the holomorphic semigroup T of angle θ bounded if $\sup_{z\in\Sigma_\theta} \|T(z)\| < \infty$. We point out that the terminology 'bounded holomorphic semigroup' is sometimes used differently.)

(b) If A is the generator of a bounded C_0-semigroup with bound $M \geqslant 1$, then $[\operatorname{Re} > 0] := \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > 0\} \subseteq \rho(A)$ and

$$\big\|(\lambda I - A)^{-1}\big\| \leqslant \frac{M}{\operatorname{Re}\lambda} \qquad (\lambda \in [\operatorname{Re} > 0]),$$

by Theorem 2.7. If $\theta \in (0, \pi/2)$ and $\lambda \in \Sigma_\theta$, then $\frac{\operatorname{Re}\lambda}{|\lambda|} \geqslant \cos\theta$, and this implies

$$\big\|(\lambda I - A)^{-1}\big\| \leqslant \frac{M}{\operatorname{Re}\lambda} \leqslant \frac{M}{\cos\theta}\frac{1}{|\lambda|}.$$

Applying this observation to the generator A of a bounded holomorphic semigroup of angle $\theta \in (0, \pi/2]$ one obtains

$$\Sigma_{\theta+\pi/2} \subseteq \rho(A) \quad \text{and} \quad \sup_{\lambda\in\Sigma_{\theta'+\pi/2}} \big\|\lambda(\lambda I - A)^{-1}\big\| < \infty \quad (\theta' \in (0, \theta)). \tag{3.1}$$

This fact has a kind of converse. Here is the complete information: a closed, densely defined operator A in X is the generator of a holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$ that is bounded on all sectors $\Sigma_{\theta'}$ with $\theta' \in (0, \theta)$ if and only if (3.1) holds (see [Kat80; IX, § 1.6], [EN00; II, Section 4a]).

3.4 The Lumer-Phillips theorem

Let H be a Hilbert space over $\mathbb{K}$. The scalar product of two elements $x, y \in H$ will be denoted by $(x \,|\, y)$, and it is defined to be linear in the first and antilinear in the second argument.

An operator A in H is called **accretive** (or **monotone**) if

$$\operatorname{Re}(Ax \,|\, x) \geqslant 0 \qquad (x \in \operatorname{dom}(A)).$$

3.16 Lemma. *Let A be an operator in H. Then A is accretive if and only if*

$$\|(\lambda I + A)x\| \geqslant \lambda \|x\| \qquad (x \in \operatorname{dom}(A)) \tag{3.2}$$

for all $\lambda > 0$.

Proof. Assume that A is accretive, and let $\lambda > 0$, $x \in \operatorname{dom}(A)$. Then

$$\|(\lambda I + A)x\|\|x\| \geqslant |\,((\lambda I + A)x \,|\, x)\,| \geqslant \operatorname{Re}((\lambda I + A)x \,|\, x) \geqslant (\lambda x \,|\, x) = \lambda \|x\|^2.$$

This shows the asserted inequality.

On the other hand, assume that the norm inequality holds, and let $x \in \operatorname{dom}(A)$. Then

$$0 \leqslant \|(\lambda I + A)x\|^2 - \lambda^2 \|x\|^2 = 2\lambda \operatorname{Re}(Ax \,|\, x) + \|Ax\|^2$$

for all $\lambda > 0$, and this implies $\operatorname{Re}(Ax \,|\, x) \geqslant 0$. □

3.17 Remark. We point out some simple consequences of the inequality (3.2), in a more general context.

If X, Y are Banach spaces, B is an operator from X to Y, and there exists $\alpha > 0$ such that

$$\|Bx\|_Y \geqslant \alpha \|x\|_X \qquad (x \in \operatorname{dom}(B)),$$

then B is injective and $\|B^{-1}y\| \leqslant \alpha^{-1}\|y\|$ for all $y \in \operatorname{ran}(B)$. If additionally $\operatorname{ran}(B) = Y$, then $B^{-1} \in \mathcal{L}(Y, X)$ and $\|B^{-1}\| \leqslant \alpha^{-1}$.

An accretive operator A satisfying $\operatorname{ran}(I + A) = H$ is called **m-accretive**. A historical note on the prefix 'm': it should be remindful of the word 'maximal'. However, 'maximal accretive operators', in the sense that there does not exist a proper accretive extension, need not be m-accretive; see [Phi59; footnote (6)].

3.18 Theorem. *(Lumer-Phillips) Let A be an operator in H. Then $-A$ is the generator of a C_0-semigroup of contractions if and only if A is m-accretive.*

For the sufficiency we need a preparation.

3.19 Lemma. *Let A be an accretive operator in H, and assume that $\lambda_0 > 0$ is such that* $\operatorname{ran}(\lambda_0 I + A) = H$. *Then* $(0, \infty) \subseteq \rho(-A)$ *(in particular, A is m-accretive), and* $\operatorname{dom}(A)$ *is dense in H.*

Proof. In view of Remark 3.17, the hypotheses imply that $\lambda_0 \in \rho(-A)$ (and $\|(\lambda_0 I + A)^{-1}\| \leqslant \lambda_0^{-1}$). Also, Remark 3.17 implies that for all $\lambda \in \rho(-A) \cap (0,\infty)$ we have $\|(\lambda I + A)^{-1}\| \leqslant \lambda^{-1}$. Assuming that $\sigma(-A) \cap (0,\infty) \neq \varnothing$ we find a sequence (λ_n) in $\rho(-A) \cap (0,\infty)$ with $\lambda := \lim_{n\to\infty} \lambda_n \in \sigma(-A) \cap (0,\infty)$. But then $\|R(\lambda_n, -A)\| \to \infty$ $(n \to \infty)$, by Remark 2.4(a), which contradicts the bound for the resolvent derived above. So we have shown that $(0,\infty) \subseteq \rho(-A)$.

Let $x \in \operatorname{dom}(A)^\perp$. Then $y := (I + A)^{-1}x \in \operatorname{dom}(A)$, and therefore

$$0 = \operatorname{Re}(x \,|\, y) = \operatorname{Re}((I + A)y \,|\, y) \geqslant \|y\|^2.$$

This implies that $y = 0$, $x = (I + A)y = 0$; therefore $\operatorname{dom}(A)^\perp = \{0\}$, i.e., $\operatorname{dom}(A)$ is dense in H. □

3.20 Remark. We note that Lemma 3.19 implies that every accretive operator $A \in \mathcal{L}(H)$ is automatically m-accretive. Indeed, $\{\lambda \in \mathbb{R};\ \lambda > \|A\|\} \subseteq \rho(A)$, by Remark 2.3(b).

Proof of Theorem 3.18. The necessity is an immediate consequence of Theorem 2.7, Remark 3.17) and Lemma 3.16.

The sufficiency follows from Lemma 3.19, Lemma 3.16 (in combination with Remark 3.17) and Theorem 2.9. □

For the remainder of this section we assume that H is a complex Hilbert space. In order to formulate a conclusion concerning the generation of holomorphic semigroups we state the following definition. For an operator A in H we define the **numerical range** $\operatorname{num}(A) := \{(Ax \,|\, x)\,;\ x \in \operatorname{dom}(A),\ \|x\| = 1\}$. We call A **sectorial** (**of angle** θ) if there exists $\theta \in [0, \pi/2)$ such that $\operatorname{num}(A) \subseteq \{z \in \mathbb{C} \setminus \{0\}\,;\ |\operatorname{Arg} z| \leqslant \theta\} \cup \{0\}$, and we call A **m-sectorial** if additionally $\operatorname{ran}(I + A) = H$.

We note that our definition of 'sectorial' is slightly more restrictive than the one used in [Kat80; p. V.3.10]. Unhappily, the notation also conflicts with a notion introduced later in [PS93; Section 3] and which meanwhile is an important concept in the functional calculus for operators.

3.21 Remarks. Let A be an operator in H.

(a) Obviously A is accretive if and only if $\operatorname{num}(A) \subseteq [\operatorname{Re} \geqslant 0]$.

(b) We note that for any angle α one has $\operatorname{num}(e^{i\alpha}A) = e^{i\alpha} \operatorname{num}(A)$. Let $\theta \in (0, \pi/2]$. Then it follows from (a) that $e^{i\alpha}A$ is accretive for all $\alpha \in (-\theta, \theta)$ if and only if $\operatorname{num}(A) \subseteq \{z \in \mathbb{C} \setminus \{0\}\,;\ |\operatorname{Arg} z| \leqslant \pi/2 - \theta\} \cup \{0\}$.

We now draw a conclusion of the Lumer-Phillips theorem for generators of holomorphic semigroups which are contractive on a sector.

3.22 Theorem. *Let A be an operator in the complex Hilbert space H, and let $\theta \in (0, \pi/2]$. Then $-A$ generates a contractive holomorphic C_0-semigroup of angle θ if and only if A is m-sectorial of angle $\pi/2 - \theta$.*

Proof. For the necessity we note that the hypothesis implies that $-e^{i\alpha}A$ generates a contractive C_0-semigroup, and therefore by Theorem 3.18, $e^{i\alpha}A$ is accretive, for all $\alpha \in (-\theta, \theta)$. Therefore Remark 3.21(b) implies that $\operatorname{num}(A) \subseteq \{z \in \mathbb{C} \setminus \{0\}\,;\ |\operatorname{Arg} z| \leqslant$

$\pi/2 - \theta\} \cup \{0\}$. As $-A$ generates a contractive C_0-semigroup one has $\operatorname{ran}(I + A) = H$, by Theorem 2.7. This shows that A is m-sectorial of angle $\pi/2 - \theta$.

For the sufficiency we first note that Theorem 3.18 implies that $-A$ generates a contractive C_0-semigroup, and therefore $\Sigma_\theta \subseteq [\operatorname{Re} > 0] \subseteq \rho(-A)$. Let $\alpha \in (-\theta, \theta)$. From Remark 3.21(b) we know that $e^{i\alpha}A$ is accretive, and therefore

$$\left\|(e^{-i\alpha}\lambda I + A)^{-1}\right\| = \left\|(\lambda I + e^{i\alpha}A)^{-1}\right\| \leqslant \frac{1}{\lambda} \qquad (\lambda > 0),$$

by Lemma 3.16. This inequality can be rewritten as

$$\left\|(\lambda I + A)^{-1}\right\| \leqslant \frac{1}{|\lambda|} \qquad (\lambda \in \Sigma_\theta).$$

Applying Theorem 3.14 we obtain the assertion. □

3.23 Remark. In the context of generators of contractive C_0-semigroups in Banach spaces, one usually considers *dissipative* instead of *accretive* operators. An operator A is called **dissipative** if $-A$ is accretive. The reason we prefer using the notion of accretive operators is that they will arise naturally in the context of forms.

Notes

The equivalence of (i), (ii), (iii) in Theorem 3.2 is due to Dunford [Dun38; Theorem 76]. Natural as the setup and proof of Theorem 3.2 may seem, it is rather surprising that a further weakening is possible: f is holomorphic if f is locally bounded, and there exist a separating set $E \subseteq X'$ such that $x' \circ f$ is holomorphic for all $x \in E$. This generalisation is due to Grosse-Erdmann [Gro92] (see also [Gro04]); an elegant short proof, based on the Theorem of Banach-Krein-Šmulian, has been given in [AN00; Theorem 3.1]. Theorem 3.5 is also due to Dunford; see Hille [Hil39; footnote to Theorem 1] (see also [HP57; Theorem 3.10.1]).

The contents of Section 3.2 are standard and can be found in most treatises on C_0-semigroups. The statement of Theorem 3.14 can be considered as standard. Its proof, however, deviates from the standard proofs. Classically, the generation theorem for holomorphic semigroups is treated by defining the semigroup as a contour integral. We refer to the literature for this kind of proof. Our proof follows the approach presented in [AEH97; Section 4] and [AE12; Section 2]. We note that the characterisation stated at the end of Remark 3.15(b) can also be proved without contour integrals; see [AEH97; Theorem 4.3]. Our treatment of the Lumer-Phillips theorem in Section 3.4 is restricted to Hilbert space. This restriction simplifies the treatment significantly in comparison to the treatment in Banach spaces. We refer to the literature for the general treatment; in our lectures we will only need the Hilbert space case.

Exercises

3.1 Define the subspace E of $c_0' = \ell_1$ by

$$E := \Big\{x = (x_n) \in \ell_1\,;\ \sum_{n=1}^{\infty} x_n = 0\Big\}.$$

Show that E is almost norming, but not norming for c_0.

3.2 Let $\Omega \subseteq \mathbb{C}$ an open set. Let (f_n) be a bounded sequence of bounded holomorphic functions $f_n\colon \Omega \to \mathbb{C}$ (i.e., $\sup_{z\in\Omega,\, n\in\mathbb{N}} |f_n(z)| < \infty$). Define the function $f\colon \Omega \to \ell_\infty$ by $f(z) := (f_n(z))_{n\in\mathbb{N}}$.

(a) Show that f is holomorphic. (Hint: Find a suitable norming subset of ℓ_∞, and use Theorem 3.2.)

(b) Assume additionally that $f_\infty(z) := \lim_{n\to\infty} f_n(z)$ exists for all $z \in \Omega$. Show that then $f\colon \Omega \to c$ is holomorphic (where c denotes the subspace of ℓ_∞ consisting of the convergent sequences), and that f_∞ is holomorphic. (Hint for the last part: The functional $c \ni (x_n) \mapsto \lim x_n \in \mathbb{C}$ belongs to c'.)

(Comment: Continuing this approach one can also show the classical result that the convergence $f_n(z) \to f_\infty(z)$ is locally uniform. The whole setup could also start with X-valued functions, thereby finally yielding an alternative proof of Theorem 3.7.)

3.3 Let X be a complex Banach space, T a holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$ on X.

(a) Show that, for each $\theta' \in (0, \theta)$, there exists $\delta > 0$ such that

$$\sup_{z\in\Sigma_{\theta'},\, \operatorname{Re} z\leqslant\delta} \|T(z)\| < \infty.$$

(b) Show Remark 3.9.

(c) Show that the estimate in Remark 3.9 can be written equivalently as follows: For each $\theta' \in (0, \theta)$, there exist $M'' \geqslant 1$, $\omega'' \in \mathbb{R}$ such that

$$\|T(z)\| \leqslant M''\mathrm{e}^{\omega''|z|} \qquad (z \in \Sigma_{\theta'}).$$

3.4 Let T be a bounded holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$.

(a) Show that there exists a strongly continuous extension (also called T) to the closure of $\Sigma_{\theta,0}$. (Hint: Show first that the extension can be defined on $\bigcup_{z\in\Sigma_\theta} \operatorname{ran}\big(T(z)\big)$.)

(b) Show that $T_{\pm\theta}$, defined by $T_{\pm\theta}(t) := T\big(\mathrm{e}^{\pm\mathrm{i}\theta}t\big)$ $(t \geqslant 0)$, are C_0-semigroups (the **boundary semigroups** of T).

(c) If $\theta = \pi/2$, then show that $T_{\pi/2}(t) := T(\mathrm{i}t)$ $(t \in \mathbb{R})$ defines a C_0-group $T_{\pi/2}$ (the **boundary group** of T).

References

[AE12] W. Arendt and A. F. M. ter Elst: From forms to semigroups. *Spectral theory, mathematical system theory, evolution equations, differential and difference equations.* Oper. Theory: Adv. Appl 221. Birkhäuser/Springer, Basel, 2012, 47–69.

[AEH97] W. Arendt, O. El Mennaoui, and M. Hieber: Boundary values of holomorphic semigroups. *Proc. Amer. Math. Soc.* **125**, 635–647 (1997).

[AN00] W. Arendt and N. Nikolski: Vector-valued holomorphic functions revisited. *Math. Z.* **234**, 777–805 (2000).

[Dun38] N. Dunford: Uniformity in linear spaces. *Trans. Amer. Math. Soc.* **44**, 305–356 (1938).

[EN00] K.-J. Engel and R. Nagel: *One-Parameter Semigroups for Linear Evolution Equations*. Springer, New York, 2000.

[Gro04] K.-G. Grosse-Erdmann: A weak critierion for vector-valued holomorphy. *Math. Proc. Cambridge Philos. Soc.* **136**, 399–411 (2004).

[Gro92] K.-G. Grosse-Erdmann: The Borel-Okada theorem revisited. Habilitationsschrift. Hagen: Fernuniversität Hagen, 1992.

[Hil39] E. Hille: Notes on linear transformations. II. Analyticity of semi-groups. *Ann. Math. (2)* **40**, 1–47 (1939).

[HP57] E. Hille and R. S. Phillips: *Functional Analysis and Semi-Groups*. Revised edition. Amer. Math. Soc., Providence, Rh. I., 1957.

[Kat80] T. Kato: *Perturbation Theory for Linear Operators*. Corrected Printing of the Second edition. Springer, Berlin, 1980.

[Phi59] R. S. Phillips: Dissipative operators and hyperbolic systems of partial differential equations. *Trans. Amer. Math. Soc.* **91**, 193–254 (1959).

[PS93] J. Prüss and H. Sohr: Imaginary powers of elliptic second order differential operators in L^p-spaces. *Hiroshima Math. J.* **23**, 161–192 (1993).