Structured eigenvalue problems and passivity of descriptor systems

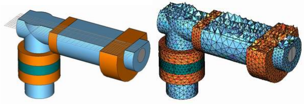
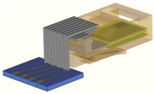
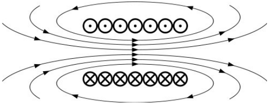
Tobias Brüll

DFG Research Center MATHEON
Mathematics for key technologies

- MATHEON Project D2 : Passivation of linear time invariant systems arising in circuit simulation and electric field computation
- Modeling of passive electric components (such as antennas, network connectors, or cables) leads to linear descriptor systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

- Due to model reduction techniques these systems are usually not passive any more (i.e., they generate energy in some sense).

With $E, A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $C \in \mathbb{C}^{p,n}$, $D \in \mathbb{C}^{p,m}$ and $x(t) \in \mathbb{C}^n$, $u(t) \in \mathbb{C}^m$, $y(t) \in \mathbb{C}^p$ the system of equations

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0, \end{aligned}$$

for $t \in \mathbb{R}$ is called a *linear continuous-time system*.
(u, x, y) is called a trajectory of the system, if the functions u, x, and y fulfill the system equations for some x_0.

With $E, A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $C \in \mathbb{C}^{p,n}$, $D \in \mathbb{C}^{p,m}$ and $x(k) \in \mathbb{C}^n$, $u(k) \in \mathbb{C}^m$, $y(k) \in \mathbb{C}^p$ the system of equations

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \\ x(0) &= x_0, \end{aligned}$$

for $k \in \mathbb{Z}$ is called a *linear discrete-time system*.
(u, x, y) is called a trajectory of the system, if the sequences u, x, and y fulfill the system equations for some x_0.

The notion of energy: The supply function

We measure the energy supplied to a linear system with the help of the *quadratic supply function* $s : \mathbb{C}^{m,p} \to \mathbb{R}$ defined by

$$s(u,y) := \begin{bmatrix} y \\ u \end{bmatrix}^* \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix},$$

where $R = R^* \in \mathbb{C}^{m,m}$, $Q = Q^* \in \mathbb{C}^{p,p}$ and $S \in \mathbb{C}^{m,p}$.

A linear continuous-time system is said to be *dissipative* with respect to the supply function s if there exists a continuous, nonnegative function $\Theta : \mathbb{C}^n \to \mathbb{R}$, called the *storage function*, such that the *dissipation inequality*

$$\Theta(x(t_1)) - \Theta(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t))dt$$

holds for all $t_0 \leq t_1$, $t_0, t_1 \in \mathbb{R}$ and all trajectories (u, x, y) of the continuous-time system.

A linear discrete-time system is said to be *dissipative* with respect to the supply function s if there exists a continuous, nonnegative function $\Theta : \mathbb{C}^n \to \mathbb{R}$, called the *storage function*, such that the *dissipation inequality*

$$\Theta(x(k_1)) - \Theta(x(k_0)) \leq \sum_{k=k_0}^{k_1-1} s(u(k), y(k))$$

holds for all $k_0 \leq k_1$, $k_0, k_1 \in \mathbb{Z}$ and all trajectories (u, x, y) of the discrete-time system.

If a system is dissipative with respect to the quadratic supply function

$$s(u, y) := \frac{1}{2} \begin{bmatrix} u \\ y \end{bmatrix}^* \begin{bmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \frac{(u^* y + y^* u)}{2} = \Re(u^* y),$$

it is also called *passive*. (This implies that $m = p$)
This supply function plays an important role in the analysis of electric circuits. There the inputs u are the currents at the current sources and the outputs y are the voltages at the current sources (or the other way round). Thus, the quantity $\Re(u^* y)$ measures the electric energy supplied to the system in one point of time.

Theorem (Brüll 2007)

Let the continuous-time system be dissipative with respect to a quadratic supply function. Assume further that $\lambda E - A$ *is regular, i.e.,* $\det(\lambda E - A) \neq 0$. *Then the corresponding transfer function defined through*

$$G(z) = C(zE - A)^{-1}B + D$$

fulfills the inequality

$$\Pi(z) := \begin{bmatrix} G^*(z) & I \end{bmatrix} \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} G(z) \\ I \end{bmatrix} \geq 0,$$

for all $z \in \mathbb{C} \setminus \sigma_f(\lambda E - A)$ *with* $\Re z \geq 0$, *where the finite spectrum is defined as* $\sigma_f(\lambda E - A) := \{\lambda \in \mathbb{C} : \det(\lambda E - A) = 0\}$.

Proof in the continuous-time case

For all $v \in \mathbb{C}^m$ and all $z \in \mathbb{C} \setminus \sigma_f(\lambda E - A)$ the triple (u, x, y) defined through

$$\begin{aligned} u(t) &= ve^{zt}, \\ x(t) &= (zE - A)^{-1} Bve^{zt}, \\ y(t) &= (C(zE - A)^{-1} B + D)ve^{zt} = G(z)u(t), \end{aligned}$$

is a trajectory of the continuous-time system. Evaluating the dissipation inequality along such trajectories with $\Re z > 0$ gives

$$\Theta(x(t_1)) - \Theta(x(t_0)) \leq v^* \Pi(z) v \int_{t_0}^{t_1} e^{2\Re zt} dt.$$

Choosing $t_1 = 0$ and performing the limit $t_0 \to -\infty$ then gives the assertion for $\Re z > 0$. For $\Re z = 0$ continuity can be used.

Theorem (Brüll 2007)

Let the discrete-time system be dissipative with respect to a quadratic supply function. Assume further that $\lambda E - A$ is regular. Then the corresponding transfer function defined through

$$G(z) = C(zE - A)^{-1}B + D$$

fulfills the inequality

$$\Pi(z) = \begin{bmatrix} G^*(z) & I \end{bmatrix} \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} G(z) \\ I \end{bmatrix} \geq 0,$$

for all $z \in \mathbb{C} \setminus \sigma_f(\lambda E - A)$ with $|z| \geq 1$, where the finite spectrum is defined as $\sigma_f(\lambda E - A) := \{\lambda \in \mathbb{C} : \det(\lambda E - A) = 0\}$.

Reversed frequency domain characterization

From the positive semi-definiteness of $\Pi(z)$

$$\Pi(z) = \begin{bmatrix} G^*(z) & I \end{bmatrix} \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} G(z) \\ I \end{bmatrix} \geq 0,$$

for all $z \in \mathbb{C} \setminus \sigma_f(\lambda E - A)$ with $\Re z \geq 0$ we can not always conclude that the system is dissipative. A counterexample with $n = 1, m = 1, p = 1, a \in \mathbb{R}$ is given by

$$\begin{aligned} \dot{x}(t) &= ax(t) + 0u(t) \\ y(t) &= x(t) \end{aligned} \qquad \text{with } s(u, y) := |u|^2 - |y|^2.$$

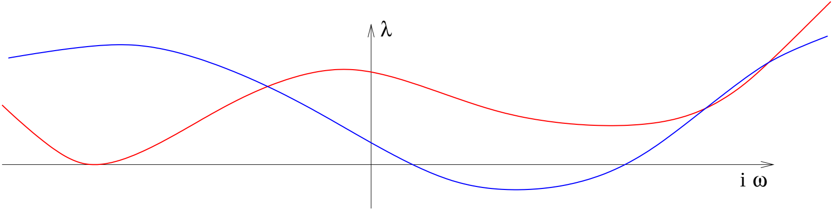
However, for controllable and observable, standard systems ($E = I$) with $Q \leq 0$ it is true (Anderson,Vongpanitlerd,1973), even if the inequality is fulfilled only for all $z = i\omega$ with $\omega \in \mathbb{R}$ and $i\omega$ not an eigenvalue of $\lambda E - A$.

How can we check if the Hermitian matrix-valued-function

$$\Pi(i\omega) = \begin{bmatrix} G^*(i\omega) & I \end{bmatrix} \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} \geq 0,$$

for all $\omega \in \mathbb{R}$?

See, e.g., (Schröder,Stykel,2007)!



A generalization of (Boyd et al.,1989)

Theorem (Brüll 2007)

Assume that $\lambda E - A$ *is a regular pencil and that* $i\omega$ *with* $\omega \in \mathbb{R}$ *is not an eigenvalue of* $\lambda E - A$. *Set*

$$\begin{aligned} Y &:= S^* + QD, \\ X_\mu &:= R + SD + D^*S^* + D^*QD - \mu I. \end{aligned}$$

Then, the Hermitian matrix $\Pi(i\omega)$ *has the eigenvalue* μ *if and only if the pencil*

$$P(\lambda) := \lambda \begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^* & C^*QC & C^*Y \\ B^* & Y^*C & X_\mu \end{bmatrix}$$

has $i\omega$ *as an eigenvalue. Since* $P(\lambda) = P(-\lambda)^*$ *we say that* P *is an even pencil.*

Theorem (Brüll 2007)

Assume that $\lambda E - A$ is a regular pencil and that $e^{i\omega}$ with $\omega \in \mathbb{R}$ is not an eigenvalue of $\lambda E - A$. Set

$$\begin{aligned} Y &:= S^* + QD, \\ X_\mu &:= R + SD + D^*S^* + D^*QD - \mu I. \end{aligned}$$

Then, the Hermitian matrix $\Pi(e^{i\omega})$ has the eigenvalue μ if and only if the matrix polynomial

$$P(\lambda) := \lambda^{\frac{1}{2}} \begin{bmatrix} 0 & -E & 0 \\ A^* & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & C^*QC & C^*Y \\ 0 & Y^*C & X_\mu \end{bmatrix} + \lambda^{-\frac{1}{2}} \begin{bmatrix} 0 & A & B \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has $e^{i\omega}$ as an eigenvalue. Since $P(\lambda) = P(\frac{1}{\lambda})^*$ we say that P is an palindromic matrix polynomial.

Theorem (Brüll 2007)

Under the same assumptions as on the previous slide we have the following.
If the Hermitian matrix $\Pi(e^{i\omega})$ *has the eigenvalue* μ *then the so called palindromic pencil*

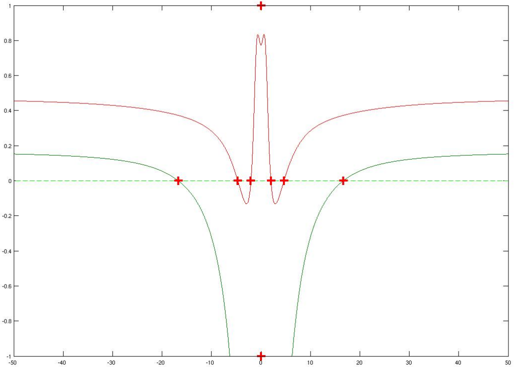
$$P(\lambda) := \lambda \begin{bmatrix} 0 & E & 0 \\ A^* & C^*QC & C^*Y \\ B^* & Y^*C & X_\mu \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ E^* & C^*QC & C^*Y \\ 0 & Y^*C & X_\mu \end{bmatrix}$$

has $e^{i\omega}$ *as an eigenvalue.*

If in addition $e^{i\omega} \neq 1$ *and the pencil* $P(\lambda)$ *has* $e^{i\omega}$ *as an eigenvalue, then we also know that* $\Pi(e^{i\omega})$ *has the eigenvalue* μ.

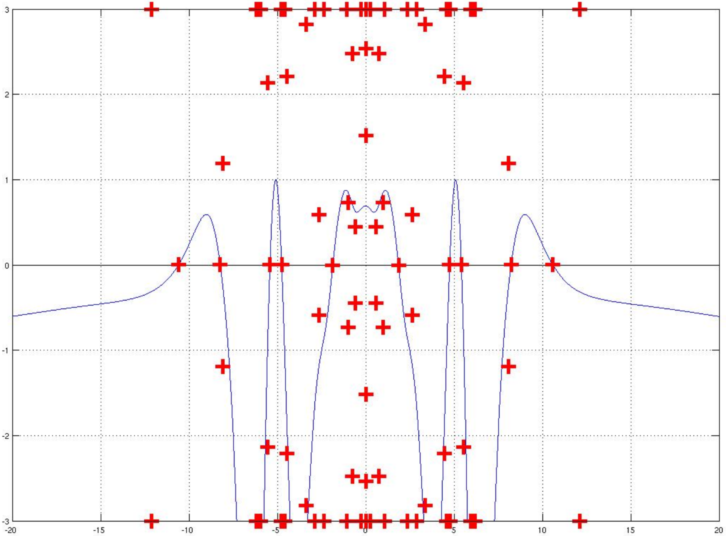
The pencil $P(\lambda)$ *always has the eigenvalue* 1.

Example 1

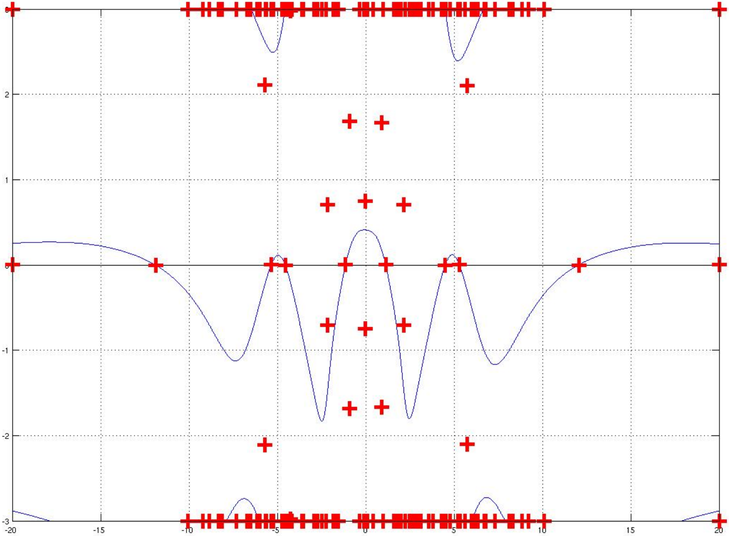


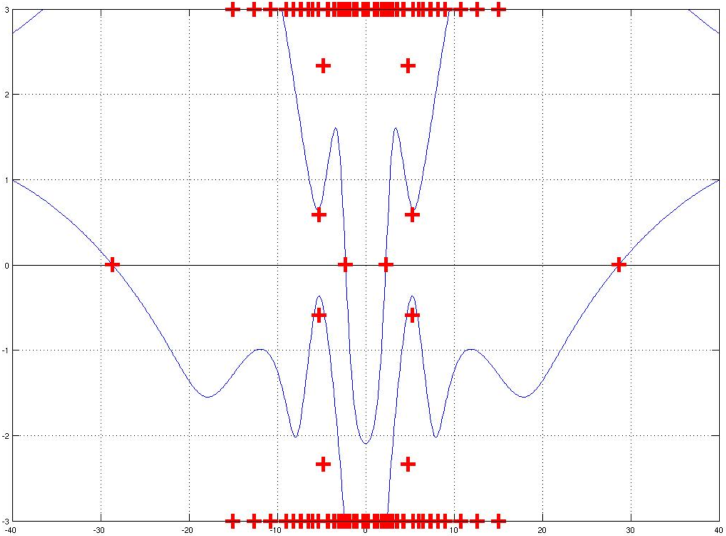
The abscissa (x-axis) represents the imaginary axis $i\omega$.
The curves represent the eigenvalues of some Popov function on the imaginary axis $\Pi(i\omega)$.
The red crosses represent the eigenvalues of the even pencil, with imaginary part plotted against the real part. They are projected into the visible area.

Example 2



Example 3

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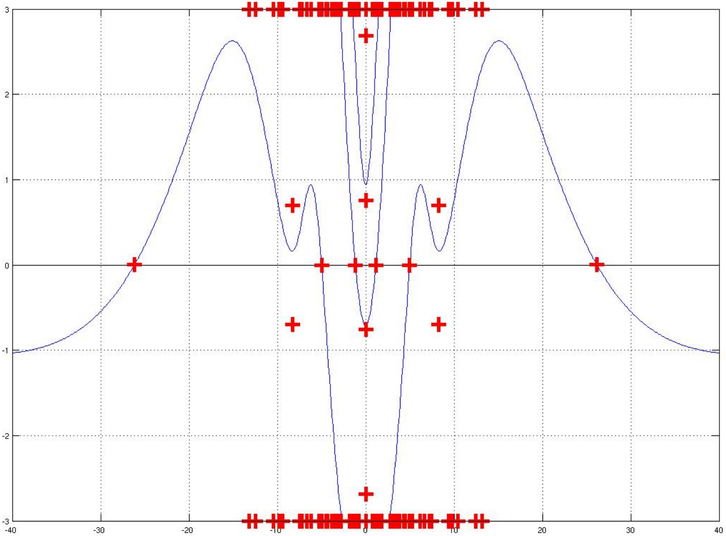
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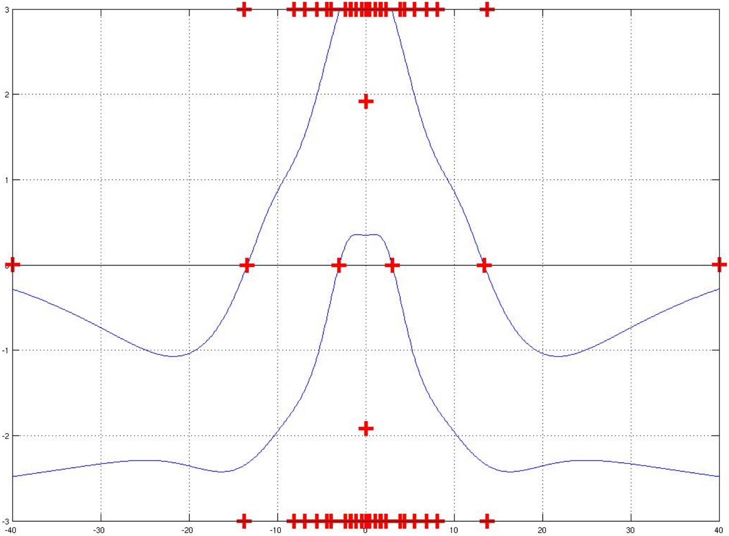
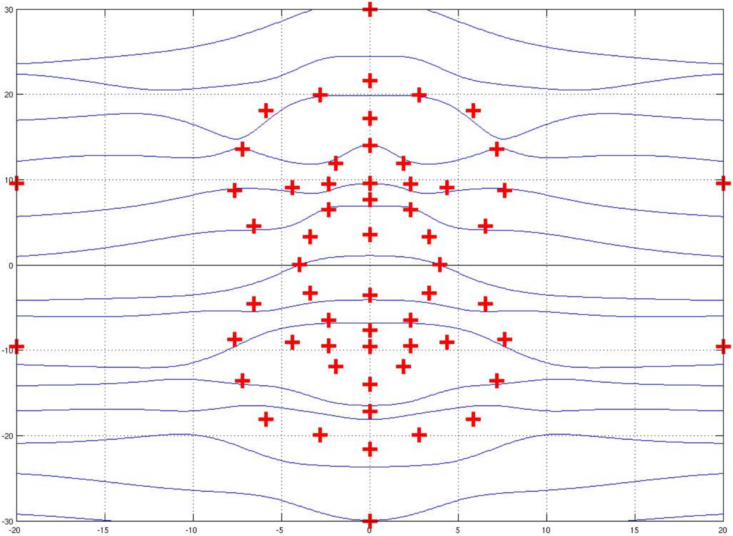
Example 6

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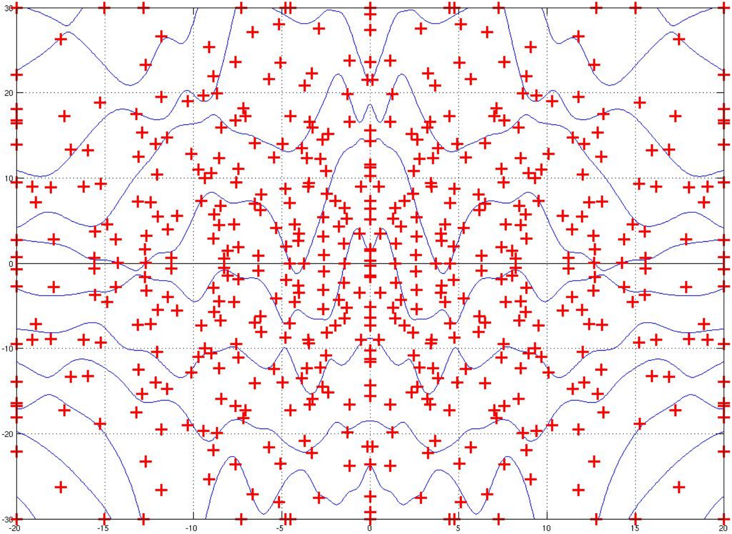
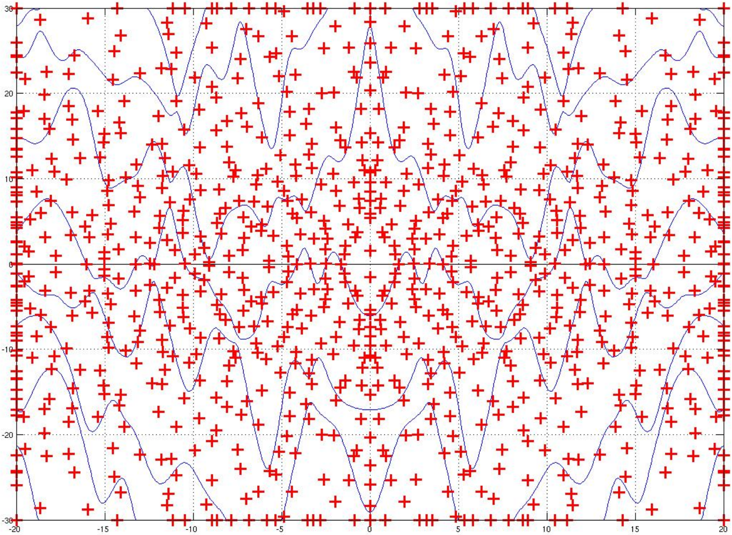
Example 8

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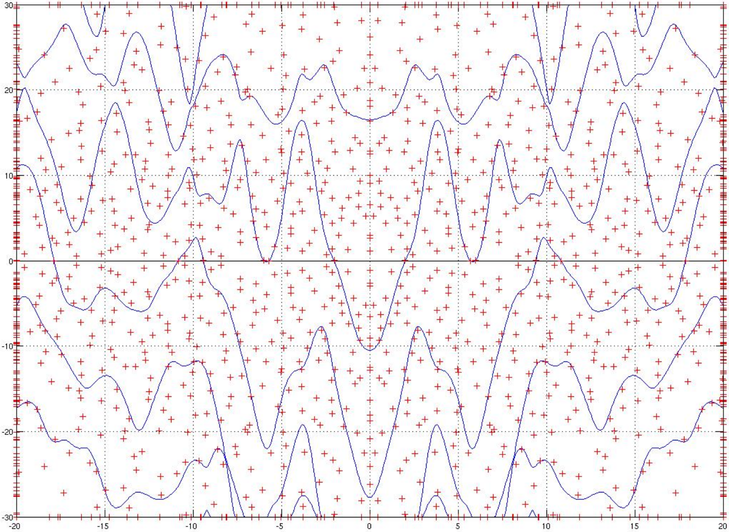
(n=500,l=p=50) Example 10



1. A property of dissipative systems that (under some assumptions) characterizes dissipativity can be checked with the help of the imaginary eigenvalues of an even pencil or the eigenvalues on the unit circle of a palindromic matrix polynomial.
2. For this check, structured eigenvalue solver seem to be the method of choice.
3. For descriptor systems it remains unsure if the dissipativity check ensures dissipativity.
4. For standard systems with $Q \nleq 0$ it remains unsure if the dissipativity check ensures dissipativity.

Thanks for your attention