I Can See Clearer Now – The Blur is Gone

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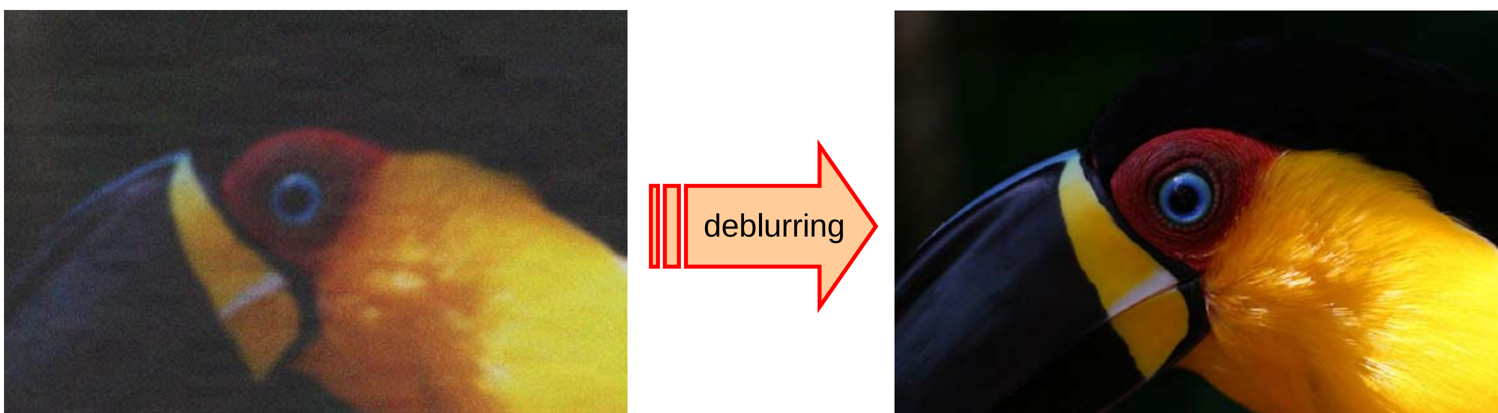
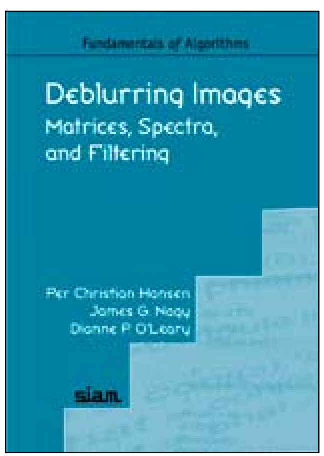


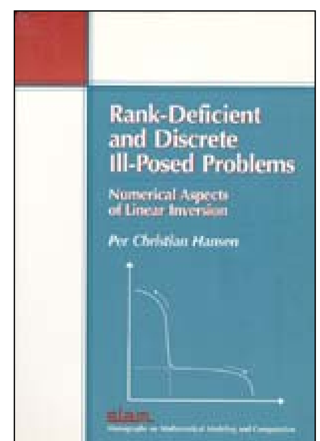
The Speaker

Per Christian Hansen

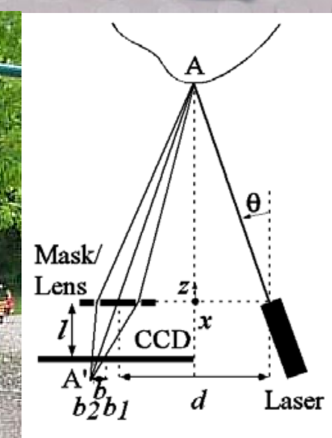
- Professor of Scientific Computing at DTU
- MSc EE 1982 – PhD Num. Anal. 1985 – Dr Techn 1996
- Key interests: large-scale computing & inverse problems

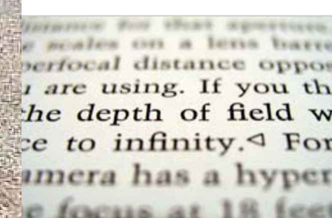
Author of two books on inverse problems:





Sources of Blurred Images





Some Types of Blur and Distortion

From the camera:

- the lens is out of focus,
- imperfections in the lens, and
- noise in the CCD and the analog/digital converter.

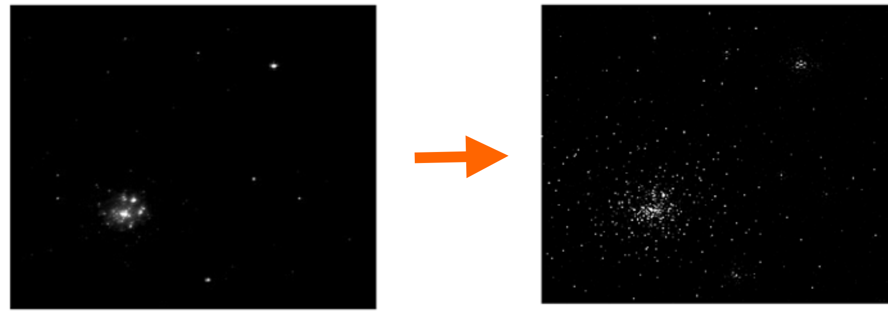
From the environments:

- motion of the object (or camera),
- fluctuations in the light's path (turbulence), and
- false light, cosmic radiation (in astronomical images).

Given a mathematical/statistical ***model*** of the blur and distortion, we can ***deblur*** the image and compute a sharper reconstruction.

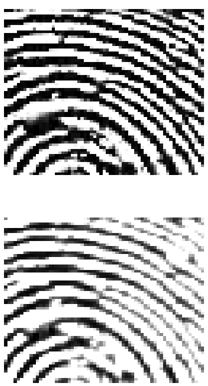
Some Applications of Deblurring

Astronomical imaging



Biometrics and surveillance

Image deblurring



Fingerprint restoration

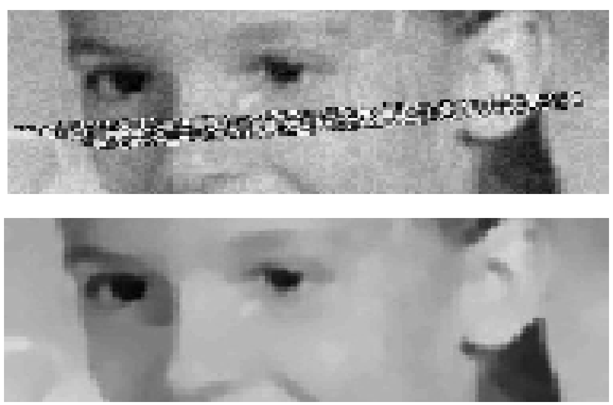
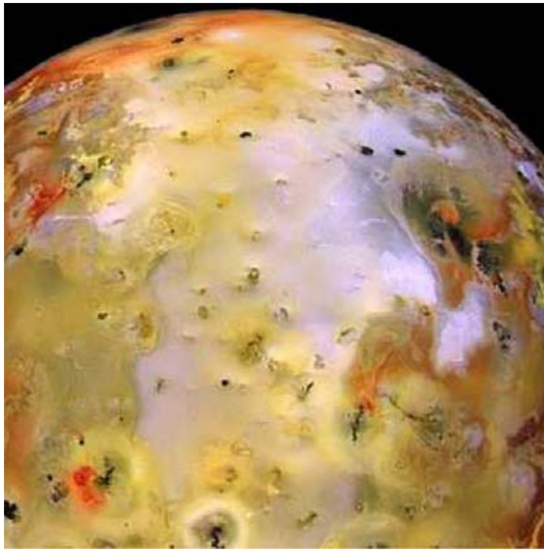


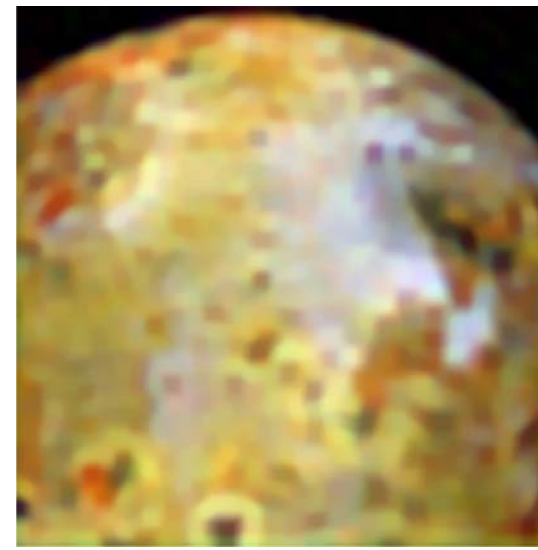
Image in-painting

Mathematics of Image Deblurring



blurring

deblurring



Io (moon of Jupiter)

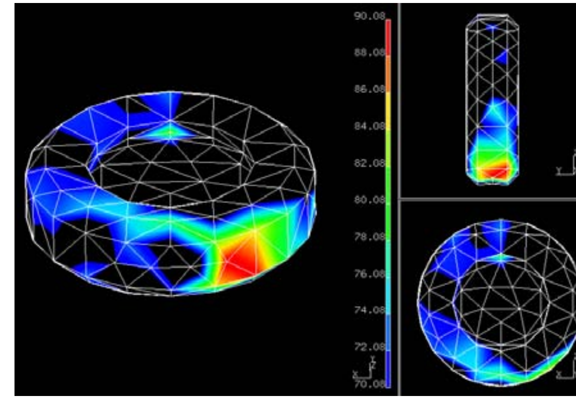
$$\int_\Omega K(\mathbf{s},\mathbf{t})\, f(\mathbf{t})\, d\Omega = g(\mathbf{s})$$

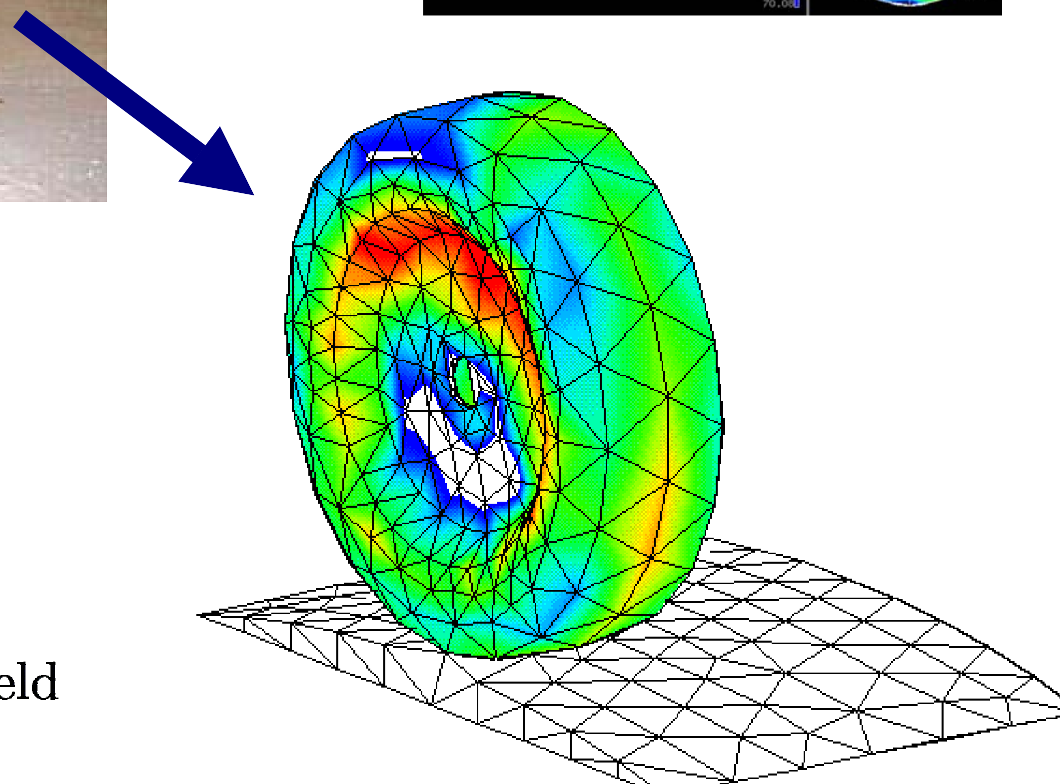
$$\begin{aligned} f(\mathbf{t}) &= \text{true scenery} \\ g(\mathbf{s}) &= \text{data (blurred image)} \\ K(\mathbf{s},\mathbf{t}) &= \text{point spread function} \end{aligned}$$

You cannot depend on your eyes when
your imagination is out of focus

– *Mark Twain*

Same Problem: Inverse Acoustics

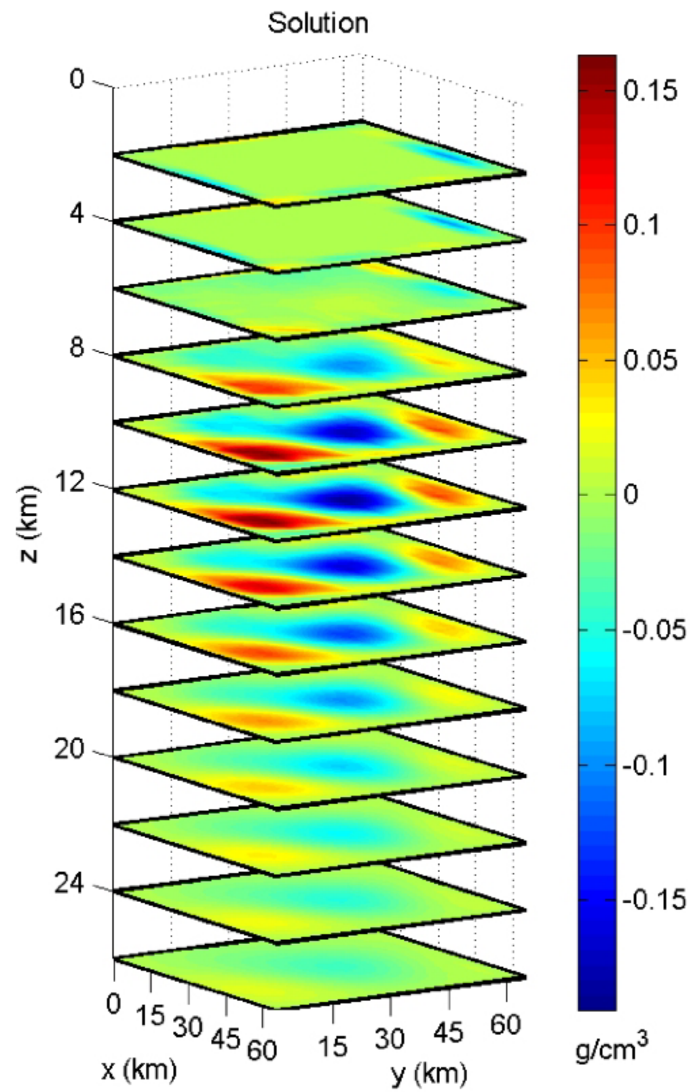




$$\int_{\Omega} K(\mathbf{s},\mathbf{t})\, f(\mathbf{t})\, d\Omega = g(\mathbf{s})$$

$$\begin{aligned} f(\mathbf{t}) &= \text{surface velocity} \\ g(\mathbf{s}) &= \text{data (pressure)} \\ K(\mathbf{s},\mathbf{t}) &= \text{acoustic dipole field} \end{aligned}$$

Same Problem: Inverse Geomagnetism



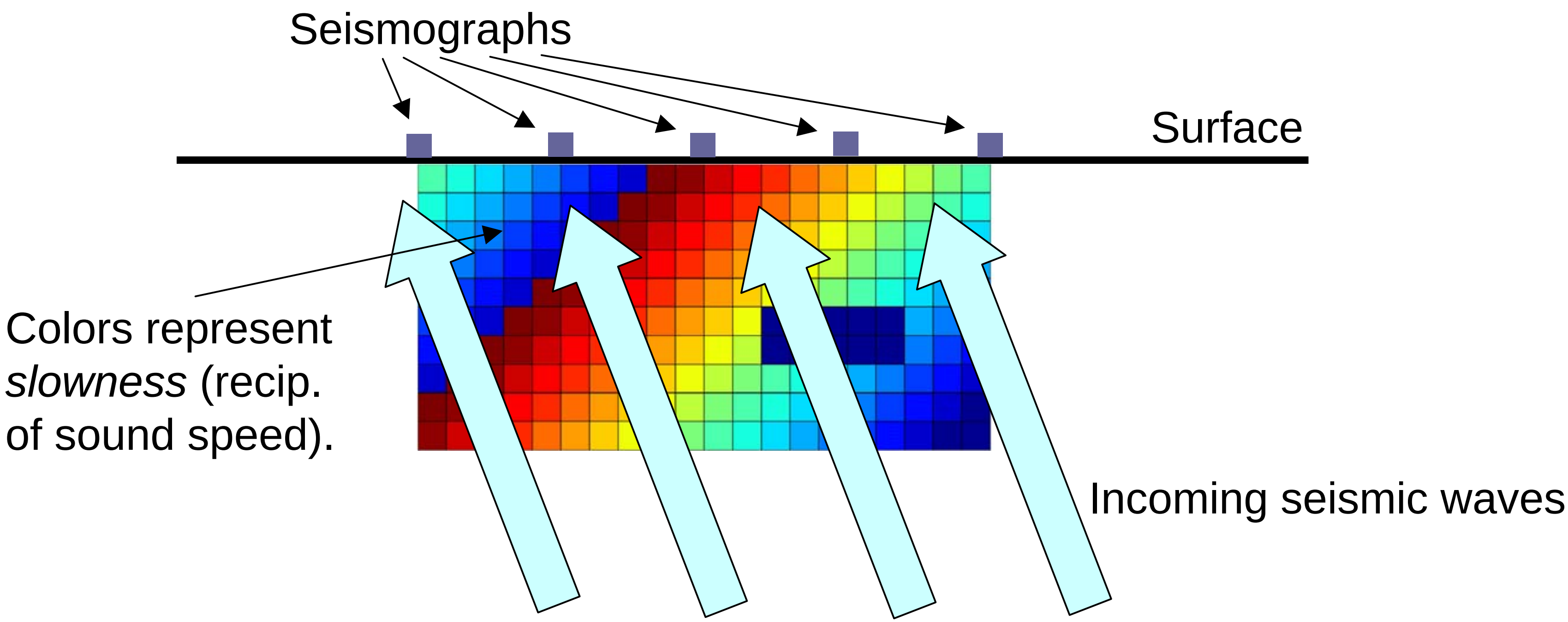
$$\int_\Omega K(\mathbf{s},\mathbf{t})\, f(\mathbf{t})\, d\Omega = g(\mathbf{s})$$

$$f(\mathbf{t}) = \text{magnetization}$$

$$g(\mathbf{s}) = \text{data (anomaly)}$$

$$K(\mathbf{s},\mathbf{t}) = \text{magnetic dipole field}$$

Same Problem: Seismic Tomography

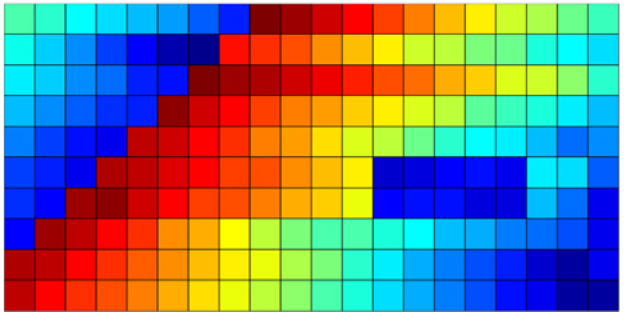


The ill-conditioned linear model

$$A\,x = b$$

gives the relationship between the material properties x and the measured travel times b.

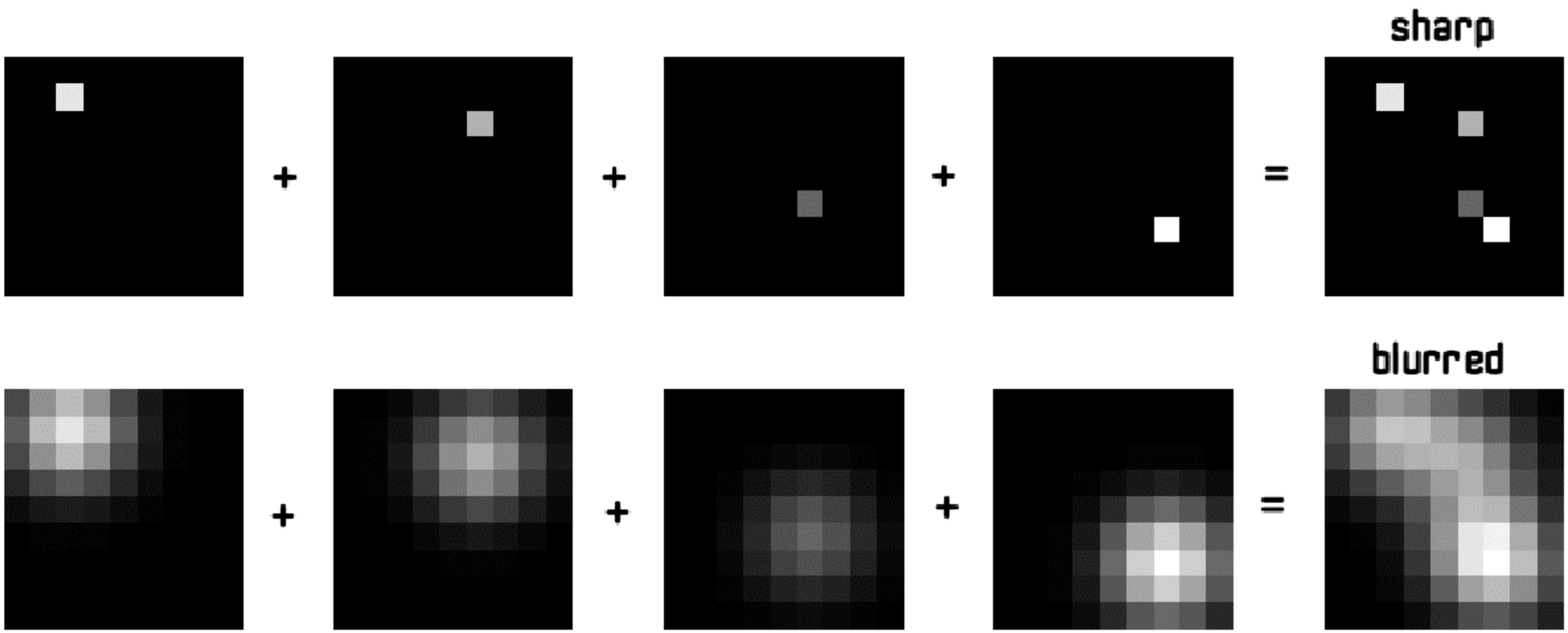
Reconstruction



The Point Spread Function

The point spread function is the image of a single bright pixel.

The blurred image is the sum of all the blurred pixels.



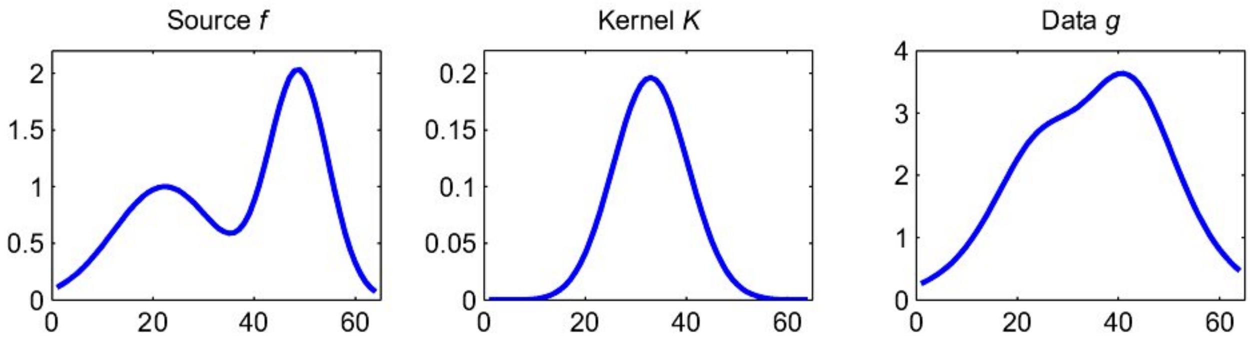
Discretization: Equations → Numbers

1-D model problem:

$$\int_0^1 K(s,t)\, f(t)\, dt = g(s)\ , \qquad 0 \le s \le 1.$$

Think of f as an unknown source, and g as the given data caused by f.

Think of K as a model for the point spread function.



Discretization yields

$$g(s_i) = h \sum_{j=1}^{n} K(s_i, t_j)\, f(t_j), \qquad i = 1, \ldots n$$

which is just a system of linear equations.

Discretization: Equations → Numbers

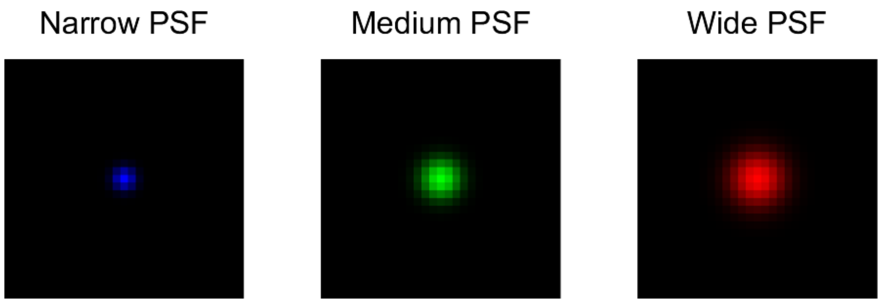
2-D model problem:

$$\int_{\Omega} K(\mathbf{s},\mathbf{t})\, f(\mathbf{t})\, d\Omega = g(\mathbf{s}) \ , \qquad \mathbf{s} \in \Omega.$$

Think of f as an unknown sharp image, and g as the given blured image (caused by blurring of f).

Think of K as a model for the point spread function.

Examples of
point spread functions

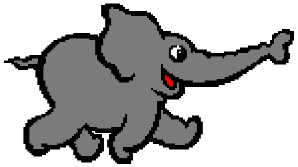


Discretization yields a LARGE system of linear equations

$$A\,x = b.$$

The matrix A is very ill conditioned, and therefore

$$\boxed{\textbf{Do not solve } A\,x = b \textbf{ !}}$$

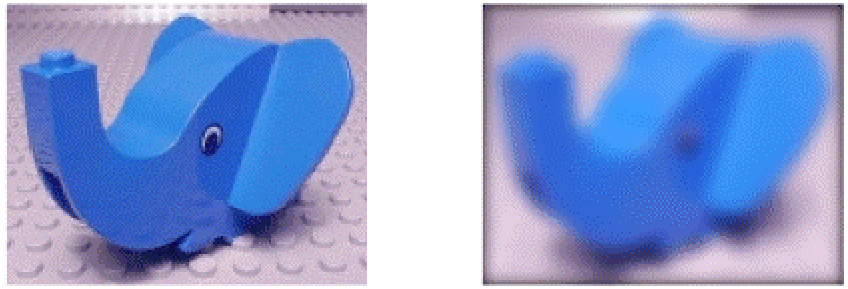


The Difficult Task of Image Deblurring

The underlying linear model:

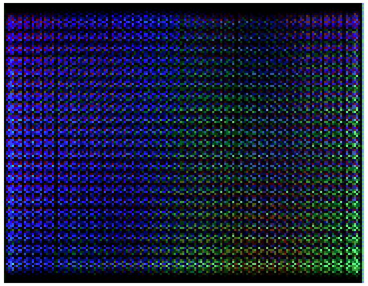
$$A\,x = b$$

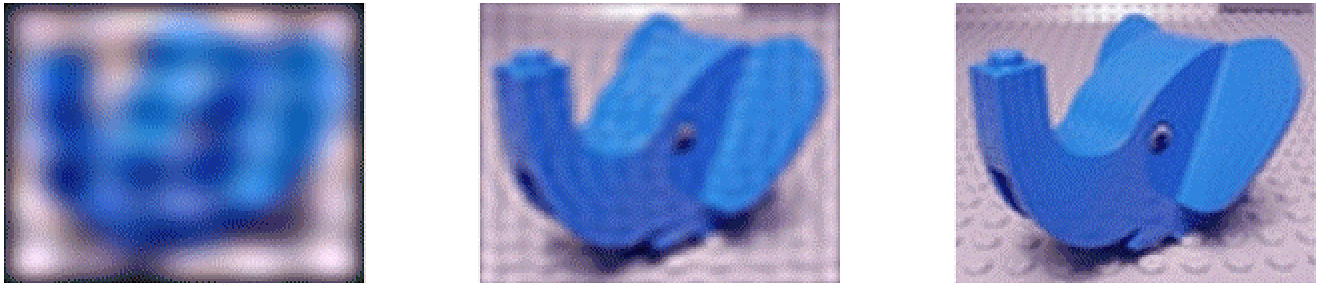
$$\mathrm{cond}(A) = \infty$$



The "naive" reconstruction:

$$x_{\mathrm{naive}} = A^{-1} b$$





Different attempts of reconstruction

Inverse Problems

Inverse problems are examples of ill-posed problems:

- the solution may not exist,
- the solution may not be unique, or
- the solution may not depend continuously on data.

Example: the world's simplest ill-posed problem: $\boldsymbol{x_1 + x_2 = 1}$.

The linear systems of equations associated with discretizations of linear inverse problems are effectively *underdetermined* – even if the system $A\,x = b$ is square or overdetermined.

Structured Matrices

The matrix *A* in image deblurring problems is often **structured**.

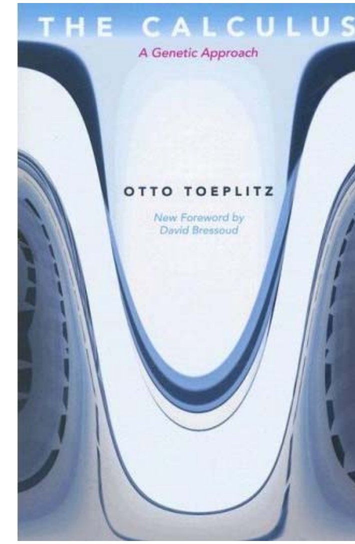
Typically, it is BTTB = block Toeplitz with Toeplitz blocks.

```
>> kron(T,T)

ans =

     9     6     3     6     4     2     3     2     1
     6     9     6     4     6     4     2     3     2
     3     6     9     2     4     6     1     2     3
     6     4     2     9     6     3     6     4     2
     4     6     4     6     9     6     4     6     4
     2     4     6     3     6     9     2     4     6
     3     2     1     6     4     2     9     6     3
     2     3     2     4     6     4     6     9     6
     1     2     3     2     4     6     3     6     9
```

Professor Toeplitz and his Matrix

Otto Toeplitz, 1881 – 1940.

Worked on linear and quadratic forms.



Given a stationary time series

$$x_1, x_2, x_3, x_4, \ldots$$

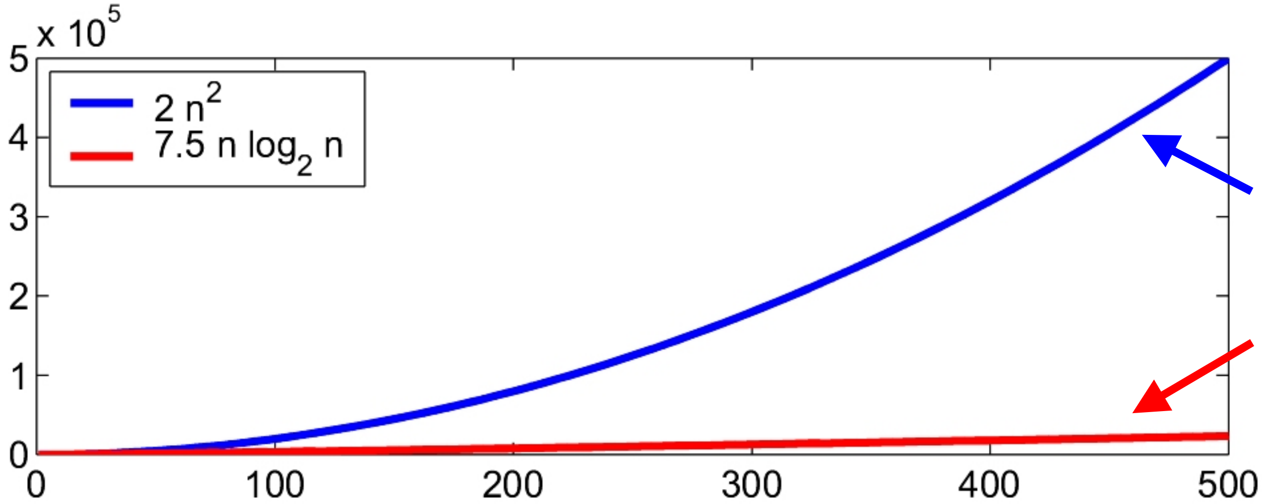
the autocorrelation matrix, with elements

$$c_{ij} = a_{i\text{-}j} = \mathrm{cov}(x_i\,,\, x_j\,)\ ,$$

is a Toeplitz matrix. →

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$

Utilization of Matrix Structure



Ordinary matrix-vector multiplication flop count.

Toeplitz matrix-vector multiplication flop count.

The FFT Algorithm

The "fathers" – published 1965.

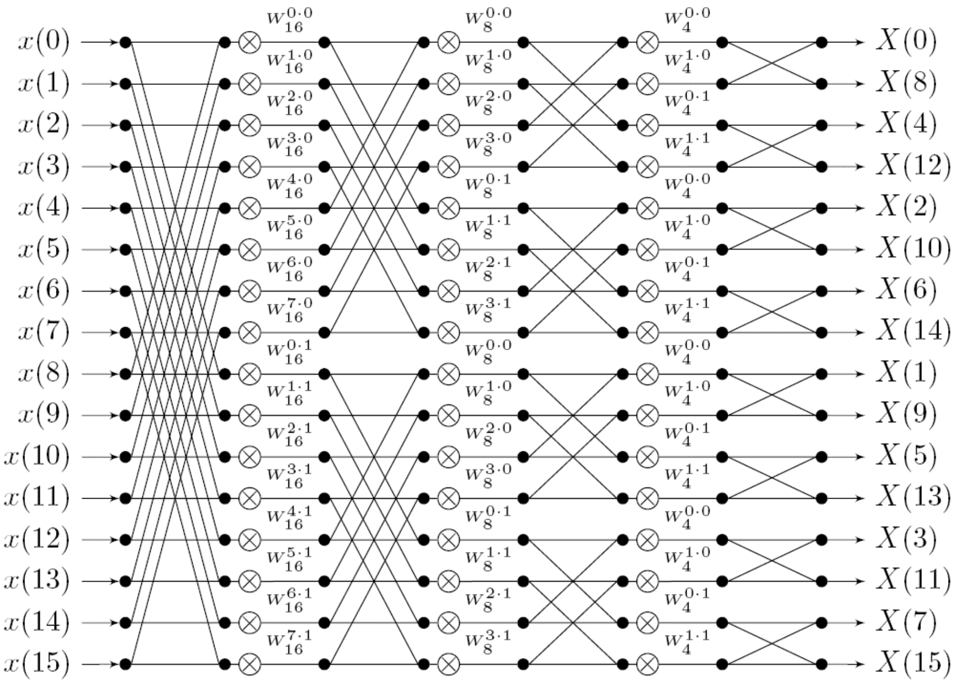

James Cooley


John Tukey

The definition

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} nk} \qquad k = 0, \dots, N-1.$$

The principle – O(n log(n)) complexity



The algorithm

```
function y = fft(x)
% FFT algorithm, n = power-of-2

n = length(x);
omega = exp(-2*pi*i/n);

if n > 2
   % Recursive divide and conquer.
   k = (0:n/2-1)';
   w = omega.^k;
   u = fft(x(1:2:n-1));
   v = w.*fft(x(2:2:n));
   y = [u+v; u-v];
else
   % The Fourier matrix.
   j = 0:n-1;
   k = j';
   F = omega.^(k*j);
   y = F*x;
end
```

FFT-Based Methods

We can immediately reconstruct the image via FFT computations:

- Periodic boundary conditions:

```
F = ifft2( fft2(G) ./ fft2(P) );
```

- with Wiener filtering

```
S = fft2(P);  % Eigenvalues.
F = ifft2( (S./(S.^2 + lambda) .* fft2(G) )
```

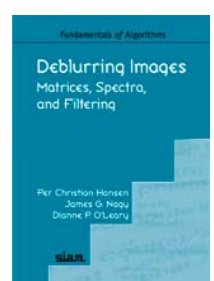
- Reflexive boundary conditions (and symmetric PSF):

```
S = dct2(P) ./ dct2(e1);  % Eigenvalues.
F = idct2( dct2(G) ./ S )
```

- with filtering

```
Q = dct2(P) ./ dct2(e1);  % Eigenvalues.
F = idct2( (S./(S.^2 + lambda) .* dct2(G) )
```

For details, see the book



Regularizing Iterations

For more general problems, we can apply Conjugate Gradients to the normal equations for the least squares problem

$$\min \|A\,x - b\|_2 \ .$$

This algorithm, called CGLS, produces a sequence of iterates $x^{(k)}$ which solve

$$\min \|A\,x - b\|_2 \qquad \text{subject to} \qquad x \in \mathcal{S}_k \ ,$$

where $\mathcal{S}_k$ is the k-dimensional Krylov subspace

$$\mathcal{S}_k = \text{span}\{A^T b\,,\ A^T A\,A^T b\,,\ (A^T A)^2 A^T b\,,\ \ldots\} \ .$$

These methods are referred to as *regularizing iterations*.

Iterative methods are based on multiplications with A and A^T (blurring).

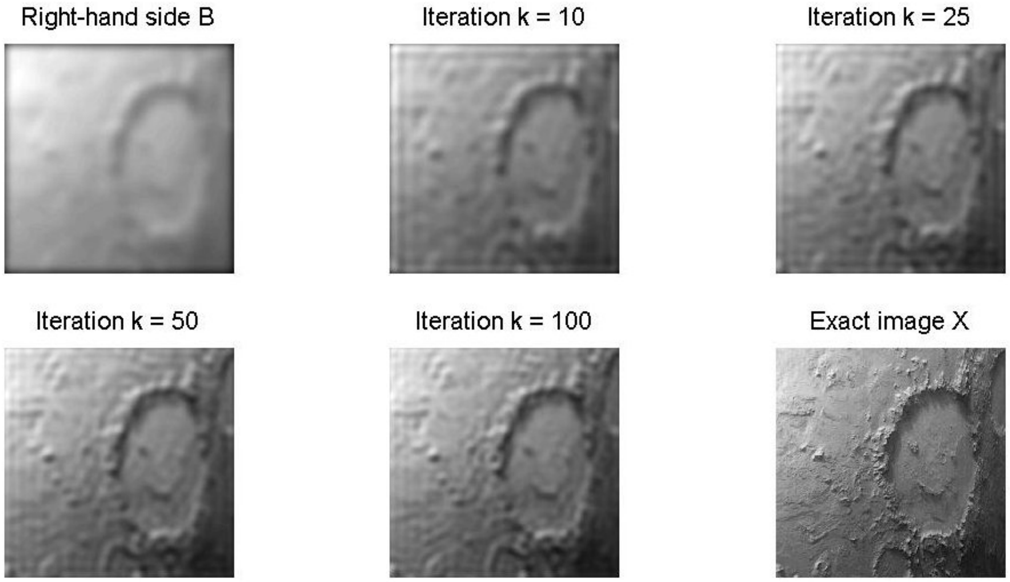
How come repeated blurings can lead to reconstruction?

$\rightarrow$ CGLS constructs a polynomial approximation to $(A^T A + \lambda^2 I)^{-1} A^T$.

Krylov Signal Subspaces

Facts about the Krylov signal subspace $\mathcal{S}_k = \mathcal{K}_k(A^T A, A^T b)$:

- b is rich in the directions of the principal left singular vectors.
- $\mathcal{S}_k$ must minimize $\|A\,x - b\|_2$ subject to $x \in \mathcal{S}_k$.
- Hence the Ritz polynomial must have zeros near large σ_i^2.
- CGLS iterate $x^{(k)}$ is rich in the principal right singular vectors.
- The iteration number k therefore tends to act as a regularization parameter.
- The Krylov subspace $\mathcal{S}_k$ provides a natural basis for regularized solutions.



Smiley Crater, Mars

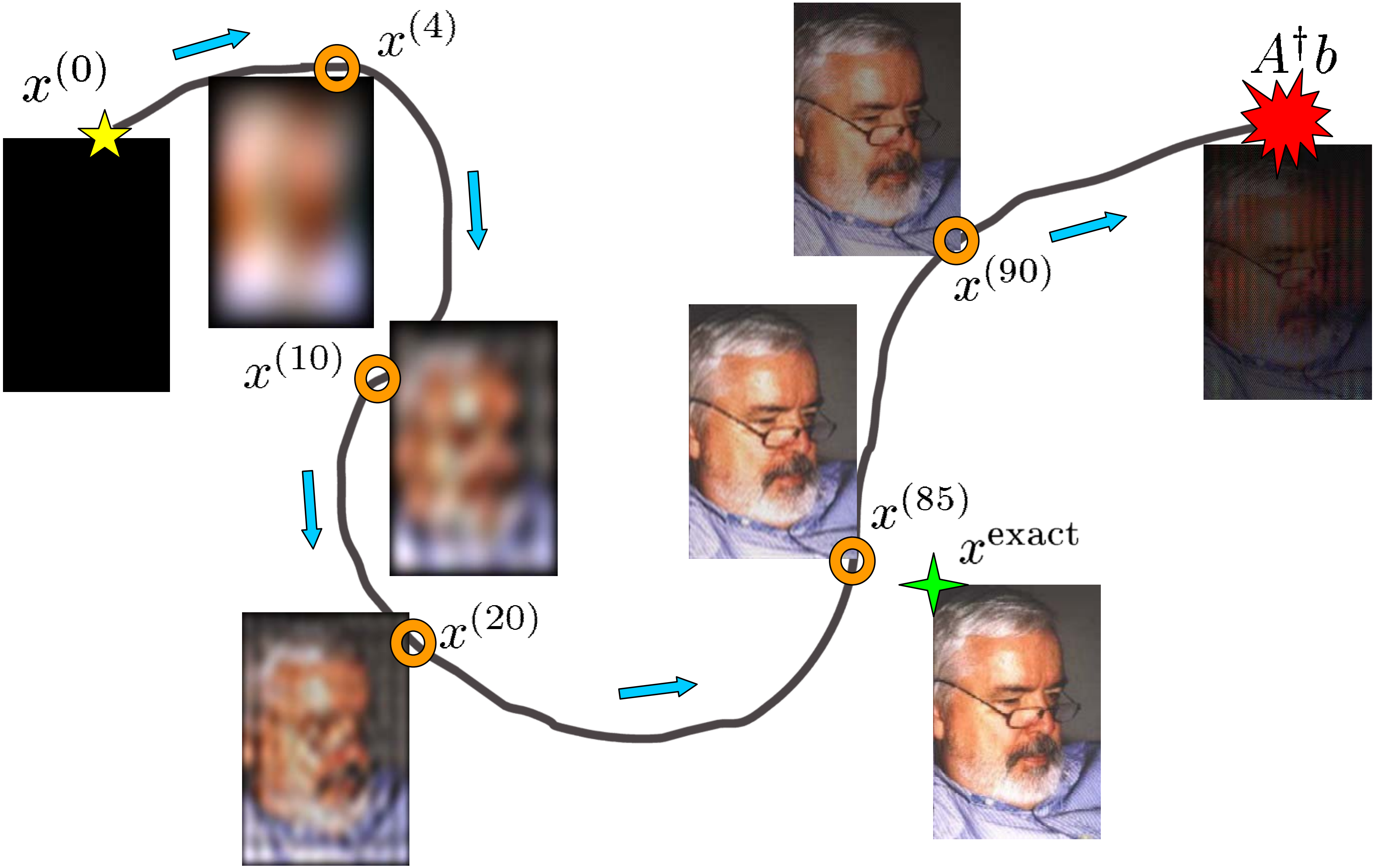
Top 10 Algorithms

J.J. Dongarra, F. Sullivan et al., *The Top 10 Algorithms*, IEEE Computing in Science and Engineering, 2 (2000), pp. 22-79.

1946: The Monte Carlo method (Metropolis Algorithm).
1947: The Simplex Method for Linear Programming.
1950: Krylov Subspace Methods (CG, CGLS, Arnoldi, etc.).
1951: Decomposition Approach to matrix computations.
1957: The Fortran Optimizing Compiler.
1961: The QR Algorithm for computing eigenvalues and –vectors.
1962: The Quicksort Algorithm.
1965: The Fast Fourier Transform algorithm.
1977: The Integer Relation Detection Algorithm.
1987: The Fast Multipole Algorithm for *N*-body simulations.

Key algorithms in image deblurring.

Semi-Convergence of CGLS



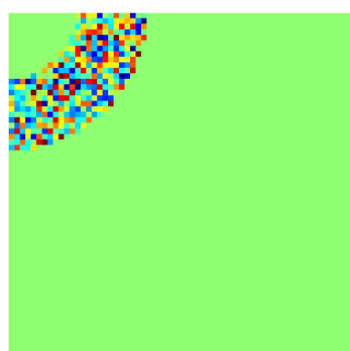
Progress of the Iterations

CGLS:
4, 10
and 25
iterations:

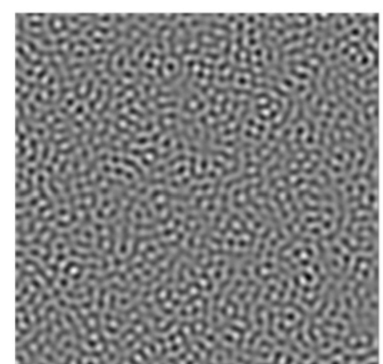
Initially, the image gets sharper – then ”freckles” start to appear.

The ”freckles” are band-pass filtered noise.

DCT spectrum



spatial domain



Away From 2-Norms

A more general Tikhonov formulation: $\min \left\{ \|A\,x - b\|_p^p + \lambda \, \|L\,x\|_q^q \right\}$.

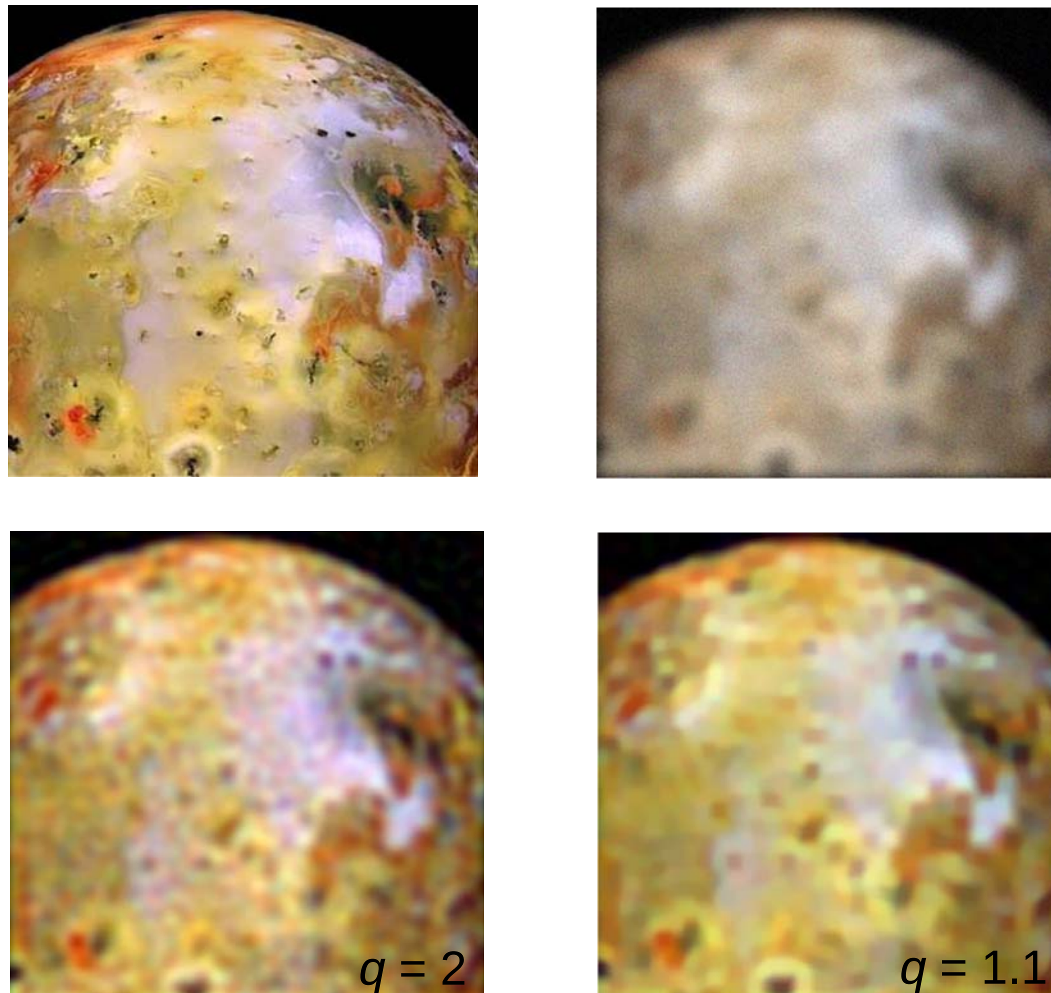
$\|A\,x - b\|_p \leftrightarrow$ noise statistics. $\|L\,x\|_q \leftrightarrow$ solution smoothness.

Important example: digital images often benefit from $q \approx 1$.

Reconstructions with $p = 2$ (Gauss. noise); $q = 2$ and $q = 1.1$.

What appears as details for $q = 2$ are "freckles."

Better reconstruction of edges for $q = 1.1$.

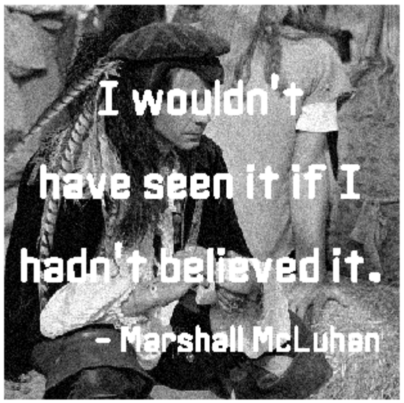


Io (moon of Saturn)

Total Variation In-Painting

$$\min \sum_{ij} \|D^{(ij)}\, x\|_2 \quad \text{s.t.} \quad \|x_I - b_I\|_2 \le \tau\sigma, \qquad \sigma = \text{noise level}$$

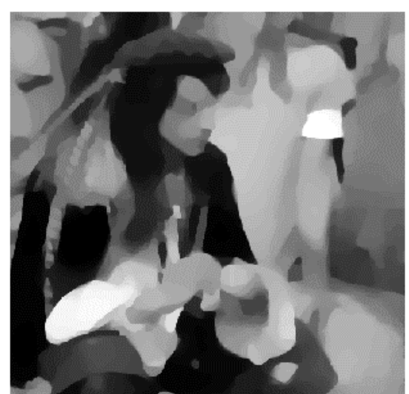
Noisy and corrupted image



TV inpainted image, $\tau = 0.85$

TV inpainted image, $\tau = 1.1$

TV inpainted image, $\tau = 1.6$

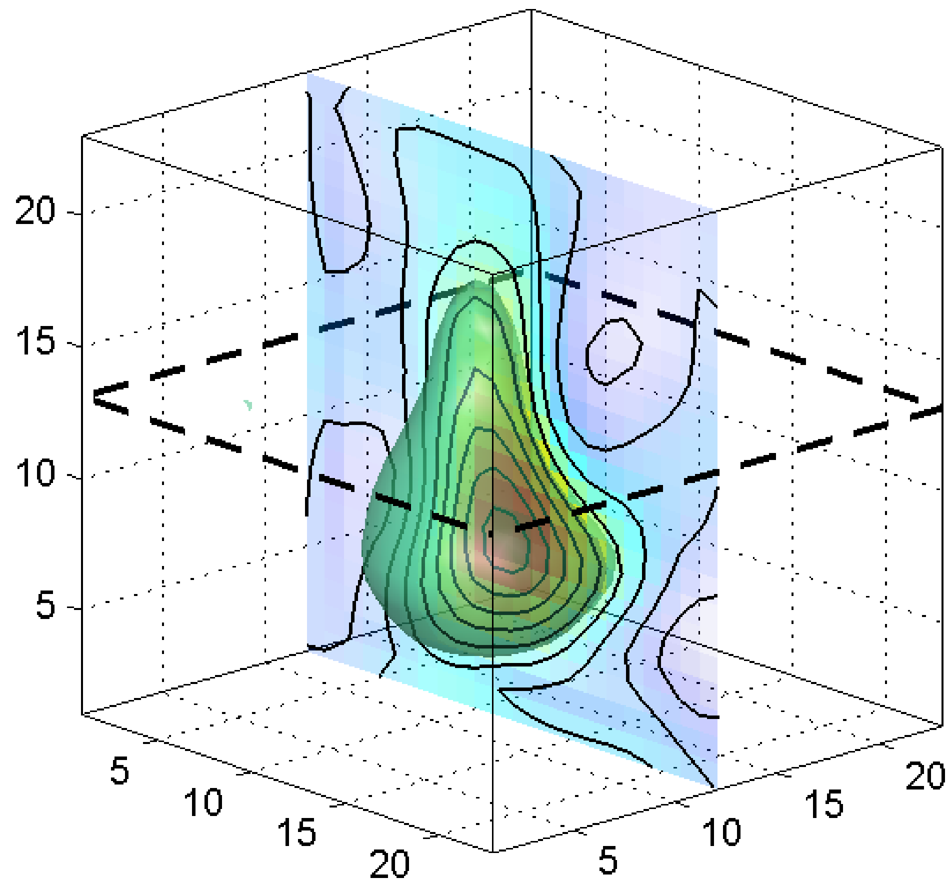


CSI: Computational Science in Imaging

Matlab and C software (working title: TV box) is almost finished.

3D Tomography in Crystallography

- Data:
  X-ray diffraction
- Reconstruction:
  orientation distribution function
- Smoothing norm:
  $|| \nabla^2 f ||^2$



Solution shows the distribution of orientations in an imperfect crystal.

Joint work with
Metals in 4D, Risø DTU