Regularization of nonlinear ill-posed problems by the exponential Euler method

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Outline

Nonlinear ill-posed problems

$F : \mathcal{D}(F) \subset X \longrightarrow Y$ continous and Fréchet differentiable
X, Y real Hilbert spaces,

$$\text{nonlinear problem: } F(x) = y$$

1. x_+ the x_0-minimum-norm solution
2. only perturbed data $y^\delta \in Y,\ \|y - y^\delta\|_Y \leq \delta$ available
3. problem ill-posed
   e.g. F compact and $\mathcal{D}(F)$ weakly closed
   $\rightsquigarrow$ regularization required

Assumptions

1. tangential cone condition

$$\left\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\right\| \leq \eta \left\|F(\tilde{x}) - F(x)\right\|, \quad \tilde{x}, x \in B_r(x_0)$$

2. source condition

$$x_0 - x_+ = J(x_+)^\gamma w, \qquad \|w\| \leq \rho, \quad J(x) := F'(x)^* F'(x)$$

3. local restriction of the derivative

$$F'(x) = R_x F'(x_+)$$
$$\|R_x - I\| \leq C_+ \|x - x_+\|, \qquad \forall x \in B_r(x_+)$$

4. w.l.o.g. $\|F'(x)\| \leq 1, \quad x \in B_r(x_0)$

e.g. Hanke, Neubauer & Scherzer (1995), Tautenhahn (1994)

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Asymptotic regularization

Showalter ode

$$\begin{aligned} \dot{u}(t) &= F'(u(t))^* \left(y^\delta - F(u(t))\right) \qquad t > 0 \\ u(0) &= x_0 \end{aligned}$$

stopping time t_* chosen by discrepancy principle ($\tau > 1$)

$$\|F(u(t_*)) - y^\delta\| \leq \tau\delta < \|F(u(t)) - y^\delta\|, \qquad 0 \leq t < t_*$$

analyzed by Tautenhahn (1994)

$$\begin{aligned} &u(t_*) \longrightarrow x_+, \\ &\|u(t_*) - x_+\| = O(\delta^{\frac{2\gamma}{2\gamma+1}}), \end{aligned} \qquad \text{as} \quad \delta \longrightarrow 0$$

Regularization with Runge-Kutta integrators

Ansatz:
Application of time integration schemes for solving

$$\dot{u}(t) = F'(u(t))^* \left(y^\delta - F(u(t))\right) \qquad t > 0$$

Böckmann & Pornsawad (2008):

1. use simplified Runge-Kutta methods
2. convergence under (severe) step size restrictions
3. numerical experiments show restrictions are due to analysis
4. no proof of optimal order

Explicit Euler

$$\dot{u}(t) = F'(u(t))^* \left(y^\delta - F(u(t))\right) \qquad t > 0$$

application of explicit Euler leads to

$$u_{n+1} = u_n + h_n F'(u_n)^* (y^\delta - F(u_n))$$

nonlinear Landweber iteration for $h_n = 1$

1. step size restriction
2. many (explicit) iterations
3. convergence and optimal order known
   Hanke, Neubauer & Scherzer (1995)

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Linearly implicit Euler

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application of implicit Euler leads to

$$u_{n+1} = u_n + h_n F'(u_{n+1})^* (y^\delta - F(u_{n+1}))$$

one Newton step with simplified Jacobian $-J(u_n) = -F'(u_n)^* F'(u_n)$ gives

$$u_{n+1} = u_n + h_n (I + h_n J(u_n))^{-1} F'(u_n)^* (y^\delta - F(u_n))$$

$\rightsquigarrow$ Newton type method with Tikhonov Philipps as inner regularization

Linearly implicit Euler

$$\begin{aligned} u_{n+1} &= u_n + h_n(I + h_n J(u_n))^{-1} F'(u_n)^* (y^\delta - F(u_n)) \\ J(u) &= F'(u)^* F'(u) \end{aligned}$$

Newton type method with Tikhonov Philipps regularization

1. large step sizes
2. one linear system in each timestep
3. convergence (rates) known
   Rieder (2001)
4. related to iteratively regularized Gauss–Newton
   Bakushinskii (1992)

Exponential Euler

$$\dot{u}(t) = F'(u(t))^* \left(y^\delta - F(u(t)) \right) \qquad t > 0$$

exponential Euler

$$\begin{aligned} u_{n+1} &= u_n + h_n \varphi(-h_n J(u_n)) F'(u_n)^* (y^\delta - F(u_n)) \\ \varphi(z) &= \frac{e^z - 1}{z} \end{aligned}$$

properties

1. solves linear problems exactly
2. "explicit" scheme
3. again: approximate Jacobian by $J(u) = F'(u)^* F'(u)$
4. equivalent to Newton method with regularized linear system

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Exponential Euler

regularization properties:

Theorem

Under suitable assumptions:

1. *exponential Euler iterates u_n converge to x_+*
2. *convergence rates are of optimal order*

Proofs: see next talk by Marlis Hochbruck

Recapitulation

nonlinear problem
$F(u) = y^\delta$

→ linearization →

linear problems
$F'(u_n)\Delta u_n = y^\delta - F(u_n)$

↓ Showalter

nonlinear ode

↓ linear regularization method

→ time integration →

nonlinear regularization method

Implementation

notation: $h = h_n$, $\quad J = J(u_n) = F'(u_n)^* F'(u_n) \quad h.psd.$

computation of $\varphi(-hJ)v$ by Krylov subspace methods

$$JV_m = V_m T_m + T_{m+1,m} v_{m+1} e_m^T$$

T_m tridiagonal, V_m orthogonal

$$\varphi(-hJ)v \approx V_m \varphi(-hT_m) e_1$$

Druskin, Khnizhnerman (1995)
Hochbruck, Lubich (1997)

Implementation

a posteriori error estimates: (van den Eshof & Hochbruck 2006)
approximation of the relative error in step m

$$\theta_m = \frac{\|w_m - w_{m-1}\|}{\|w_m\|}, \qquad w_m := \varphi(-hT_m)e_1$$

approximation of the error ϵ_m

$$\epsilon_m \lesssim \frac{\theta_m}{1-\theta_m}\,\|w_m\|$$

accuracy $\epsilon_m = O(\delta)$ sufficient

Implementation

properties

1. no linear systems
2. matrix free implementation
   only functions providing $F'(u_n)v$, $F'(u_n)^*v$ required
3. low accuracy $\rightsquigarrow$ low dimensional Krylov subspaces
4. fast computation of $\varphi(-hT_m)$ by Padé approximation

Example I: parameter identification in pde's (Rieder 1999)

reconstruction of x in

$$\begin{aligned} -\Delta x + yx &= v \quad \text{in } \Omega \\ x &= w \quad \text{on } \partial\Omega \end{aligned}$$

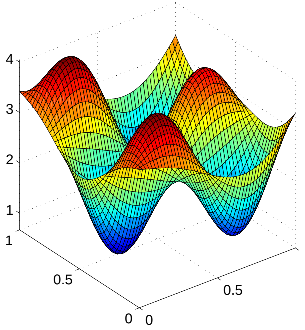
$\Omega = (0,1)^2$, v, w and y^δ known

exact solution

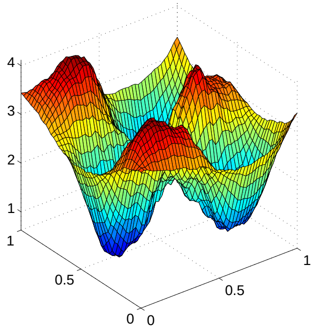
$$x_+(\xi,\eta) = 1.5\sin(2\pi\xi)\sin(3\pi\eta) + 3\left((\xi-0.5)^2 + (\eta-0.5)^2\right) + 2$$

x_0 chosen such that source condition is satisfied with $\gamma = \frac{1}{2}$

Example I: parameter identification in pde's

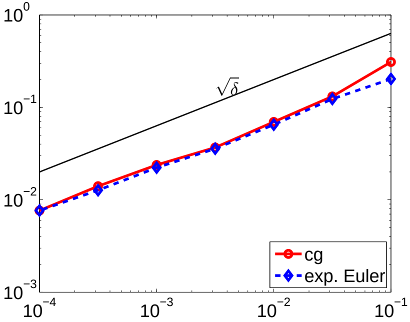


coefficient $x_+(\xi,\eta)$

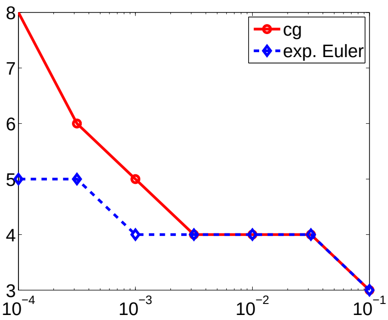


reconstruction $u^{\delta}_{n_*}(\xi,\eta)$
(rel. error 5%,
noise $\delta = 10^{-2.5} \approx 0.0032$)

Example I: parameter identification in pde's



reconstruction error



number of outer iterations

as functions of the perturbation parameter δ
(exponential Euler method vs. cg-REGINN)

Example II: groundwater hydrology (Hanke 1997)

$$\begin{aligned} -\mathrm{div}(x \,\mathrm{grad}\, y) &= f \quad \text{in } \Omega \\ y &= g \quad \text{on } \partial\Omega \end{aligned}$$

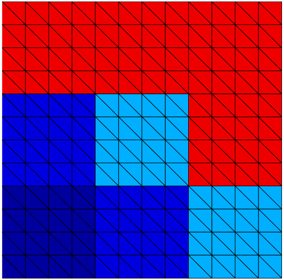
in $\Omega = [0,6]^2$, mixed Dirichlet–Neumann boundary data

$$y(\xi,0) = 100, \quad y_\xi(6,\eta) = 0, \quad (xy_\xi)(0,\eta) = -500, \quad y_\eta(\xi,6) = 0$$

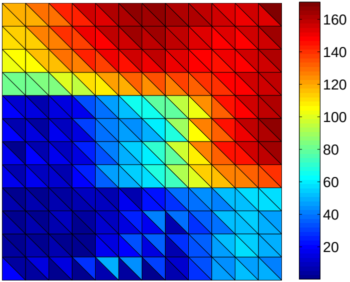
right-hand side

$$f(\xi,\eta) = \begin{cases} 0 & 0 < \eta < 4, \\ 137 & 4 < \eta < 5, \\ 274 & 5 < \eta < 6. \end{cases}$$

Example II: groundwater hydrology



diffusivity x_+



reconstruction $u_{n_*}^{\delta}$
(rel. error 18%, noise $\delta = 10^{-3}$)

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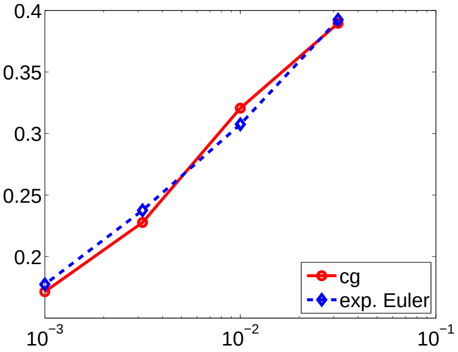
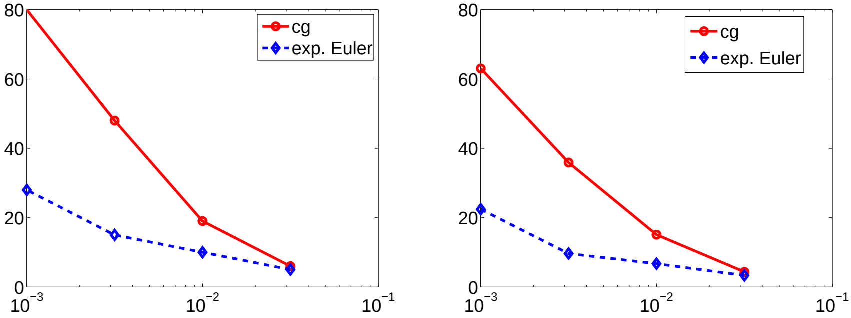


reconstruction error as function of the perturbation parameter δ (exponential Euler method vs. cg-REGINN)

Example II: groundwater hydrology



number of outer iterations cpu-time

as functions of the perturbation parameter δ
(exponential Euler method vs. cg-REGINN)

Conclusion

so far:

1. new regularization method introduced
2. method fits in known class of regularizations
3. regularizing properties stated
4. numerical experiments reflect theory
5. regularization method is competitive

to do:

1. show convergence and optimal order ($\rightsquigarrow$ following talk)

References

M. Hochbruck, M. Hönig and A. Ostermann.
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M2NA, submitted.

M. Hochbruck, M. Hönig and A. Ostermann.
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technical report.

Preprints available at `www.am.uni-duesseldorf.de`