Convergence of the exponential Euler iteration for nonlinear ill-posed problems

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Outline

Motivation

Nonlinear ill posed problem, time integration of odes

Convergence

Convergence rates

Summary

Motivation for this work

- I have never worked on ill-posed problems so far

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- I have never worked on ill-posed problems so far
- many regularization methods can be viewed as time integration schemes applied to the Showalter ode
- Tautenhahn 94: regularization properties of exact solution of the Showalter ode
- proofs of Tautenhahn based on variation-of-constants formula
- recent results on exponential integrators (...) for the solution of nonlinear evolution equations
  (H., Ostermann 05; H., Ostermann, Schweitzer 08)
  construction and analysis based on variation-of-constants formula

Nonlinear problem – Showalter ode

- nonlinear problem (exact – perturbed)

$$F(x) = y, \qquad F(u) = y^\delta, \qquad \|y - y^\delta\| \leq \delta$$

  assume that a solution x_+ of unperturbed problem exists

- Showalter ode for $u = u(t) = u^\delta(t)$

$$u' = F'(u)^* \big(y^\delta - F(u)\big) =: G(u)$$

- time integration – regularization method

$$u_{n+1} = u_n + h_n \varphi(-h_n J_n) G(u_n) \approx u(t_{n+1}), \quad t_{n+1} = t_n + h_n$$

  - $J_n = F'(u_n)^* F'(u_n) \approx -G'(u_n)$
  - φ suitable function (filter)
  - h_n step size, t_{n_*} regularization parameter

Convergence?

$$u_{n+1} = u_n + h_n \varphi(-h_n J_n) G(u_n), \qquad G(u) = F'(u)^* \big(y^\delta - F(u)\big)$$

stiff ode	ill posed problem
$u' = G(u)$	$F(u) = y^\delta$
$u_n \approx u(t_n)$	$u_{n_*} \approx x_+$
all $t_n \in [0, T]$	t_{n_*} stopping time
$\|J_n\|$ large	$\|J_n\|$ small
$h_n \to 0$	$h_n \to \infty$
$\|h_n J_n\|$ large	$\|h_n J_n\|$ large

Methods

$$u_{n+1} = u_n + h_n \varphi(-h_n J_n) G(u_n), \qquad G(u) = F'(u)^* \left(y^\delta - F(u)\right)$$

$\varphi(z)$	stiff ode	ill-posed problem
	$u' = G(u)$	$F(u) = y^\delta$
1	explicit Euler	Landweber, $h_n = 1$
$\dfrac{1}{1-z}$	linearly implicit Euler	Levenberg-Marquardt
$\dfrac{e^z - 1}{z}$	exponential Euler	asymptotic regularization

Some results – convergence / rates

- Landweber: Hanke, Neubauer, Scherzer, Num. Math. 95 ($h_n = 1$), conv. + optimal rates
- Levenberg-Marquardt: Hanke, IP 97, conv. (variable step sizes h_n chosen appropriately)
- iteratively regularized methods: Jin, Tautenhahn 08 conv. + optimal rates (schemes cannot be written as ode solver)
- (inexact) Newton methods: Rieder 99 conv. + rates
- asymptotic regularization: Tautenhahn 94 conv. + optimal rates
- today: conv. + optimal rates for exponential Euler

Stopping – discrepancy principle

- continuous case (Tautenhahn)

$$\left\| F(u(t_{n_*})) - y^\delta \right\| \leq \tau\delta < \left\| F(u(t)) - y^\delta \right\| , \qquad 0 \leq t < t_{n_*}$$

- discrete case (standard)

$$\left\| F(u_{n_*}) - y^\delta \right\| \leq \tau\delta < \left\| F(u_n) - y^\delta \right\| , \qquad \text{for all } n < n_*.$$

Convergence – assumptions

(1) tangential cone condition

$$\|F(\widetilde{x}) - F(x) - F'(x)(\widetilde{x} - x)\| \leq \eta \, \|F(\widetilde{x}) - F(x)\|$$

for all $\widetilde{x}, x \in B_r(x_0)$

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technical assumptions omitted

Convergence – continuous case

Theorem (Tautenhahn 94)

- stopping time $t_{n_*} < \infty$
- $u(t_{n_*}) \to x_+$, $\delta \to 0$

sketch of proof:

- $\left\| F(u(t_{n_*})) - y^\delta \right\|$ is monotonically decreasing
- $\left\| F(u(t_{n_*})) - y^\delta \right\| = \tau\delta$ has a unique solution
- convergence follows as in Hanke, Neubauer, Scherzer 95

Convergence – discrete case

Theorem

- stopping index $n_* < \infty$
- $u_{n_*} \longrightarrow x_+$, $\delta \longrightarrow 0$

sketch of proof:

- as in Hanke 97: $\{\|u_n - x_+\|\}_n$ monotonically decreasing
- step sizes h_n chosen based on

$$\|\varphi(-h_n K_n)\Delta F_n\| = \mu \|\Delta F_n\|, \quad K_n = F'(u_n)F'(u_n)^*,$$

where $\Delta F_n = y^\delta - F(u_n)$ are uniquely defined

Hanke: same step size selection for $\varphi(z) = (1-z)^{-1}$

Convergence rates – further assumptions

(2) source condition: there exists $\gamma \in (0, 1/2]$ and w s.t.

$$x_0 - x_+ = J_+^\gamma w, \qquad \|w\| \leq \rho, \quad J_+ := F'(x_+)^* F'(x_+)$$

(3) local restriction of the derivative

$$\begin{aligned} &F'(x) = R_x F'(x_+) \\ &\|R_x - I\| \leq C_+ \|x - x_+\|, \qquad \forall x \in B_r(x_+) \end{aligned}$$

(4) w.l.o.g. $\|F'(x)\| \leq 1, \quad x \in B_r(x_0)$

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e.g. Hanke, Neubauer, Scherzer 95; Tautenhahn 94

Convergence rates – continuous case

Theorem (Tautenhahn 94)
there exists a constant $c_* = c_*(\gamma, \tau, \eta)$ such that the error $e(t_{n_*}) = u(t_{n_*}) - x_+$ satisfies

$$\|e(t_{n_*})\| \leq c_* \rho^{1/(2\gamma+1)} \delta^{2\gamma/(2\gamma+1)}$$

for ρ sufficiently small

sketch of proof

- v.o.c. formula for exact solution of ode:

$$e(t) = \mathrm{e}^{-tJ_+} e(0) + \int_0^t \mathrm{e}^{-(t-s)J_+} A_+^* (y^\delta - y) \mathrm{d}s$$
$$+ \int_0^t \mathrm{e}^{-(t-s)J_+} r(s) \mathrm{d}s$$

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source condition: $e(0) = J(x_+)^\gamma w$, $\|w\| \leq \rho$
$A_+ = F'(x_+)$, $J_+ = A_+^* A_+$

Proof of convergence rates (Tautenhahn)

notation: $\varepsilon(t) = \|e(t)\|$, $\alpha(t) = \|A_+ e(t)\|$
verify that

$$\varepsilon(t) \leq \frac{\rho}{(1+t)^\gamma} + c_1\sqrt{t}\alpha(t) + c_2 \int_0^t \frac{\varepsilon(t-s)\alpha(t-s)}{\sqrt{s+1}} \mathrm{d}s$$

$$\alpha(t) \leq \frac{\rho}{(1+t)^{\gamma+1/2}} + c_1\alpha(t) + c_2 \int_0^t \frac{\varepsilon(t-s)\alpha(t-s)}{(s+1)} \mathrm{d}s$$

to conclude

$$\varepsilon(t) \leq c_* \frac{\rho}{(1+t)^\gamma}, \qquad \alpha(t) \leq c_* \frac{\rho}{(1+t)^{\gamma+1/2}},$$

Convergence rates – discrete case

Theorem

let the step sizes satisfy

- $h_j \le h_{j+1}$, for all $j \ge 0$
- $h_j \le c_h (1 + t_j)^{\gamma - \epsilon}$ for some $\epsilon > 0$

then there is a constant c_* such that

$$\|u_{n_*} - x_+\| \le c_* \rho^{1/(2\gamma+1)} \delta^{2\gamma/(2\gamma+1)}$$

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step size restriction in Jin, Tautenhahn 08: $h_j \le h_{j+1} + \Delta h$

Proof of convergence rates

discrete v.o.c. formula for $e_n = u_n - x_+$

$$\begin{aligned} e_n =& \mathrm{e}^{-t_n J_+} e_0 \\ &+ \sum_{j=0}^{n-1} h_j \mathrm{e}^{-(t_n - t_{j+1}) J_+} A_+^* \big(r_j + \varphi(-h_j K_+)(y^\delta - y) \big) \end{aligned}$$

- major problem hidden in r_j because numerical scheme uses $J_n = J(u_n)$ but analysis requires J_+
- $\|\varphi(-h_n K_n) - \varphi(-h_n K_+)\| \le c h_n \|e_n\| \, \|A_+ e_n\|$

$A_+ = F'(x_+)$, $J_+ = A_+^* A_+$, $K_+ = A_+ A_+^*$, $K_n = A_n A_n^*$

Proof of convergence rates, cont'd

verify

$$\begin{aligned}\|e_n\| \le& \frac{\rho}{(1+t_n)^\gamma} + \left(\sqrt{t_n/2} + h_{\max}\right)\delta \\ &+ c_1 \sum_{j=0}^{n-1} h_j \frac{(1+t_j)^{\gamma-\epsilon}}{\sqrt{1+t_n-t_{j+1}}} \|e_j\| \|A_+ e_j\| \\ \|A_+ e_n\| \le& \frac{\rho}{(1+t_n)^{\gamma+1/2}} + \delta \\ &+ c_2 \sum_{j=0}^{n-2} h_j \frac{(1+t_j)^{\gamma-\epsilon}}{1+t_n-t_{j+1}} \|e_j\| \|A_+ e_j\| \\ &+ c_3 h_{n-1} \|e_{n-1}\| \|A_+ e_{n-1}\|\end{aligned}$$

and induction to conclude

$$\|e_n\| \le c_* \frac{\rho}{(1+t_n)^\gamma}, \qquad \|A_+ e_n\| \le c_* \frac{\rho}{(1+t_n)^{\gamma+1/2}}$$

Difficulties and a magic moment

- variable step sizes (stability)
- Jacobian $J_n = J(u_n)$ changes from step to step
- discrete sums cannot be bounded directly by integrals arising in continuous analysis

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- variable step sizes (stability)
- Jacobian $J_n = J(u_n)$ changes from step to step
- discrete sums cannot be bounded directly by integrals arising in continuous analysis
- magic moment: telescopic identity

$$\sum_{j=0}^{n-1} h_j \mathrm{e}^{-(t_n - t_{j+1})K_+} K_+ \varphi(-h_j K_+) = I - \mathrm{e}^{-t_n K_+}$$

- magic moment does not happen for other φ functions (e.g. for Levenberg-Marquardt)

Summary

- exponential Euler regularization scheme converges for suitably chosen step sizes
- optimal convergence rates shown under mild step size restrictions

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H., Hönig, Ostermann 08

- A convergence analysis of the exponential Euler iteration for nonlinear ill-posed problems
- Regularization of nonlinear ill-posed problems by exponential integrators

Technical Reports, available at `www.am.uni-duesseldorf.de`