On an unsymmetric eigenvalue problem governing free vibrations of fluid-solid structures

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This is joint work with Heinrich Voss

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Outline

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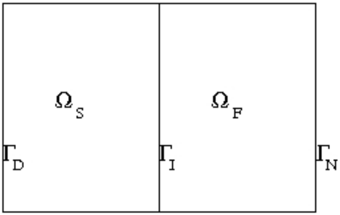
Problem definition

Vibrations of fluid-solid structures can be modelled in terms of solid displacement and fluid pressure and one obtains the classical form of an eigenproblem.

$$
\begin{aligned}
\text{Div}\,\sigma(u) + \lambda \rho_s u &= 0 \text{ in } \Omega_s,\\
\Delta p + \frac{\lambda}{c^2} p &= 0 \text{ in } \Omega_f,\\
\sigma(u)\, n - p\, n &= 0 \text{ on } \Gamma_I,\\
\nabla p n - \lambda \rho_f u n &= 0 \text{ on } \Gamma_I,\\
u &= 0 \text{ on } \Gamma_D,\\
\nabla p\, n &= 0 \text{ on } \Gamma_N,
\end{aligned}
$$



where

- u: solid displacement
- p: fluid pressure
- λ: eigenparameter
- $\sigma(u)$: linearized stress tensor
- ρ_s, ρ_f: densities of solid and fluid

Problem definition, cntd.

This eigenvalue problem can be given an unsymmetric variational formulation which can be discretized by the Finite-Element method and one obtains the unsymmetric matrix eigenproblem

$$Kx := \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} =: \lambda Mx, \tag{1}$$

where

- $K_s, M_s \in \mathbb{R}^{s\times s}$ are symmetric positive definite stiffness and mass matrices of the solid,
- $K_f, M_f \in \mathbb{R}^{f\times f}$ are symmetric stiffness and mass matrices of the fluid, where K_f is semi positive-definite and M_f positive definite,
- $C \in \mathbb{R}^{s\times f}$ is due to the coupling effects between fluid and solid,
- $x_s \in \mathbb{R}^s$ is the solid displacement vector, and
- $x_f \in \mathbb{R}^f$ the fluid pressure vector.

This talks considers the properties of eigenproblem (1) and discusses ways how to use the symmetry of K_s,K_f,M_s, and M_f to adapt symmetric eigensolvers to the given problem.

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Properties

Some properties can easily be derived:

Lemma

- *(1) can be symmetrized by* $T := \begin{pmatrix} M_s^{-1}K_s & M_s^{-1}C \\ 0 & I \end{pmatrix}$, *i.e.*

$$T^T Kx = \lambda T^T Mx$$

  is a symmetric eigenvalue problem.
- *(1) has only real non-negative eigenvalues.*
- *If* $x := \begin{pmatrix} x_s \\ x_f \end{pmatrix}$ *is a right eigenvector of (1) corresponding to the eigenvalue* λ,

  then $\hat{x} := \begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}$ *is a left eigenvector.*
- *Right eigenvectors can be chosen orthonormal with respect to* $\tilde{M} := \begin{pmatrix} K_s & 0 \\ 0 & M_f \end{pmatrix}$,

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Properties, cntd.

Lemma

Let $\lambda_j(A,B)$ denote the j smallest eigenvalue of the eigenproblem $Ax = \lambda Bx$ regarding the multiplicity of eigenvalues. Then it holds that

$$\begin{aligned}
\lambda_j(K,M) &\leq \lambda_j(K_s,M_s),\ j=1,\dots,s\\
\lambda_{s+f+1-j}(K,M) &\geq \lambda_{s+1-j}(K_s,M_s),\ j=1,\dots,s\\
\lambda_j(K,M) &\leq \lambda_j(K_f,M_f),\ j=1,\dots,f\\
\lambda_{s+f+1-j}(K,M) &\geq \lambda_{f+1-j}(K_f,M_f),\ j=1,\dots,f.
\end{aligned}$$

Proof:
Let $E_s := \text{span}\{e_1,\dots,e_s\}$ where $e_j \in \mathbb{R}^{s+f}$ denotes the jth unit vector containing a 1 in its jth component and zeros elsewhere. Then it holds that

$$\lambda_j(K,M) = \min_{\dim V=j}\ \max_{x\in V, x\neq 0} \frac{x^T T^T K x}{x^T T^T M x} \leq \min_{\dim V=j, V\subset E_s}\ \max_{x\in V, x\neq 0} \frac{x^T T^T K x}{x^T T^T M x}$$

$$= \min_{\dim W=j, W\subset \mathbb{R}^s}\ \max_{y\in W, y\neq 0} \frac{y^T K_s M_s^{-1} K_s y}{y^T K_s y} = \lambda_j(K_s,M_s).$$

The second inequality is obtained analogously from the maxmin characterization, and the third and fourth inequalities follow in a similar way.

An inverse-free Rayleigh functional

A Rayleigh quotient for fluid-solid eigenproblems is given immediately by its symmetrized version. As it involves inverse matrices it is numerically less valuable and we are interested in an inverse-free analogon.

For a given right eigenvector $\begin{pmatrix} x_s \\ x_f \end{pmatrix}$ corresponding to the eigenvalue λ it holds

$$\lambda = \frac{\begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}^T \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix}}{\begin{pmatrix} \lambda x_s \\ x_f \end{pmatrix}^T \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix}}$$

This suggests to define a Rayleigh functional p for some general $s + f$-dimensional vector by the requirement

$$p(x_s, x_f) = \frac{p(x_s, x_f) x_s^T K_s x_s + p(x_s, x_f) x_s^T C x_f + x_f^T K_f x_f}{p(x_s, x_f) x_s^T M_s x_s - x_f^T C^T x_s + x_f^T M_f x_f}$$

which leads to

$$p(x_s, x_f)^2 x_s^T M_s x_s + p(x_s, x_f)(x_f^T M_f x_f - x_s^T K_s x_s - 2 x_s^T C x_f) - x_f^T K_f x_f = 0.$$

We therefore choose the unique positive root of this equation as Rayleigh functional.

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Rayleigh functional and properties

Definition

$$p(x_s, x_f) := \begin{cases} q(x_s, x_f) + \sqrt{q(x_s, x_f)^2 + \frac{x_f^T K_f x_f}{x_s^T M_s x_s}} & \text{if} \quad x_s \neq 0 \\ \frac{x_f^T K_f x_f}{x_f^T M_f x_f} & \text{if} \quad x_s = 0 \end{cases} \tag{2}$$

where

$$q(x_s, x_f) := \frac{x_s^T K_s x_s - x_f^T M_f x_f + 2 x_s^T C x_f}{2 x_s^T M_s x_s}$$

is called Rayleigh functional of the fluid-solid vibration eigenvalue problem (1).

Well-known properties for symmetric eigenproblems can be generalized to the fluid-solid eigenproblem using the Rayleigh functional (2):

Lemma

Any eigenvector x of (1) is a stationary point of p, i.e. $\nabla p(x) = 0$.

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Rayleigh functional and properties, cntd.

As for symmetric eigenproblems one can derive variational formulations.

Proposition

The following variational characterizations hold

i) *(Minmax characterization)*

$$\lambda_k = \min_{dim S_k = k} \max_{0 \neq x \in S_k} p(x) = \max_{dim S_k = n+1-k} \min_{0 \neq x \in S_k} p(x).$$

ii) *(Rayleigh's principle)*

$$\begin{aligned} \lambda_k &= \min\{p(x) : x^T \tilde{M} x_j = 0, j = 1, \dots, k-1\} \\ &= \max\{p(x) : x^T \tilde{M} x_j = 0, j = k+1, \dots, s+f\} \end{aligned}$$

These variational characterizations can be proved by the following lemma.

Lemma

Assume that $x = \sum_{i \in I} \alpha_i x_i$ is a linear combination of some eigenvectors indexed by I. Then the Rayleigh functional is bounded by

$$\min_{i \in I} \lambda_i \leq p(x) \leq \max_{i \in I} \lambda_i.$$

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Rayleigh functional iteration

This Rayleigh functional can be used to define a Rayleigh functional iteration:

Require: initial vector $x^{(0)}$, $k = 0$

repeat
  $p^{(k)} = p(x^{(k)})$
  $x^{(k+1)} = (K - p^{(k)}M)^{-1}Mx^{(k)}$
  Normalize $x^{(k+1)}$ by $\tilde{M}$
  $k \leftarrow k + 1$
until convergence

As done for the Rayleigh quotient iteration for symmetric eigenproblems, local cubical convergence can be shown.

Proposition

The iterates $p^{(k)}$ and $x^{(k)}$ converge locally cubically towards to an eigenvalue λ and a corresponding eigenvector x.

Rayleigh functional iteration converges fast, but is highly sensitive with respect to the given initial vector. To overcome this drawback, one can consider projection methods where the projection space V is extended by the direction of the Rayleigh functional iteration. To reduce the computational effort we wish to solve

$$(K - p^{(k)}M)x^{(k+1)} = Mx^{(k)} \tag{3}$$

only approximately by some iterative method.

Jacobi-Davidson method

However, close to the desired eigenvector, the expanded subspace is very sensitive to inexact solves of (3) and hence we will replace the expansion direction by $t := x^{(k)} + \alpha x^{(k+1)}$ with α determined such that ${x^{(k)}}^T t = 0$ which leads to the equivalent so called correction equation

$$(I - \frac{Mxx^T}{x^TMx})(K - \theta M)(I - \frac{xx^T}{x^Tx})t = -(K - \theta M)x, \quad t^Tx = 0$$

for a given eigenpair approximation (θ, x). Iterative projection methods of this type were introduced by Sleipen and van der Vorst and are known as Jacobi-Davidson method. For fluid-solid eigenproblems we still have to ensure real eigenvalues of the projected eigenproblem and therefore we use structure preserving projection matrices

$$V = \begin{pmatrix} V_s & 0 \\ 0 & V_f \end{pmatrix}$$

and obtain

$$V^TKV = \begin{pmatrix} V_s^TK_sV_s & V_s^TCV_f \\ 0 & V_f^TK_fV_f \end{pmatrix} \text{ and } V^TMV = \begin{pmatrix} V_s^TM_sV_s & 0 \\ -V_f^TC^TV_s & V_f^TM_fV_f \end{pmatrix}.$$

Proposition

Assume that V has maximal rank l. Then it holds

$$\lambda_k(K, M) \leq \lambda_k(V^TKV, V^TMV) \text{ for } k = 1, \dots, l.$$

Jacobi-Davidson method, cntd.

Require: Initial basis $V = \begin{pmatrix} V_s & 0 \\ 0 & V_f \end{pmatrix}$, $V_s^T V_s = V_f^T V_f = I$, $m = 1$

determine preconditioner $L \approx (K - \sigma M)^{-1}$, σ close to the first desired eigenvalue

while $m \leq$ number of wanted eigenvalues **do**

compute the m smallest eigenvalue θ_m and corresponding eigenvector $y = \begin{pmatrix} y_s^T & y_f^T \end{pmatrix}^T$ of the projected problem

$$\begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix} = \theta \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C V_s & V_f^T M_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix}.$$

determine Ritz vector $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and residual $r = (K - \theta_m M)x$

if $\|r\| / \|x\| < \epsilon$ **then**

accept approximate mth eigenpair (θ_m, x); increase $m \leftarrow m + 1$; reduce search space

determine new preconditioner $L \approx (K - \theta_m M)^{-1}$ if necessary

choose approximation (θ_m, x) to next eigenpair

compute residual $r = (K - \theta_m M)x$

end if

compute approximate solution $t = \begin{pmatrix} t_s^T & t_f^T \end{pmatrix}^T$ of the correction equation

$$(I - \frac{Mxx^T}{x^T Mx})(K - \theta M)(I - \frac{xx^T}{x^T x})t = r, x^T t = 0$$

orthogonalize $v_s = t_s - V_s V_s^T t_s$, $v_f = t_f - V_f V_f^T t_f$

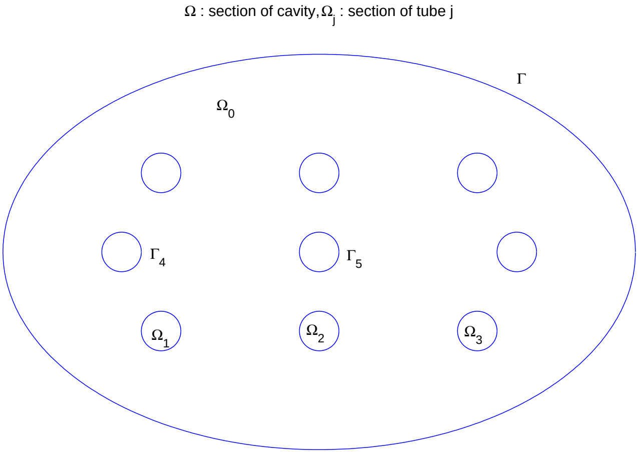
expand search space $V_s \leftarrow [V_s, v_s / \|v_s\|]$, $V_f \leftarrow [V_f, v_f / \|v_f\|]$ and update proj. problem

end while

Outline

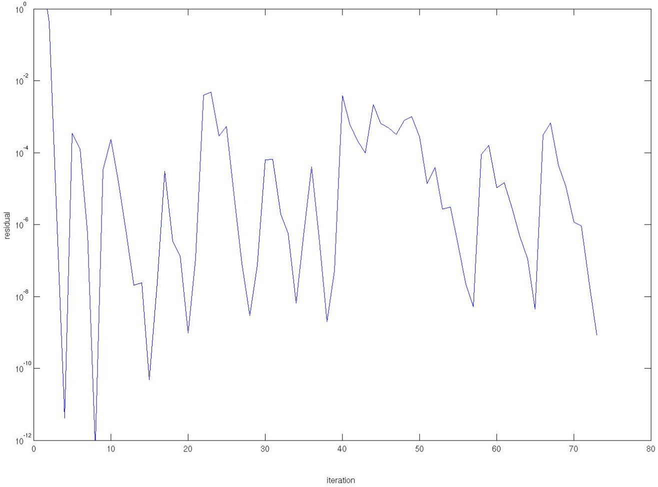
Numerical Results

We consider the 2D-example of 9 tubes immersed in a fluid. The model has been discretized by the FEM with 9162 degrees of freedom. The task is to find the 10 smallest eigenvalues by the Jacobi-Davidson method.



Numerical Results: JD method for general unsymmetric eigenproblems

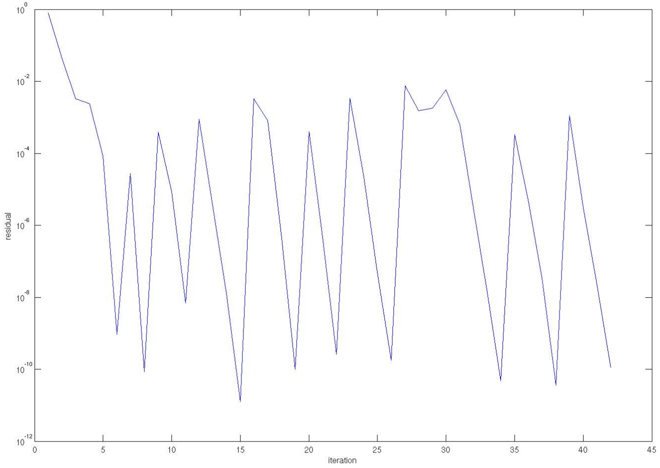
Convergence history for Jacobi-Davidson method for general nonsymmetric eigenproblems



required iterations for 10 smallest eigenvalues: 73

Numerical Results: JD method for fluid-solid interaction eigenproblems

Convergence history for fluid-solid interaction Jacobi-Davidson method



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