

On stabilized discretization and GMRES convergence for a convection-diffusion model problem

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1. The convection-diffusion model problem

We consider a model problem that has been studied by many authors, e.g. [Fischer, Ramage, Sylvester, Wathen - 1999], [Ernst - 2000], [Elman, Ramage - 2001, 2002], [Liesen, Strakoš - 2005]:

$$\begin{aligned} -\nu \nabla^2 u + w \cdot \nabla u &= 0 & \text{in } \Omega &= (0, 1) \times (0, 1), \\ u &= g & \text{on } \partial\Omega, \end{aligned}$$

where

- ν : scalar diffusion coefficient
- $w = [0, 1]^T$: velocity field (wind)
- g : given by discontinuous inflow (Raithby) boundary conditions

$$\begin{aligned} u(x, 0) = u(1, y) &= 1 & \text{for } \frac{1}{2} \leq x \leq 1 \text{ and } 0 \leq y < 1, \\ u(x, y) &= 0 & \text{elsewhere on } \partial\Omega \end{aligned}$$



1. The convection-diffusion model problem

We use **Streamline Upwind Petrov Galerkin (SUPG) discretization** ([Brooks, Hughes - 1979]) to suppress non-physical oscillations. The idea is to add an artificial diffusion term in the local flow direction. The modified coefficient matrix C has the form

$$C_{i,j} = \nu \langle \nabla \phi_j, \nabla \phi_i \rangle + \langle w \cdot \nabla \phi_j, \phi_i \rangle + \hat{\delta} \langle w \cdot \nabla \phi_j, w \cdot \nabla \phi_i \rangle,$$

where $\phi_j, j = 1, 2, \dots$ are the bilinear finite element nodal basis functions for an N by N grid with spacing $h = 1/(N + 1)$ and $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product on Ω .

Here, $\hat{\delta}$ is the **stabilization parameter**. Stabilization parameters are chosen as

$$\hat{\delta} = \frac{\delta h}{\|w\|},$$

for some positive **tuning parameter** $0 < \delta < 1$.



1. The convection-diffusion model problem

For linear finite element discretization in a one-dimensional constant coefficient problem it can be shown that the choice

$$\delta_0 = \frac{1}{2} \left(\coth(P_h) - \frac{1}{P_h} \right), \quad \text{i.e.} \quad \hat{\delta}_0 = \frac{h}{2\|w\|} \left(\coth(P_h) - \frac{1}{P_h} \right)$$

where $P_h \equiv h/(2\nu)$ is the mesh Peclet number, yields the **exact solution at the node points** [Brooks, Hughes - 1982].

In two dimensions one frequently uses the simplified value [Elman, Ramage - 2001, 2002]

$$\delta_0 = \frac{1}{2} \left(1 - \frac{1}{P_h} \right), \quad \text{i.e.} \quad \hat{\delta}_0 = \frac{h}{2\|w\|} \left(1 - \frac{1}{P_h} \right)$$

In the following we call this choice of stabilization **optimal stabilization**. Note that with large mesh Peclet numbers we have $\delta_0 \approx 1/2$.



1. The convection-diffusion model problem

Consider an example where we use

- A 6×6 grid (linear system dimension 36)
- The small diffusion coefficient $\nu = 0.00025$
- Hence large Peclet number $P_h = 285$ and optimal tuning parameter $\delta_0 = 0.498$ close to 0.5
- **GMRES** to solve the resulting linear systems
- The zero initial guess

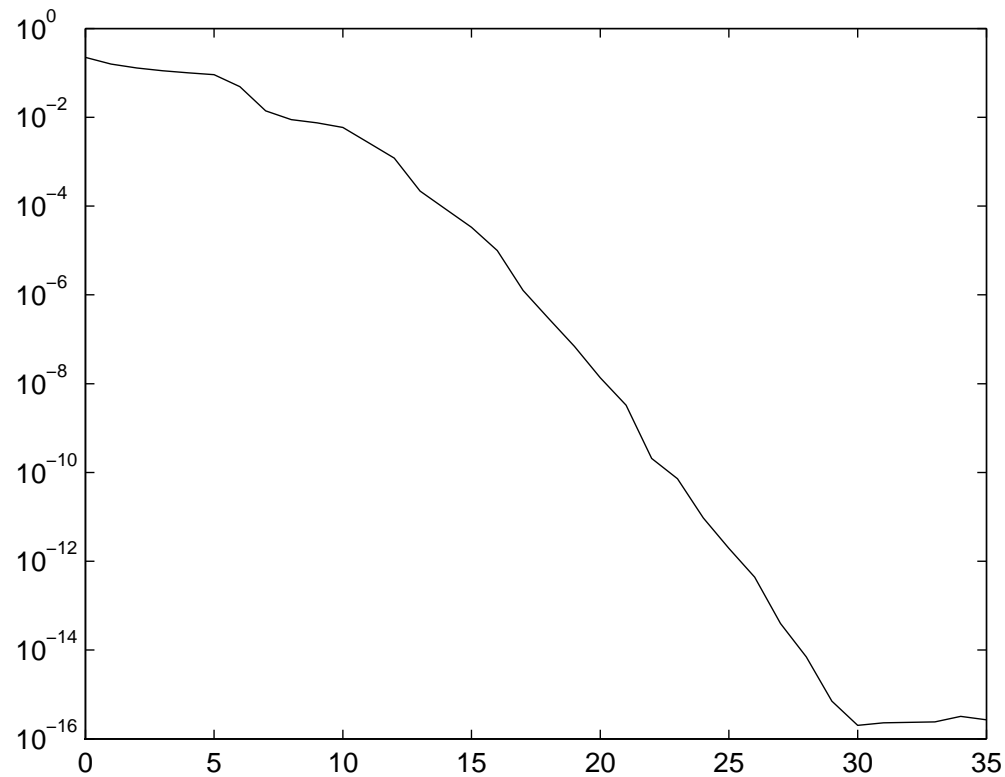
In addition to the optimal tuning parameter $\delta_0 = 0.498$ we consider also

$$\delta_1 = 0.398, \quad \delta_2 = 0.298, \quad \delta_2^+ = 0.698, \quad \delta_3^+ = 0.798.$$



1. The convection-diffusion model problem

GMRES for the convection-diffusion model problem

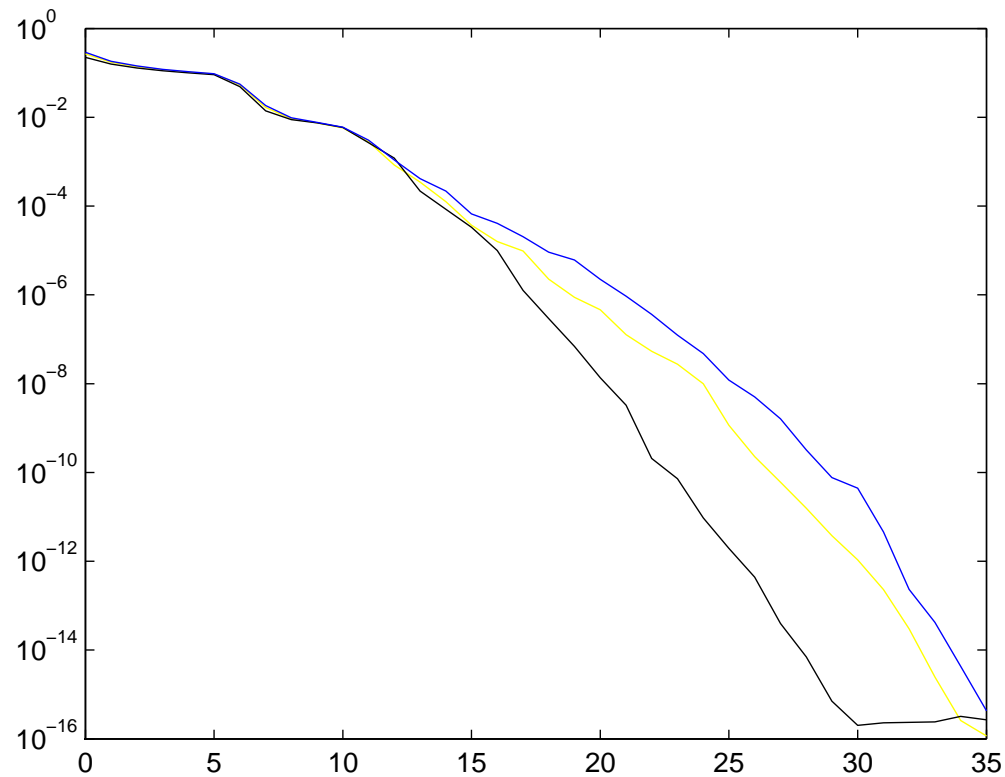


Black: δ_0 .



1. The convection-diffusion model problem

GMRES for the convection-diffusion model problem

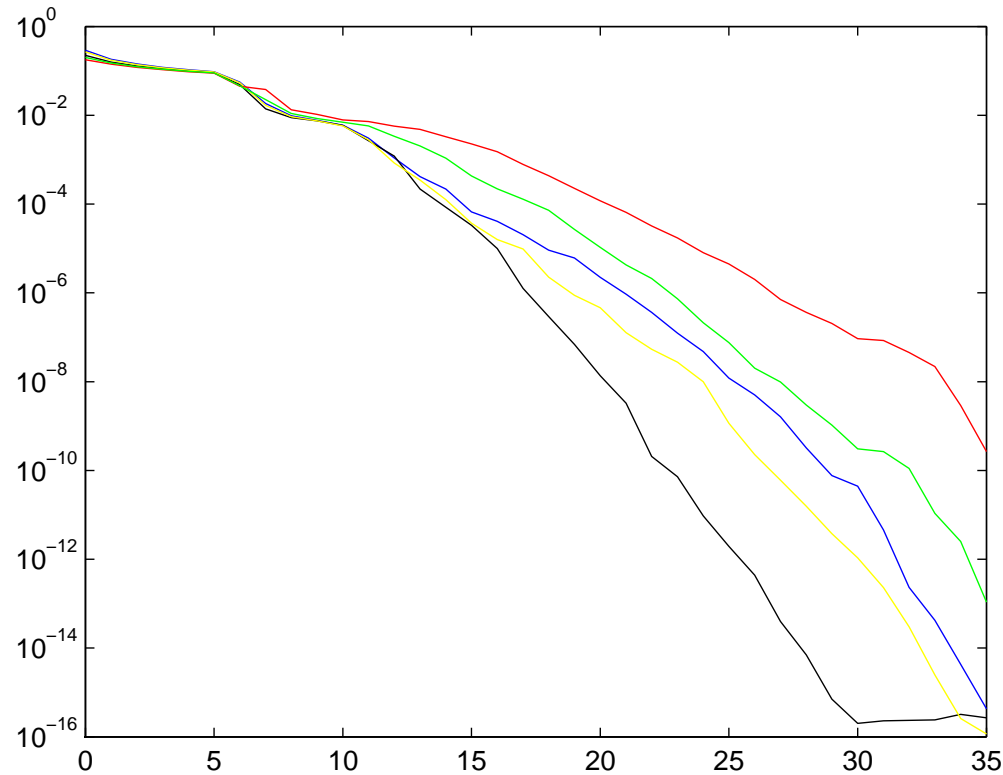


Black: δ_0 , yellow: δ_2^+ , blue: δ_3^+ .



1. The convection-diffusion model problem

GMRES for the convection-diffusion model problem



Black: δ_0 , red: δ_2 , green: δ_1 , yellow: δ_2^+ , blue: δ_3^+ .



1. The convection-diffusion model problem

The fact that **GMRES converges worse for worse stabilization** shows a fascinating and rather lucky relation between discretization and behavior of an iterative solver.

This fact has been observed in [Fischer, Ramage, Silvester, Wathen - 1999], but has not been fully explained. (Explanation of the behavior of GMRES for δ_0 has been more or less completed [Liesen, Strakoš - 2005], [DT - 2005].)

In this talk we offer an explanation. In particular, we will present **lower bounds for $\delta \neq \delta_0$** .



2. Properties of the system matrices

With the **discrete sine transformation** from [Liesen, Strakoš - 2005], system matrices \mathbf{A} are block diagonal with tridiagonal Toeplitz blocks of dimension $N \times N$:

$$\mathbf{A} x = \begin{pmatrix} \mathbf{T}_1 & & & \\ & \mathbf{T}_2 & & \\ & & \ddots & \\ & & & \mathbf{T}_N \end{pmatrix} x = \begin{pmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(N)} \end{pmatrix} = b,$$

where

$$\mathbf{T}_j = \begin{pmatrix} \lambda_j & \mu_j & & & \\ \gamma_j & \lambda_j & \mu_j & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_j & \lambda_j & \mu_j \\ & & & \gamma_j & \lambda_j \end{pmatrix}.$$



2. Properties of the system matrices

Assuming $P_h = h/(2\nu) \gg 1$ we can write the entries in simplified form as

$$\lambda_j = 2\delta h c_j + \mathcal{O}(\nu),$$

$$\mu_j = -\delta h c_j + \frac{h}{2} c_j + \mathcal{O}(\nu),$$

$$\gamma_j = -\delta h c_j - \frac{h}{2} c_j + \mathcal{O}(\nu),$$

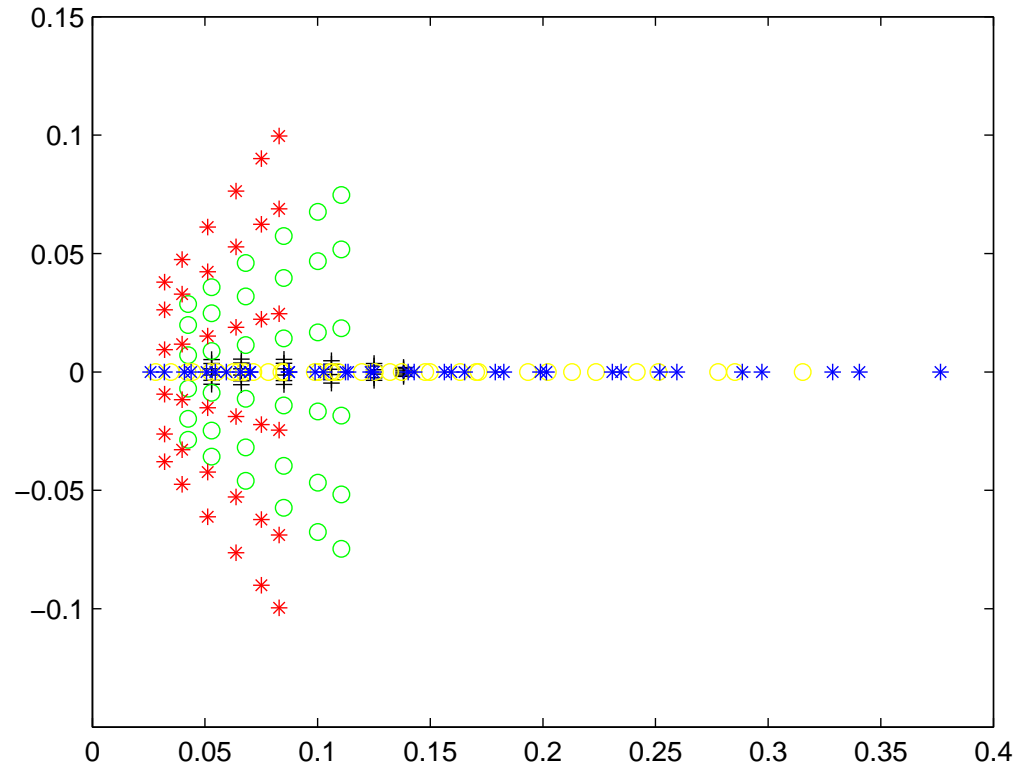
where $c_j = 2 + \cos jh\pi$. Note that for $\delta_0 \approx 1/2$ we have $\mu_j \approx 0$.

The **eigenvalues** of the whole matrix are given by

$$\sigma_{jk} = \lambda_j + 2\sqrt{\mu_j \gamma_j} \cos kh\pi, \quad 1 \leq j, k \leq N.$$



2. Properties of the system matrices



Spectra for δ_0 (black cross), δ_2 (red star), δ_1 (green circle), δ_2^+ (yellow circle) and δ_3^+ (blue star).



2. Properties of the system matrices

The **eigenvector matrix** of T_j is given by

$$W_j = \text{diag}(\zeta_\delta^{-\frac{1}{2}}, \zeta_\delta^{-1}, \dots, \zeta_\delta^{-\frac{N}{2}})V, \quad V_{i,j} = \sin(jih\pi),$$

where

$$\zeta_\delta = \frac{\mu_j}{\gamma_j} = \frac{1 - \frac{1}{2\delta}}{1 + \frac{1}{2\delta}} + \mathcal{O}(\nu).$$

Lemma 1 The **condition number** κ of the eigenvector basis formed of eigenvectors $w_{k,j}$, $1 \leq k, j \leq N$, equals

$$\kappa = \left| \frac{1 - \frac{1}{2\delta}}{1 + \frac{1}{2\delta}} \right|^{\frac{1-N}{2}}.$$

For $\delta_0 \approx 1/2$ we obtain a huge κ . The matrix with the **worst** conditioned eigenvector basis gives the **best** GMRES convergence.



3. Analysis for the $(n - 1)$ st step

Let A_δ be the system matrix with tuning parameter δ . We will use the well-known equality (e.g. [Ipsen - 2000])

$$\|r_k\| = \left\| e_1^T \cdot [b_\delta, A_\delta b_\delta, \dots, A_\delta^k b_\delta]^+ \right\|^{-1}$$

and an analysis similar to [Liesen, Tichy - 2004]. For $k = n - 1$ the Moore-Penrose pseudo-inverse translates to

$$[b_\delta, A_\delta b_\delta, \dots, A_\delta^{n-1} b_\delta]^+ = [b_\delta, A_\delta b_\delta, \dots, A_\delta^{n-1} b_\delta]^{-1}.$$

Let the **spectral decomposition** of A_δ be

$$A_\delta = W_\delta \Sigma_\delta W_\delta^{-1},$$

and write the right hand side in the **eigenvector basis** as

$$b_\delta = \sum_{j=1}^n \theta_j w_j.$$



3. Analysis for the $(n - 1)$ st step

We have

$$A_\delta^k b_\delta = W_\delta \Theta_\delta (\sigma_1^k, \dots, \sigma_n^k)^T,$$

where $\Theta_\delta = \text{diag}(\theta_1, \dots, \theta_n)$. Hence

$$[b_\delta, A_\delta b_\delta, \dots, A_\delta^{n-1} b_\delta] = W_\delta \Theta_\delta M_\delta,$$

where M_δ is the **Vandermonde** matrix whose i th row is $(1, \sigma_i, \sigma_i^2, \dots, \sigma_i^n)$.

For the one but last residual norm we obtain

$$\|r_{n-1}\| = \|e_1^T \cdot [b_\delta, A_\delta b_\delta, \dots, A_\delta^{n-1} b_\delta]^{-1}\|^{-1} = \|e_1^T \cdot M_\delta^{-1} \Theta_\delta^{-1} W_\delta^{-1}\|^{-1}.$$

First we take a closer look at the matrix $\Theta_\delta^{-1} W_\delta^{-1}$. This matrix is also block diagonal,

$$\Theta_\delta^{-1} W_\delta^{-1} = \text{diag}(\Theta_1^{-1} W_1^{-1}, \dots, \Theta_N^{-1} W_N^{-1}).$$



3. Analysis for the $(n - 1)$ st step

Recall that

$$W_j = \text{diag}(\zeta_\delta^{-\frac{1}{2}}, \zeta_\delta^{-1}, \dots, \zeta_\delta^{-\frac{N}{2}})V, \quad V_{i,j} = \sin(jih\pi), \quad \zeta_\delta = \frac{1 - \frac{1}{2\delta}}{1 + \frac{1}{2\delta}} + \mathcal{O}(\nu).$$

With $b^{(j)} = (\beta_1, \dots, \beta_N)^T$ and $V^{-1} = \frac{2}{N+1}V$, the components of $b^{(j)}$ in the eigenvector basis W_j are

$$\begin{aligned} \begin{pmatrix} \theta_1 \\ \dots \\ \theta_N \end{pmatrix} &= W_j^{-1} \begin{pmatrix} \beta_1 \\ \dots \\ \beta_N \end{pmatrix} = \frac{2}{N+1} V \text{diag}(\zeta_\delta^{\frac{1}{2}}, \zeta_\delta, \dots, \zeta_\delta^{\frac{N}{2}}) \begin{pmatrix} \beta_1 \\ \dots \\ \beta_N \end{pmatrix} \\ &= \frac{2}{N+1} \sum_{i=1}^N \beta_i \zeta_\delta^{\frac{N+1}{2}i} \begin{pmatrix} \sin ih\pi \\ \dots \\ \sin iNh\pi \end{pmatrix}. \end{aligned}$$



3. Analysis for the $(n - 1)$ st step

Putting the expressions for θ_i in the diagonal matrix Θ_j ,

$$\begin{aligned}\Theta_j^{-1}W_j^{-1} &= \text{diag} \left(\sum_{i=1}^N \beta_i \zeta_\delta^{\frac{i}{2}} \sin ih\pi, \dots, \sum_{i=1}^N \beta_i \zeta_\delta^{\frac{i}{2}} \sin iNh\pi \right)^{-1} V \text{diag} (\zeta_\delta^{\frac{1}{2}}, \zeta_\delta, \dots, \zeta_\delta^{\frac{N-1}{2}}) \\ &= \frac{1}{\beta_1} \hat{V}_\delta^{(j)} \text{diag} (1, \zeta_\delta^{\frac{1}{2}}, \zeta_\delta, \dots, \zeta_\delta^{\frac{N-1}{2}}),\end{aligned}$$

where

$$(\hat{V}_\delta^{(j)})_{i,k} \equiv \frac{\beta_1 \sin(ikh\pi)}{\sum_{m=1}^N \zeta_\delta^{\frac{m-1}{2}} \beta_m \sin(imh\pi)}.$$

Note that $|\zeta_\delta| = \left| \frac{1 - \frac{1}{2\delta}}{1 + \frac{1}{2\delta}} \right| \ll 1$ and hence the denominators in $(\hat{V}_\delta^{(j)})_{i,j}$ are restricted to the first few summands \Rightarrow

$$(\hat{V}_\delta^{(j)})_{i,k} \approx \frac{\beta_1 \sin(ikh\pi)}{\beta_1 \sin(ih\pi)}.$$



3. Analysis for the $(n - 1)$ st step

$$\Theta_{\delta}^{-1} W_{\delta}^{-1} \approx \begin{pmatrix} \frac{1}{\beta_1^{(1)}} \mathbf{E} & & & \\ & \frac{1}{\beta_1^{(2)}} \mathbf{E} & & \\ & & \ddots & \\ & & & \frac{1}{\beta_1^{(N)}} \mathbf{E} \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

where $\frac{1}{\beta_1^{(j)}}$ is the first component of the right hand side corresponding to block j .

From this form it can be seen that **the matrix $\Theta_{\delta}^{-1} W_{\delta}^{-1}$ is only slightly dependent on the choice of δ** . In particular, it seems **the influence of the conditioning of the eigenvector matrix W_{δ} is eliminated** by taking into account the right hand side through the matrix Θ_{δ} .



3. Analysis for the $(n - 1)$ st step

Having analyzed $\Theta_\delta^{-1}W_\delta^{-1}$ in

$$\|r_{n-1}\| = \left\| e_1^T \cdot M_\delta^{-1} \Theta_\delta^{-1} W_\delta^{-1} \right\|^{-1},$$

we proceed with $e_1^T \cdot M_\delta^{-1}$. It is well known there exist **exact formulae** for the entries of inverses of Vandermonde matrices (e.g. [Muir - 1960]),

$$e_1^T \cdot M_\delta^{-1} = (-1)^{n-1} \left(\frac{\prod_{k \neq 1} \sigma_k}{\prod_{k \neq 1} (\sigma_1 - \sigma_k)}, \frac{\prod_{k \neq 2} \sigma_k}{\prod_{k \neq 2} (\sigma_2 - \sigma_k)}, \dots, \frac{\prod_{k \neq n} \sigma_k}{\prod_{k \neq n} (\sigma_n - \sigma_k)} \right).$$

The behavior of the expression on the right can be explained with

Lemma 2. The **difference between two eigenvalues** of block T_j is

$$\sigma_{jk} - \sigma_{jl} = c_1 (2 + \cos jh\pi) \sqrt{1 - \frac{1}{4\delta^2}} (\cos kh\pi - \cos lh\pi).$$

\Rightarrow for $\delta_0 \approx 1/2$ we have $\sigma_{jk} - \sigma_{jl} \approx 0$ and $\left| \frac{\prod_{k \neq i} \sigma_k}{\prod_{k \neq i} (\sigma_i - \sigma_k)} \right| \rightarrow \infty$.



3. Analysis for the $(n - 1)$ st step

Lemma 3. The **product of a pair of eigenvalues** σ_{jk} and σ_{jl} satisfying $k + l = N + 1$ equals

$$\sigma_{jk} \cdot \sigma_{jl} = c_1^2 (2 + \cos jh\pi)^2 \left(1 - \left(1 - \frac{1}{4\delta^2} \right) \cos^2 kh\pi \right).$$

The function $|\sigma_{jk}(\delta) \cdot \sigma_{jl}(\delta)|$ is slightly decreasing in δ .

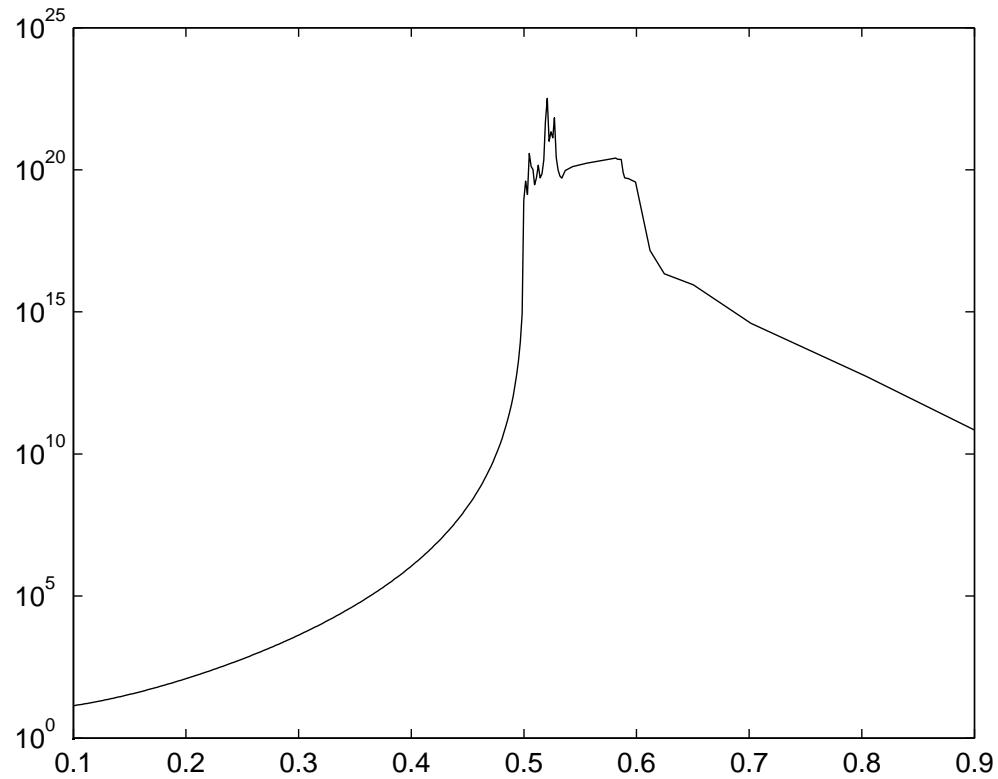
We obtain for the function

$$m(\delta) \equiv \|e_1^T \cdot M_\delta^{-1}\| = \left\| \left(\frac{\prod_{k \neq 1} \sigma_k}{\prod_{k \neq 1} (\sigma_1 - \sigma_k)}, \dots, \frac{\prod_{k \neq n} \sigma_k}{\prod_{k \neq n} (\sigma_n - \sigma_k)} \right) \right\|.$$

the following behavior:



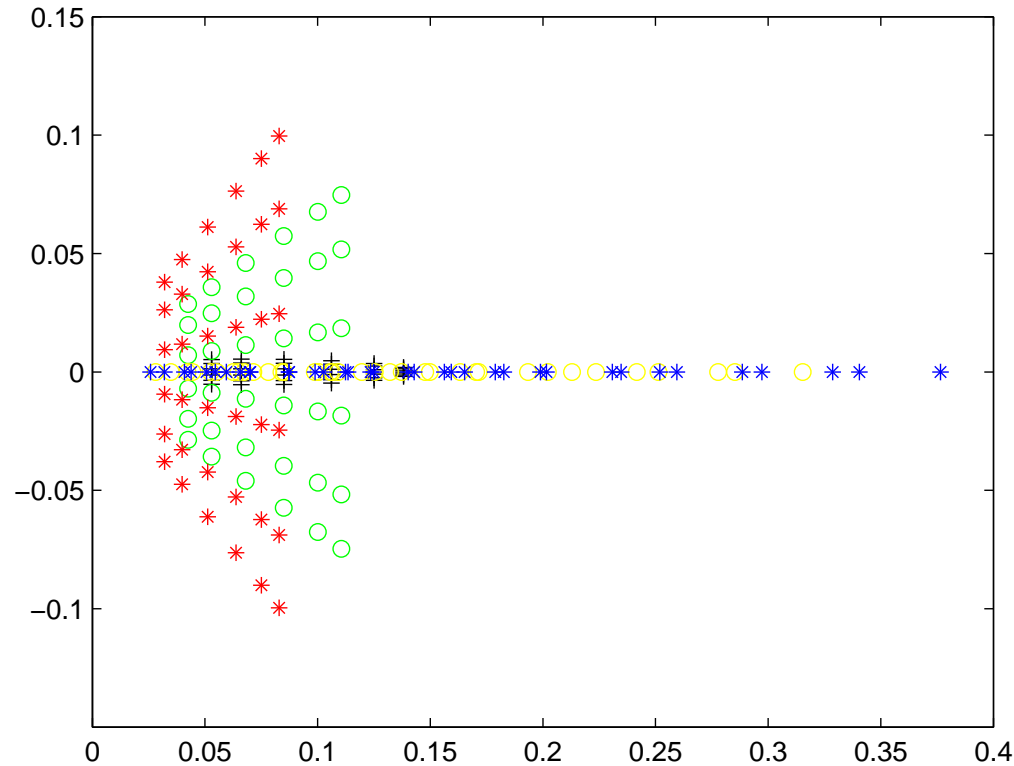
3. Analysis for the $(n - 1)$ st step



The function $m(\delta) \equiv \|e_1^T \cdot M_\delta^{-1}\|$ with $N = 6$.



3. Analysis for the $(n - 1)$ st step



Spectra for δ_0 (black cross), δ_2 (red star), δ_1 (green circle), δ_2^+ (yellow circle) and δ_3^+ (blue star).



3. Analysis for the $(n - 1)$ st step

All together, we have

$$\|r_{n-1}\|^{-1} = \|e_1^T \cdot M_\delta^{-1} \Theta_\delta^{-1} W_\delta^{-1}\|$$

$$\approx \left\| e_1^T \cdot M_\delta^{-1} \begin{pmatrix} \frac{1}{\beta_1^{(1)}} \mathbf{E} & & & \\ & \frac{1}{\beta_1^{(2)}} \mathbf{E} & & \\ & & \ddots & \\ & & & \frac{1}{\beta_1^{(N)}} \mathbf{E} \end{pmatrix} \right\|, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

We can show that convergence is indeed governed by $\|e_1^T \cdot M_\delta^{-1}\|$. Hence **the main reason for convergence delay** by step $n - 1$ with non-optimal stabilization **are the relatively large distances between eigenvalues**. An example where eigenvalues do govern convergence behavior even though eigenvectors are very ill-conditioned, i.e. matrices are highly non-normal!



4. Bounds for the general step

In practice GMRES processes will often have converged before step $n - 1$. For a step $k - 1 < n - 1$ we must deal with

$$\|r_k\|^{-1} = \left\| e_1^T \cdot [b_\delta, A_\delta b_\delta, \dots, A_\delta^{k-1} b_\delta]^+ \right\| = \left\| e_1^T \cdot \left(W_\delta \Theta_\delta M_\delta \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\|, \quad (1)$$

and **explicit formulae for the entries of the Vandermonde inverse M_δ^{-1} cannot be exploited.**

As a tool to explain the behavior of (1) **we will use the UL-decomposition of $M^{-1} = UL$ [Golub, van Loan - p. 186]:**

$$L_{i,j} = \prod_{l=1, l \neq j}^i (\sigma_j - \sigma_l)^{-1} \quad \Rightarrow$$



4. Bounds for the general step

$$L = \begin{pmatrix} 1 & & & & \\ \frac{1}{\sigma_1 - \sigma_2} & & & & \\ \frac{1}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} & \frac{1}{(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3)} & \frac{1}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} & & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

$$U_{i,j} = \epsilon_{j-i}(\sigma_1, \dots, \sigma_{j-1}), \quad \text{where} \quad \epsilon_k(\sigma_1, \dots, \sigma_l) = \sum \sigma_{\nu_1} \cdot \sigma_{\nu_2} \cdots \sigma_{\nu_k} \quad \Rightarrow$$

$$U = \begin{pmatrix} 1 & -\sigma_1 & \sigma_1\sigma_2 & -\sigma_1\sigma_2\sigma_3 & \cdots \\ & 1 & -\sigma_2 - \sigma_1 & \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 & \cdots \\ & & 1 & -\sigma_1 - \sigma_2 - \sigma_3 & \cdots \\ & & & 1 & \cdots \\ & & & & \ddots \end{pmatrix},$$



4. Bounds for the general step

Then

$$\begin{aligned} \|r_k\|^{-1} &= \left\| e_1^T \cdot \left(W_\delta \Theta_\delta M_\delta \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\| = \left\| e_1^T \cdot \left(W_\delta \Theta_\delta L_\delta^{-1} U_\delta^{-1} \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\| \\ &= \left\| e_1^T \cdot \left(W_\delta \Theta_\delta L_\delta^{-1} \begin{pmatrix} I_k \\ 0 \end{pmatrix} (U_\delta^{(k)})^{-1} \right)^+ \right\| = \left\| e_1^T \cdot U_\delta^{(k)} \left(W_\delta \Theta_\delta L_\delta^{-1} \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\|, \end{aligned}$$

where $U_\delta^{(k)}$ denotes the left upper $k \times k$ block of U_δ .

With

$$e_1^T \cdot U_\delta^{(k)} = [1, -\sigma_1, \sigma_1 \cdot \sigma_2, \dots, (-1)^{k-1} \prod_{j=1}^{k-1} \sigma_j],$$

we have isolated **eigenvalue products**. To **isolate** important **eigenvalue differences** in L_δ we use a scaling matrix $\Pi_\delta^{(k)}$:



4. Bounds for the general step

With an ordering of eigenvalues such that the largest entries of L_δ are on the diagonal, let

$$\Pi_\delta^{(k)} = \text{diag}(L_\delta) = \left(\text{diag}\left(1, \prod_{j \neq 1}^2 |\sigma_1 - \sigma_j|, \prod_{j \neq 2}^3 |\sigma_2 - \sigma_j|, \dots, \prod_{j \neq k-1}^k |\sigma_{k-1} - \sigma_j|\right) \right)^{-1}$$

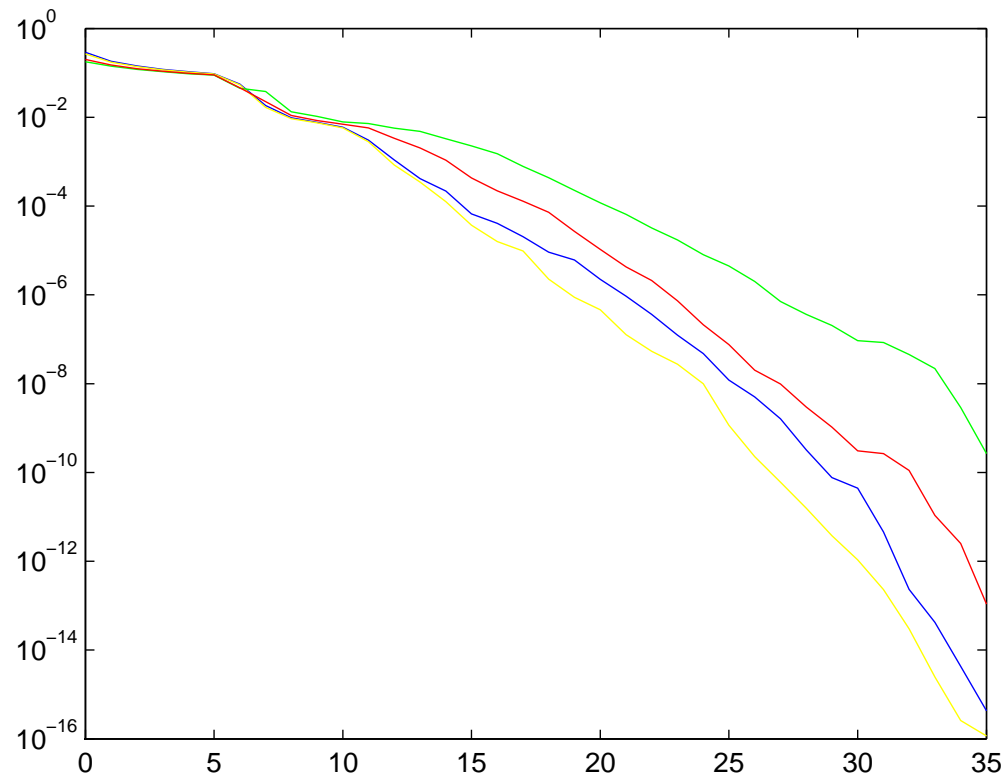
Now, with s_{\min} denoting the minimal singular value, we have $\|r_k\| =$

$$\begin{aligned} \left\| e_1^T \cdot U_\delta^{(k)} \left(W_\delta \Theta_\delta L_\delta^{-1} \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\|^{-1} &= \left\| e_1^T \cdot U_\delta^{(k)} \Pi_\delta^{(k)} \left(W_\delta \Theta_\delta L_\delta^{-1} \Pi_\delta^{(n)} \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right)^+ \right\|^{-1} \\ &\geq s_{\min} \left(W_\delta \Theta_\delta L_\delta^{-1} \Pi_\delta^{(n)} \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right) \left\| e_1^T \cdot U_\delta^{(k)} \Pi_\delta^{(k)} \right\|^{-1} . \end{aligned}$$



4. Bounds for the general step

GMRES for the convection-diffusion model problem

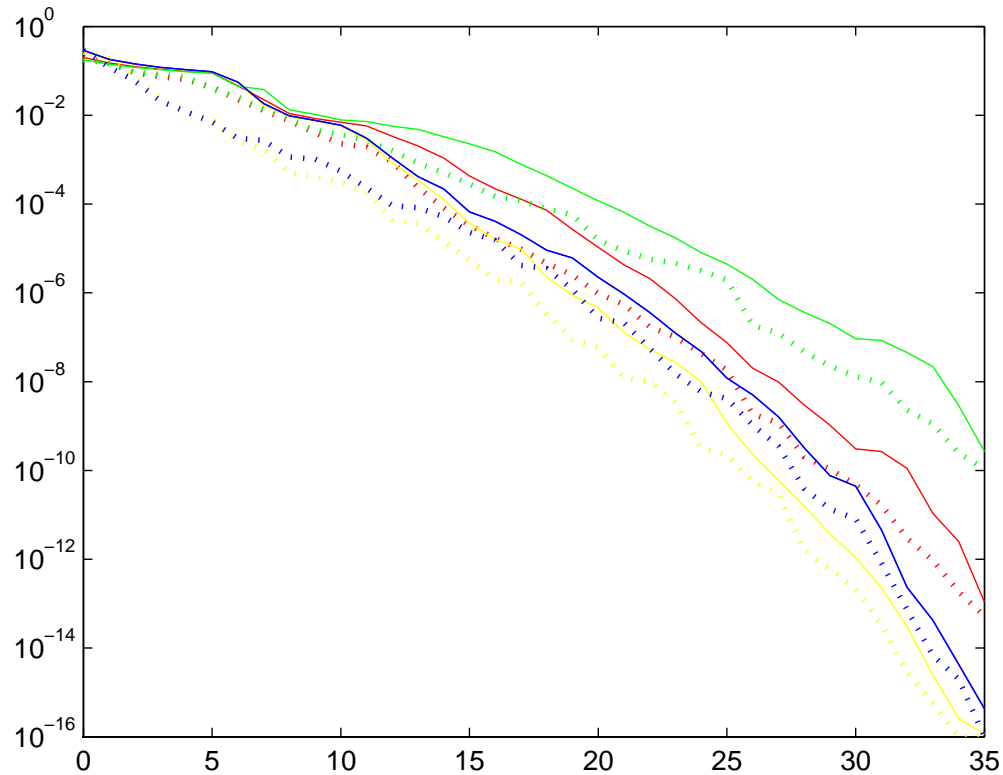


Red: δ_2 , green: δ_1 , yellow: δ_2^+ , blue: δ_3^+ .



4. Bounds for the general step

GMRES for the convection-diffusion model problem

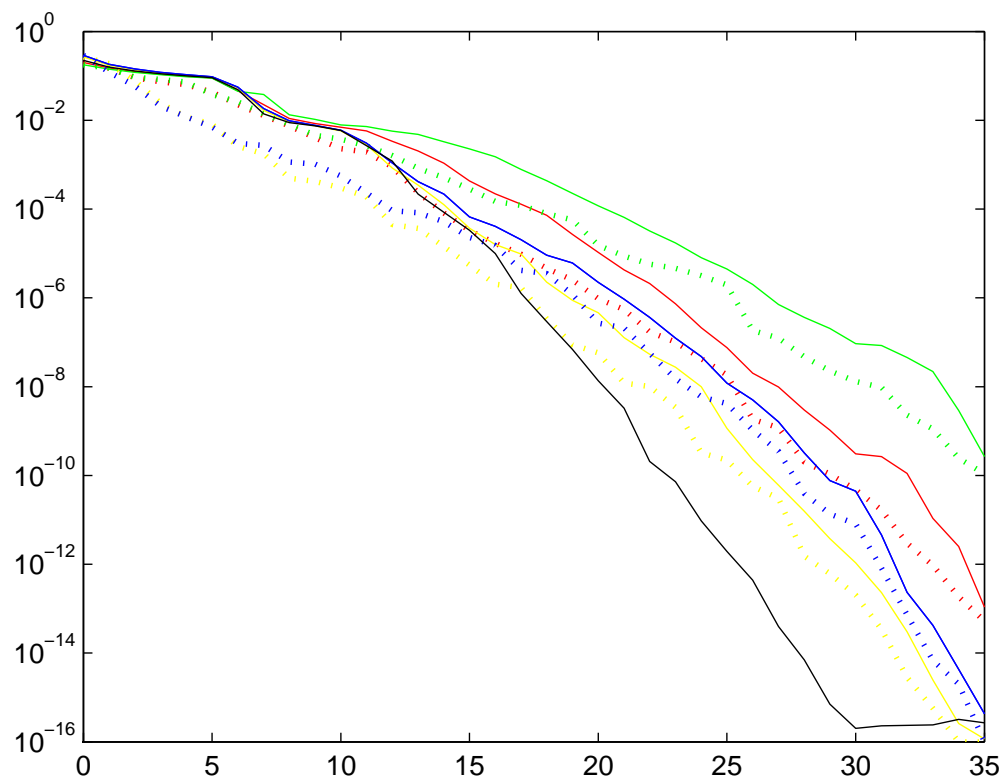


Red: δ_2 , green: δ_1 , yellow: δ_2^+ , blue: δ_3^+ .



4. Bounds for the general step

GMRES for the convection-diffusion model problem



Black: δ_0 , red: δ_2 , green: δ_1 , yellow: δ_2^+ , blue: δ_3^+ .



4. Bounds for the general step

We can extend to the less accurate but more descriptive bound

$$\begin{aligned}\|r_k\| &\geq s_{\min} \left(W_\delta \Theta_\delta L_\delta^{-1} \Pi_\delta^{(n)} \begin{pmatrix} I_k \\ 0 \end{pmatrix} \right) \left\| e_1^T \cdot U_\delta^{(k)} \Pi_\delta^{(k)} \right\|^{-1} \\ &\geq \left\| (\Pi_\delta^{(n)})^{-1} L_\delta \right\|^{-1} \left\| \Theta_\delta^{-1} W_\delta^{-1} \right\|^{-1} \left\| e_1^T \cdot U_\delta^{(k)} \Pi_\delta^{(k)} \right\|^{-1} .\end{aligned}$$

$\Theta_\delta^{-1} W_\delta^{-1}$ is nearly independent from the choice of δ ; the matrix $(\Pi_\delta^{(n)})^{-1} L_\delta$ is lower triangular with a clear decay when moving away from the diagonal. Essential is that $\left\| e_1^T \cdot U_\delta^{(k)} \Pi_\delta^{(k)} \right\|$ where

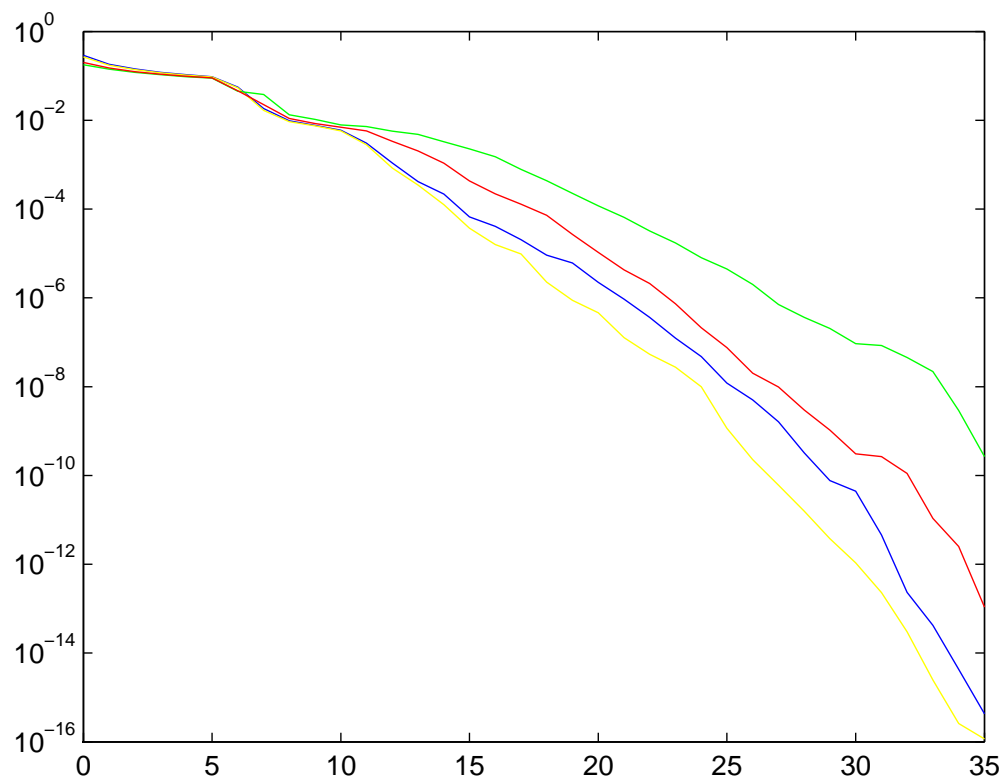
$$\Pi_\delta^{(k)} = \text{diag} \left(1, \frac{1}{\prod_{j \neq 1}^2 |\sigma_1 - \sigma_j|}, \frac{1}{\prod_{j \neq 2}^3 |\sigma_2 - \sigma_j|}, \dots, \frac{1}{\prod_{j \neq k-1}^k |\sigma_{k-1} - \sigma_j|} \right)$$

is heavily decreasing when growing away from δ_0 ; hence the **main reason for convergence delay are again growing distances between eigenvalues.**



4. Bounds for the general step

GMRES for the convection-diffusion model problem

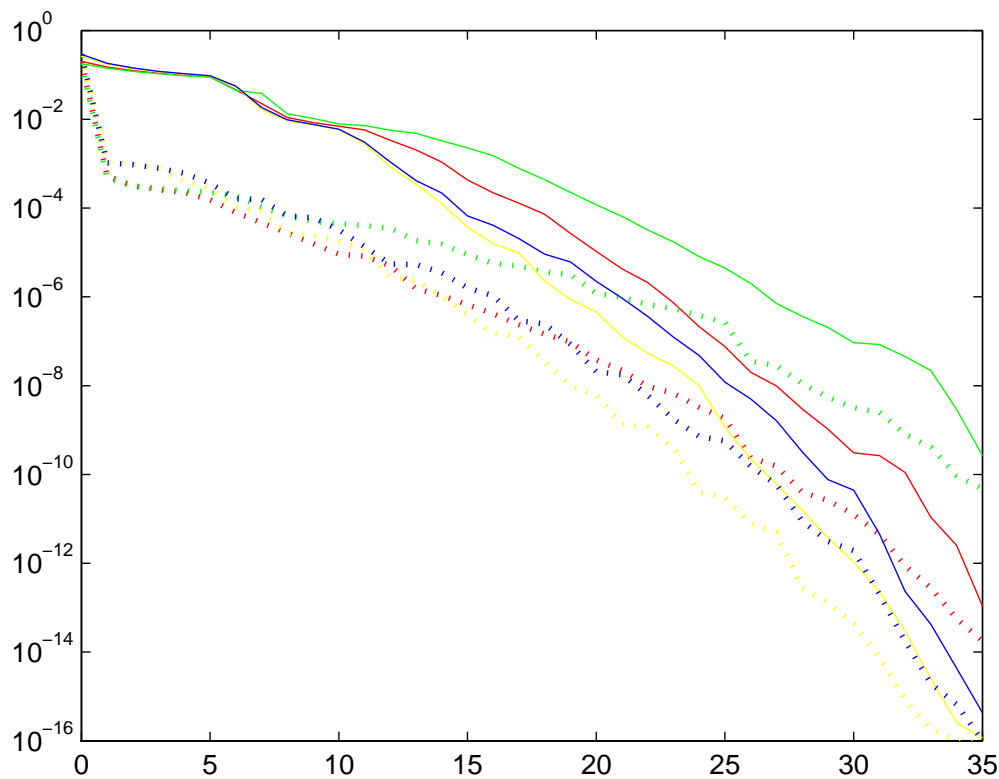


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4. Bounds for the general step

GMRES for the convection-diffusion model problem

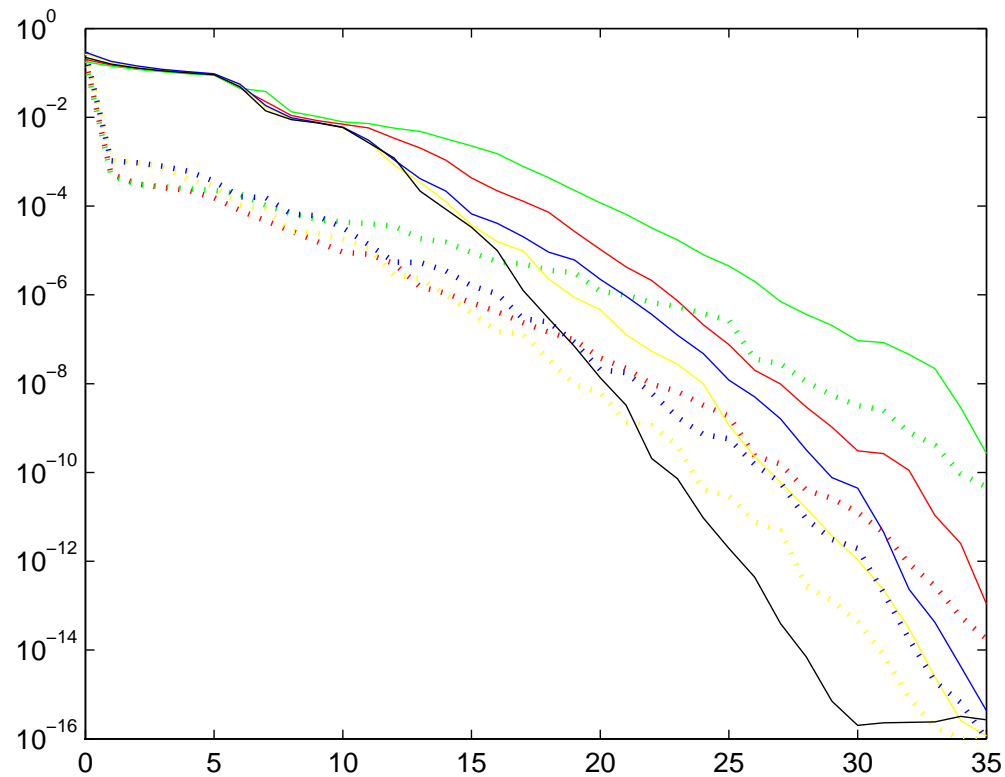


Red: δ_2 , green: δ_1 , yellow: δ_2^+ , blue: δ_3^+ .



4. Bounds for the general step

GMRES for the convection-diffusion model problem



Black: δ_0 , red: δ_2 , green: δ_1 , yellow: δ_2^+ , blue: δ_3^+ .



5. Conclusions

Thank you for your attention.

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