Linear equations in quaternionic variables

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Basic definitions for quaternions

$\mathbb{H} := \mathbb{R}^4$ ... the skew field of quaternions
Let

$$x = (x_1, x_2, x_3, x_4), \quad y = (y_1, y_2, y_3, y_4) \in \mathbb{H}.$$

Then

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$$

and

$$\begin{aligned} xy &= (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4,\ x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3, \\ &\quad x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2,\ x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1). \end{aligned}$$

- The first component x_1 ... the real part of x, denoted by $\Re x$.
- The second component x_2 ... the imaginary part of x, denoted by $\Im x$.
- $x = (x_1, 0, 0, 0)$ will be identified with $x_1 \in \mathbb{R}$
- $x = (x_1, x_2, 0, 0)$ will be identified with $x_1 + \mathrm{i}x_2 \in \mathbb{C}$
- The conjugate of x will be defined by $\overline{x} = (x_1, -x_2, -x_3, -x_4)$
- The absolute value of x will be defined by $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$
- The inverse quaternion is defined as $x^{-1} = \dfrac{\overline{x}}{|x|^2}$ for $x \in \mathbb{H}\backslash\{0\}$

Classes of equivalence

Two quaternions x and y are called equivalent, $\mathbf{x} \sim \mathbf{y}$,
if there is $h \in \mathbb{H}\backslash\{0\}$ such that $y = h^{-1}xh$.

$$[x] = \{y \in \mathbb{H} : \; y = h^{-1}xh \;\text{ for }\; h \in \mathbb{H}\backslash\{0\}\} \quad \ldots \quad \text{equivalence class of } x$$

Lemma. Two quaternions x and y are equivalent if and only if

$$\Re x = \Re y \quad \text{and} \quad |x| = |y|.$$

$$a \in \mathbb{R} \implies [a] = \{a\}$$
$$c \in \mathbb{C} \implies c \in [c], \; \overline{c} \in [c]$$

Corollary. Let $x = (x_1, x_2, x_3, x_4)$. Then

$$\widetilde{x} = (x_1, \sqrt{x_2^2 + x_3^2 + x_4^2}, 0, 0) = x_1 + |x_v|\,\mathrm{i} \in [x]$$

is the only complex element in $[x]$ with non negative imaginary part.

Quaternionic linear mappings

A mapping

$$L : \mathbb{H} \to \mathbb{H}$$

is called a quaternionic linear mapping over $\mathbb{R}$ if

$$L(\gamma x + \delta y) = \gamma L(x) + \delta L(y) \text{ for all } \; x, y \in \mathbb{H}, \; \gamma, \delta \in \mathbb{R}.$$

Let us remark that

$$L(0) = 0\,,$$

$$L(\alpha x) \neq \alpha L(x)\,, \qquad L(x\alpha) = L(x)\alpha \quad \text{for } \alpha \in \mathbb{H} \setminus \mathbb{R}$$

The four unit vectors in $\mathbb{H}$ are denoted by

$$e_1 := (1,0,0,0) =: \mathbf{1},\; e_2 := (0,1,0,0) =: \mathbf{i},\; e_3 := (0,0,1,0) =: \mathbf{j},\; e_4 := (0,0,0,1) =: \mathbf{k}.$$

Definition A quaternionic linear mapping L is called singular if there is $x \neq 0$ with $L(x) = 0$.
The mapping L is called non singular if it is not singular.

Theorem Let L be a quaternionic linear mapping.
Then, there exists a matrix $\mathbf{M} \in \mathbb{R}^{4\times 4}$ such that

$$L(x) = \mathbf{M}\mathbf{x},$$

where $x,\ L(x)$ have to be identified with the corresponding column vectors $\mathbf{x}, \mathbf{M}\mathbf{x}$ and (with the same identification)

$$\mathbf{M} := (L(e_1), L(e_2), L(e_3), L(e_4))\,.$$

Lemma A quaternionic linear mapping L is singular if and only if

$$\det \mathbf{M} = 0\,.$$

Definition Let $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}$.
Let us introduce two mappings $\tau_1, \tau_2 : \mathbb{H} \longrightarrow \mathbb{R}^{4\times 4}$ by

$$\tau_1(a) \ := \ (ae_1, ae_2, ae_3, ae_4) := \mathbf{A} := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix},$$

$$\tau_2(a) \ := \ (e_1a, e_2a, e_3a, e_4a) := \widetilde{\mathbf{A}} := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix}.$$

Denote by $\mathbb{H}_{\mathbb{R}}$ all matrices of the type $\mathbf{A}$, by $\mathbb{H}_{\mathbb{P}}$ all matrices of the type $\widetilde{\mathbf{A}}$.
$\mathbb{H}_{\mathbb{P}}$ are called pseudo-quaternions.

Remark $\mathbb{H}_{\mathbb{R}}$ is isomorphic to $\mathbb{H}$.

Lemma Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}$, $b = (b_1, b_2, b_3, b_4) \in \mathbb{H}$, $col(a) = (a_1, a_2, a_3, a_4)^{\mathrm{T}}$ be the column vector consisting of the four components of a.

Then for $a, b, c \in \mathbb{H}$ we have

(1) $\tau_2(ab) = \tau_2(b)\tau_2(a)$,

(2) $\tau_1(a)\tau_2(b) = \tau_2(b)\tau_1(a)$,

(3) $col(ab) = \tau_1(a)col(b) = \tau_2(b)col(a)$,

(4) $col(abc) = \tau_1(a)\tau_1(b)col(c) = \tau_2(c)\tau_2(b)col(a)$,

(5) $col(abc) = \tau_1(a)\tau_2(c)col(b)$.

Theorem Let $L^{(1)}(x) := axb$ for arbitrary $a, b, x \in \mathbb{H}$.
Then, there is a matrix $\mathbf{M} \in \mathbb{R}^{4\times 4}$ such that

$$Ł^{(1)}(x) := axb = \mathbf{M}\mathbf{x}, \quad \text{and} \quad \mathbf{M} = \tau_1(a)\tau_2(b)\,.$$

Corollary Let $L^{(m)}(x) := \sum_{j=1}^{m} a^{(j)} x b^{(j)}$ for arbitrary $a^{(j)}, b^{(j)}, x \in \mathbb{H}$ be given.
Then, there is a matrix $\mathbf{M} \in \mathbb{R}^{4\times 4}$ such that

$$L^{(m)}(x) \;:=\; \sum_{j=1}^{m} a^{(j)} x b^{(j)} = \mathbf{M}\mathbf{x}, \;\text{where}$$

$$\mathbf{M} \;:=\; \sum_{j=1}^{m} \mathbf{M}_j, \quad \text{and} \quad \mathbf{M}_j := \tau_1(a^{(j)})\tau_2(b^{(j)})\,.$$

The singular cases are exactly those with vanishing determinant of $\mathbf{M}$.
Apart from special cases it seems, however, to be impossible to characterize the singular cases just by the given quaternions $a^{(j)}, b^{(j)},\ j = 1, 2, \ldots, m$.

Linear equations in one variable

Let $m \in \mathbb{N}$, $\mathbf{a} = (a^{(1)}, a^{(2)}, \ldots, a^{m})$, $\mathbf{b} = (b^{(1)}, b^{(2)}, \ldots, b^{(m)}) \in \mathbb{H}^m$,
$a^{(j)}, b^{(j)} \in \mathbb{H}\backslash\{0\}$, $j = 1, \ldots, m$, $e \in \mathbb{H}$ be given

? $x \in \mathbb{H}$:

$$L^{(m)}(x) := \sum_{j=1}^{m} a^{(j)} x b^{(j)} = e$$

$L^{(m)} : \mathbb{H} \to \mathbb{H}$ is the quaternionic linear mapping.

Assumption: $a^{(1)} = b^{(m)} = 1$, $a, b, c, d, x \in \mathbb{H}$

$$L^{(1)}(x) := x\,, \quad L^{(2)}(x) := ax + xb\,, \quad L^{(3)}(x) := ax + cxd + xb\,, \quad \ldots$$

$c, d \in \mathbb{H} \setminus \{\mathbb{R}\}$ $\Longrightarrow$ **cxd** ... **middle terms**

Lemma

Let $b = (b_1, b_2, b_3, b_4),\ c = (c_1, c_2, c_3, c_4)$.

Equation $\mathbf{bxc = e}$ has a unique solution for all choices of e if and only if $bc \neq 0$.

In this case the solution is

$$x = b^{-1} e c^{-1}$$

Corollary

Let a, b, c, d, e be given quaternions, $abcd \neq 0$.

Let $\widetilde{\mathbf{L}} := \mathbf{axb + cxd}$, and solve $\widetilde{\mathbf{L}}(\mathbf{x}) = \mathbf{e}$, $x \in \mathbb{H}$.

The equation $\widetilde{L}(x) = e$ has a unique solution for all choices of e if and only if

$$\Re(a^{-1}c) + \Re(bd^{-1}) \neq 0 \quad \text{or} \quad \sum_{j=2}^{4} ((a^{-1}c)_j^2 - (bd^{-1})_j^2) \neq 0,$$

where the subscript j defines the component number.

Sylvester's equation in quaternions:

$$L^{(2)}(x) := ax + xb = e\,,\ a \notin \mathbb{R}\,,\ b \notin \mathbb{R}$$

Let $\mathbf{L^{(2)}(x) := ax + xb}$, $\quad a, b \notin \mathbb{R}\,.$
Then, $L^{(2)}$ is singular if and only if

$$(1) \qquad |a| = |b| \qquad \text{and} \qquad \Re(a) + \Re(b) = 0\,.$$

or in other words if and only if $a \ \sim \ -b\,.$

The nullspace of $L^{(2)}$ is

$$\mathcal{N} = \begin{cases} \{0\} & \text{if (1) is not valid}\,, \\ \mathbb{H} & \text{if (1) is valid and } \ a, b \in \mathbb{R}\,, \\ 2-\text{dim} \ \text{subspace of} \ \mathbb{H} & \text{if (1) is valid and } a \notin \mathbb{R} \text{ or } b \notin \mathbb{R}\,. \end{cases}$$

The equivalent matrix representation:

$$L^{(2)}(x) := ax + xb = \mathbf{M}\mathbf{x}\,, \qquad \text{where}$$

$$\mathbf{M} = \tau_1(a)\tau_2(1) + \tau_1(1)\tau_2(b) = \begin{pmatrix} s_1 & -s_2 & -s_3 & -s_4 \\ s_2 & s_1 & -d_4 & d_3 \\ s_3 & d_4 & s_1 & -d_2 \\ s_4 & -d_3 & d_2 & s_1 \end{pmatrix}$$

and

$$s = a + b = (s_1, s_2, s_3, s_4)\,, \qquad d = a - b = (d_1, d_2, d_3, d_4)\,.$$

The conditions (1) are equivalent to det $\mathbf{M} = 0$, where

$$\det(\mathbf{M}) = s_1^2(|s|^2 + d_2^2 + d_3^2 + d_4^2) + (s_2 d_2 + s_3 d_3 + s_4 d_4)^2\,.$$

Theorem Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$.
The equation $\mathbf{L^{(2)}(x) = e}$ has a unique solution for all choices of e if and only if

$$a_1 + b_1 \neq 0 \qquad \text{or} \quad \sum_{j=2}^{4} (a_j^2 - b_j^2) \neq 0\,.$$

In this case the solution is

$$x = f_l^{-1}(e + a^{-1}e\bar{b}), \quad f_l := 2\Re b + a + |b|^2 a^{-1} \quad \text{if } a \neq 0$$

or

$$x = (e + \overline{a}eb^{-1})f_r^{-1}, \quad f_r := 2\Re a + b + |a|^2 b^{-1} \quad \text{if } b \neq 0.$$

Remark Numerical computations:
If $|a| > 0$ but close to zero, avoid the first formula containing a^{-1}.
If $|a| \leq |b|$ the use of the second formula is recommended, otherwise use the first one.

Linear equations of general type in one quaternionic variable

Theorem Let $\mathbf{L^{(m)}(x) := \sum_{j=1}^{m} a^{(j)} x b^{(j)}}$ for arbitrary $a^{(j)}, b^{(j)}, x \in \mathbb{H}$ be given.

Then $\mathbf{L^{(m)}(x) = e}$ is equivalent to the $(4 \times 4)-$matrix equation

$$\left(\sum_{j=1}^{n} \tau_1(a^{(j)}) \tau_2(b^{(j)}) \right) col(x) = col(e).$$

Corollary The linear function $L^{(m)}$ is singular if and only if

$$\det \left(\sum_{j=1}^{n} \tau_1(a^{(j)}) \tau_2(b^{(j)}) \right) = 0\,.$$

Banach's fixed point theorem $\Longrightarrow$ **sufficient condition for nonsingularity**

Theorem Let $\mathbf{L^{(m)}(x)} := \sum_{\mathbf{j=1}}^{\mathbf{m}} \mathbf{a^{(j)} x b^{(j)}}$ with $m \geq 3$, $a^{(j)}, b^{(j)} \in \mathbb{H}$, $a^{(j)} b^{(j)} \neq 0$ for all $j = 1, 2, \ldots, m$. If there is j_0, $1 \leq j_0 \leq m$, such that

$$\kappa := \frac{\sum_{j \neq j_0}^{m} |a^{(j)}||b^{(j)}|}{|a^{(j_0)}||b^{(j_0)}|} \; < \; 1 \,,$$

then $L^{(m)}$ is nonsingular.

Corollary Let $\mathbf{L^{(m)}(x)} := \sum_{\mathbf{j=1}}^{\mathbf{m}} \mathbf{a^{(j)} x b^{(j)}}$ with $m \geq 3$, $a^{(j)}, b^{(j)} \in \mathbb{H}$, $a^{(j)} b^{(j)} \neq 0$ for all $j = 1, 2, \ldots, m$. If there is a constant $k > 0$ and an index j_0, $1 \leq j_0 \leq m$, such that $|a^{(j)}||b^{(j)}| \leq k$ for all $j \neq j_0$ and $|a^{(j_0)}||b^{(j_0)}| \geq k^2$, then $L^{(m)}$ is nonsingular if $k > m - 1$. For $k = m - 1$ this is not necessarily true.

Theorem Let $f_n(x) := \sum_{j=1}^{n} a^{(j)} x b^{(j)}$ with quaternionic entries.
If $n \geq 3$ the equation $f_n(x) = c$ cannot be, in general, treated by using only quaternionic algebra.

Proof The mapping f_n is a linear mapping over $\mathbb{R}$ and as such it is representable in the isomorphic form $f_n(x) = \mathbf{A}\mathbf{x}$, where $\mathbf{x}$ is now to be understood as a (4×1) column vector and $\mathbf{A}$ is a real (4×4) matrix.
This equation could be treated by means of quaternionic algebra only if $\mathbf{A}$ would be isomorphic to a quaternion. For $n \geq 3$ it is not the case, since

$$a^{(j)} x b^{(j)} = \mathbf{A}_j x = \tau_1(a^{(j)}) \tau_2(b^{(j)}) x \,,$$

where $\tau_1(a^{(j)}) \tau_2(b^{(j)})$ is (in general) not isomorphic to $\mathbb{H}$.

For $n \leq 2$ it is possible to make some premultiplications such that $\mathbf{A}$ remains in $\mathbb{H}$.

Remark The term "cannot be treated" means the following:
It is neither possible to detect whether the equation has a solution at all nor it is possible to solve that equation if one knows that there is a solution.

Linear systems in quaternionic variables

$$\begin{aligned} xa + by &= f, \\ cx + dy &= g \end{aligned}$$

We apply the column operator *col* and obtain

$$\begin{aligned} col(xa) + col(by) &= \tau_2(a)col(x) + \tau_1(b)col(y) &= col(f)\,, \\ col(cx) + col(dy) &= \tau_1(c)col(x) + \tau_1(d)col(y) &= col(g)\,. \end{aligned}$$

In real terms:

$$\mathbf{A}\mathbf{z} = \mathbf{h}\,,$$

where

$$\mathbf{A} = \begin{pmatrix} \tau_2(a) & \tau_1(b) \\ \tau_1(c) & \tau_1(d) \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} col(x) \\ col(y) \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} col(f) \\ col(g) \end{pmatrix}.$$

Example

$$\begin{aligned} x\mathbf{k} + \mathbf{j}y &= (-11, 11, 3, -5), \\ \mathbf{i}x + (\mathbf{1} + \mathbf{k})y &= (-5, 0, 9, 16) \end{aligned}$$

In real terms:

$$\mathbf{A}\mathbf{z} \;=\; \mathbf{h}\,,$$

where

$$\mathbf{A} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} \begin{pmatrix} -11 \\ 11 \\ 3 \\ -5 \end{pmatrix} \\ \begin{pmatrix} -5 \\ 0 \\ 9 \\ 16 \end{pmatrix} \end{pmatrix}.$$

Solution:

$$x = (1, 2, 3, 4)\,, \quad y = (5, 6, 7, 8)\,.$$

Theorem The quaternionic $n \times n$ system

$$\sum_{k=1}^{n} L_{n_{jk}}^{(jk)}(x_k) = c^{(j)}, \quad j = 1, 2, \ldots, n\,,$$

is equivalent to the real $(4n \times 4n)$ system

$$\mathbf{A}\mathbf{z} \;=\; \mathbf{c}\,,$$

where

$$\mathbf{A} := (\mathbf{a}_{jk}),\;\; j,k = 1,2,\ldots,n, \quad \mathbf{z} := \begin{pmatrix} col(x_1) \\ col(x_2) \\ \vdots \\ col(x_n) \end{pmatrix}, \quad \mathbf{c} := \begin{pmatrix} col(c^{(1)}) \\ col(c^{(2)}) \\ \vdots \\ col(c^{(n)}) \end{pmatrix},$$

and where

$$\mathbf{a}_{jk} = \sum_{p=1}^{n_{jk}} \tau_1(a_p^{(jk)})\tau_2(b_p^{(jk)}) \in \mathbb{R}^{4\times 4}\,.$$

The Kronecker product for quaternionic systems

Let

$$\mathbf{L}(\mathbf{X}) := \mathbf{AXB}\,,\ \ \mathbf{A} \in \mathbb{H}^{J\times K}\,,\ \ \mathbf{X} \in \mathbb{H}^{K\times L}\,,\ \ \mathbf{B} \in \mathbb{H}^{L\times M}\,,$$

$$\text{i.e.} \quad \mathbf{L} : \mathbb{H}^{K\times L} \longrightarrow \mathbb{H}^{J\times M}\,.$$

It may be regarded as a linear real mapping $\mathbf{L} : \mathbb{R}^{4KL} \longrightarrow \mathbb{R}^{4JM}$.

Thus, there is a real matrix $\mathbf{\Pi}(\mathbf{A}, \mathbf{B}) \in \mathbb{R}^{4JM\times 4KL}$ such that

$$col(\mathbf{L}(\mathbf{X})) \; = \; \mathbf{\Pi}(\mathbf{A}, \mathbf{B})col(\mathbf{X})\,.$$

$\mathbf{\Pi}(\mathbf{A}, \mathbf{B})$ ... Kronecker product for $\mathbf{AXB}$

Conclusions

- We study the quaternionic linear system of the form

$$L^{(m)}(x) := \sum_{j=1}^{m} a^{(j)} x b^{(j)} = e$$

  for arbitrary $a^{(j)}, b^{(j)}, e, x \in \mathbb{H}$:
  one equation in one variable, Sylvester's equation, $m \geq 3$.

- By making use of a fixed point theorem, we obtained sufficient conditions for non singularity.

- The general case of a linear quaternionic system is treated.

- An analog of the Kronecker product for quaternionic systems of the form $\mathbf{AXB}$ is given.

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