The Spectrum of a Glued Matrix

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Outline

1. What is a glued matrix?
2. Why care about glued matrices?
3. Basic mathematical formulation of gluing
4. Rational functions and secular equations
5. Approximative distribution of eigenvalues
6. Summary

Literature

This talk is essentially advertising for the following papers:

- I. Dhillon, B. Parlett, and C. V. *Glued matrices and the MRRR algorithm*, *SIAM J. Sci. Comp.*, 27(2):496-510, 2005.
- B. Parlett and C. V. *The spectrum of a glued matrix* To appear in *SIAM J. Matrix Anal. Appl.*

What is a glued matrix?

1. What is a glued matrix?

Starting point: a block-diagonal matrix

$$\left[\begin{array}{cccc|cccc}
\ddots & \ddots & \ddots & & & & & \\
& * & * & * & & & & \\
& & * & * & \mathbf{0} & & & \\ \hline
& & & \mathbf{0} & * & * & & \\
& & & & * & * & * & \\
& & & & & \ddots & \ddots & \ddots
\end{array}\right] \tag{1}$$

- (1,1) and (2,2) blocks: unreduced symmetric tridiagonals T_1, T_2, each with simple eigenvalues
- if $T_1 = T_2$, then each eigenvalue from the spectrum gets replicated
- Q: what happens when one makes the block-diagonal matrix unreduced?

Rank-1 and rank-2 gluing, glue $\gamma \neq 0$

$$G_1 = \left[\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & & & & \\ & * & * & * & & & & \\ & & * & * + \gamma & \gamma & & & \\ \hline & & & \gamma & * + \gamma & * & & \\ & & & & * & * & * & \\ & & & & & \ddots & \ddots & \ddots \end{array}\right], \qquad (2)$$

$$G_2 = \left[\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & & & & \\ & * & * & * & & & & \\ & & * & * & \gamma & & & \\ \hline & & & \gamma & * & * & & \\ & & & & * & * & * & \\ & & & & & \ddots & \ddots & \ddots \end{array}\right]. \qquad (3)$$

Why care about glued matrices?

1. What is a glued matrix?
2. Why care about glued matrices?

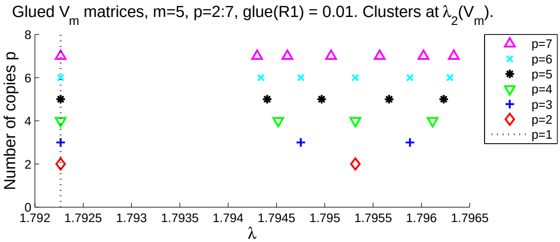
A preliminary answer

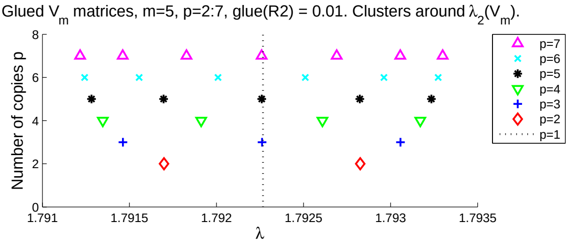
First answer (now): good test matrices, eigenvalues can agree to hundreds of figures

Table 1: Repeated refinement of eigenvalues of the second-smallest cluster of a glued matrix (5 copies of one tridiagonal).

Index	Level 0	Level 1	Level 21
6	0.253805817096679	1.0524021997591E-13	1.14585459648E-312
7	0.253805817096679	1.0524021997591E-13	1.14585459649E-312
8	0.253805817096679	1.0524021997591E-13	1.14585459651E-312
9	0.253805817096679	1.0524021997591E-13	1.14585459653E-312
10	0.253805817096679	1.0524021997591E-13	1.14585459654E-312

A second answer: interesting spectral properties


Glued V_m matrices, m=5, p=2:7, glue(R1) = 0.01. Clusters at $\lambda_2(V_m)$.


Glued V_m matrices, m=5, p=2:7, glue(R2) = 0.01. Clusters around $\lambda_2(V_m)$.

Issues to investigate

Issues to investigate:

- Differences between rank-1 and rank-2 gluing
- Dependence on T, on p, and on γ
- Interlacing properties
- Eigenvalue repetition

Basic mathematical formulation of gluing

1. What is a glued matrix?
2. Why care about glued matrices?
3. Basic mathematical formulation of gluing

Rank-1 and rank-2 updates

Definition

For an unreduced tridiagonal T, $p \geq 2$, and $\gamma \neq 0$, let $T^{(p)} := \mathrm{diag}(T, T, \ldots, T)$ and

$$\begin{aligned} G_1(T,p,\gamma) &:= T^{(p)} + \gamma(\hat{E}_1 + \hat{E}_n)(\hat{E}_1 + \hat{E}_n)^t, \qquad (4) \\ G_2(T,p,\gamma) &:= T^{(p)} + \gamma(\hat{E}_n \hat{E}_1^t + \hat{E}_1 \hat{E}_n^t). \qquad (5) \end{aligned}$$

Auxiliary matrices (each with $p-1$ columns and np rows)

$$\hat{E}_n := \begin{bmatrix} e_n & & & \\ 0 & e_n & & \\ & 0 & \ddots & \\ & & \ddots & \ddots & \\ & & & \ddots & e_n \\ & & & & 0 \end{bmatrix}, \quad \hat{E}_1 := \begin{bmatrix} 0 & & & \\ e_1 & 0 & & \\ & e_1 & \ddots & \\ & & \ddots & \ddots & \\ & & & \ddots & 0 \\ & & & & e_1 \end{bmatrix}. \qquad (6)$$

Interlacing

Inductive construction of $G_1(T, p+1, \gamma)$: rank-1 modification of the direct sum S_1 of $G_1(T, p, \gamma)$ and T (same for G_2)

Theorem

Let $\gamma > 0$, and $\bar{n} := (p+1) \cdot n$. Then

$$\lambda_i\left(G_1(T, p+1, \gamma)\right) \in \begin{cases} [\lambda_i(S_1), \lambda_{i+1}(S_1)], & i \neq \bar{n}, \\ [\lambda_n(S_1), \lambda_n(S_1) + 2\gamma], & i = \bar{n}. \end{cases} \tag{7}$$

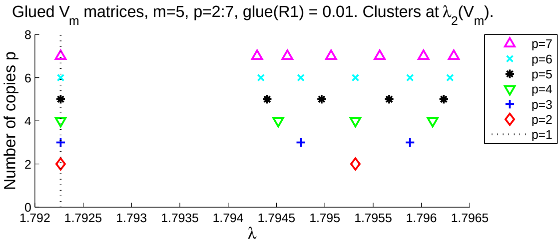
$$\lambda_i\left(G_2(T, p+1, \gamma)\right) \in [\lambda_{i-1}(S_2), \lambda_{i+1}(S_2)], \; i \neq 1, \bar{n}. \tag{8}$$

Conclusion: eigenvalue repetition for G_1

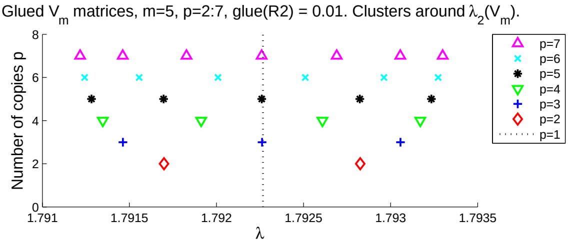
Theorem

Any eigenvalue λ of a real unreduced $n \times n$ symmetric tridiagonal T is also an eigenvalue of the rank-1 glued matrix $G_1(T, p, \gamma)$, for any γ and for $p \geq 2$.

Re-inspection: interlacing & eigenvalue repetition



Glued V_m matrices, m=5, p=2:7, glue(R1) = 0.01. Clusters at $\lambda_2(V_m)$.



Glued V_m matrices, m=5, p=2:7, glue(R2) = 0.01. Clusters around $\lambda_2(V_m)$.

Eigenvalue repetition for rank-2 gluing

More complicated scenario for G_2:
need M'': obtained from M by deleting the first/last rows/columns

Theorem

Let λ, v denote an eigenpair of a real unreduced $n \times n$ symmetric tridiagonal T.

- *λ is an eigenvalue of $G_2(T, p, \gamma)$ for all $\gamma \neq 0$ if and only if (λ, w'') is an eigenpair of $[G_2(T, p-2, \gamma)]''$.*
- *If the previous condition holds for a certain p, then λ also is eigenvalue of $G_2(T, q, \gamma)$, for $q = p + (p-1), p + 2(p-1), \ldots$.*

(Proof omitted, see SIMAX paper)

Rational functions and secular equations

1. What is a glued matrix?
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4. Rational functions and secular equations

Spectral decomposition

Let $T = V \Lambda V^t$ then

$$\begin{aligned} G_1 &= V^{(p)} \left[\Lambda^{(p)} + \gamma (F+L)(F+L)^t\right] V^{(p)t}, && (9) \\ G_2 &= V^{(p)} \left[\Lambda^{(p)} + \gamma (LF^t + FL^t)\right] V^{(p)t}. && (10) \end{aligned}$$

where

$$F = V^{(p)t} \hat{E}_1 = \begin{bmatrix} 0 & & \\ f & \cdot & \\ & \cdot & 0 \\ & & f \end{bmatrix}, \quad L := V^{(p)t} \hat{E}_n = \begin{bmatrix} l & & \\ 0 & \cdot & \\ & \cdot & l \\ & & 0 \end{bmatrix}.$$

('F/L represent first/last eigenvector rows')

Some determining determinants

$$\begin{aligned} G_1 &= V^{(p)}\left[\Lambda^{(p)} + \gamma(F+L)(F+L)^t\right] V^{(p)t}, && (11)\\ G_2 &= V^{(p)}\left[\Lambda^{(p)} + \gamma(LF^t + FL^t)\right] V^{(p)t}. && (12)\end{aligned}$$

For any ζ that is not an eigenvalue of T:

$$\begin{aligned}\det[G_1 - \zeta I] &= \det\left[\Lambda^{(p)} - \zeta I\right] \det\left[I_{(p-1)} + \gamma(\Lambda^{(p)} - \zeta I)^{-1}(F+L)(F+L)^t\right]\\ &= \det[\Lambda - \zeta I]^p \det\left[I_{(p-1)} + \gamma(F+L)^t(\Lambda^{(p)} - \zeta I)^{-1}(F+L)\right],\end{aligned}$$

and

$$\begin{aligned}\det[G_2 - \zeta I] &= \det\left[\Lambda^{(p)} - \zeta I\right] \det\left[I_{np} + \gamma(\Lambda^{(p)} - \zeta I)^{-1}(LF)\begin{pmatrix} F^t \\ L^t \end{pmatrix}\right]\\ &= \det[\Lambda - \zeta I]^p \det\left[I_{2(p-1)} + \gamma\begin{pmatrix} F^t \\ L^t \end{pmatrix}(\Lambda^{(p)} - \zeta I)^{-1}(LF)\right].\end{aligned}$$

Secular equations

$$
\begin{aligned}
0 &= \Gamma_1(T,p,\gamma)(\zeta) := \det\left[I_{(p-1)} + \gamma(F+L)^t(\Lambda^{(p)} - \zeta I)^{-1}(F+L)\right],\\
0 &= \Gamma_1(T,p,\gamma)(\zeta) := \det\left[I_{2(p-1)} + \gamma\begin{pmatrix} F^t \\ L^t \end{pmatrix}(\Lambda^{(p)} - \zeta I)^{-1}(LF)\right].
\end{aligned}
$$

Need auxiliary rational functions $\rho_{ff}, \rho_{fl}, \rho_{ll}$:

$$
\rho_{xy}(\zeta) := x^t(\Lambda - \zeta I)^{-1}y = \sum_{j=1}^{n} \frac{x_j y_j}{\lambda_j - \zeta}. \tag{13}
$$

Surprising finding:

$$
\begin{aligned}
\Gamma_1(T,p,\gamma)(\zeta) &= \det\left[\mathrm{Toep}_{p-1}\left(\gamma\rho_{fl}, 1 + \gamma(\rho_{ff} + \rho_{ll}), \gamma\rho_{fl}\right)\right] \qquad (14)\\
\Gamma_2(T,p,\gamma)(\zeta) &= \det\left[\mathrm{Toep}_{p-1}\left(\gamma\rho_{fl}, 1 - \gamma^2(\rho_{ff}\rho_{ll} - \rho_{fl}^2), \gamma\rho_{fl}\right)\right.\\
&\qquad \left. - (\gamma\rho_{fl})^2 e_1 e_1^t\right]. \qquad (15)
\end{aligned}
$$

Approximative distribution of eigenvalues

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Toeplitz matrices of rational functions

$$\begin{aligned}\Gamma_1(T,p,\gamma)(\zeta) &= \det\left[\mathrm{Toep}_{p-1}\left(\gamma\rho_{fl}, 1+\gamma(\rho_{ff}+\rho_{ll}), \gamma\rho_{fl}\right)\right]\\ \Gamma_2(T,p,\gamma)(\zeta) &= \det\left[\mathrm{Toep}_{p-1}\left(\gamma\rho_{fl}, 1-\gamma^2(\rho_{ff}\rho_{ll}-\rho_{fl}^2), \gamma\rho_{fl}\right)\right.\\ &\qquad \left. -(\gamma\rho_{fl})^2 e_1 e_1^t\right].\end{aligned}$$

Need spectral decomposition of a tridiagonal Toeplitz matrix:

$$\mathrm{Toep}\,(a,b,a) = SDS^t,\; D = \mathrm{diag}\left(b + 2a\,\cos(\frac{j\pi}{m+1})\right), \qquad (16)$$

and S is orthonormal with

$$s_{ij} = \sqrt{\frac{2}{m+1}}\sin\left(\frac{ij\,\pi}{m+1}\right), i,j = 1,\ldots,m. \qquad (17)$$

In our case, the eigenvalues of Γ_1, Γ_2 are rational functions of ζ!

Distribution of zeros for rank-1 gluing

Express determinant of Toeplitz matrix as product of eigenvalues.

Theorem

Those eigenvalues of $G_1(T, p, \gamma)$ that are not eigenvalues of T are zeros of the rational function

$$\Gamma_1(T, p, \gamma) = \prod_{k=1}^{p-1} \{1 + \gamma\big(\rho_{ff}(\zeta) + \rho_{ll}(\zeta) + 2\rho(\zeta)\cos(k\pi/p)\big)\}. \quad (18)$$

Corollary

When λ_j is isolated, then the eigenvalues of G_1 close to λ_j, other than λ_j itself, are to first order in γ,

$$\lambda_j + \gamma\{f_j^2 + l_j^2 + 2(-1)^{n-j}\,|f_j l_j|\cos(k\pi/p)\},\ k = 1, 2, \cdots, p-1. \quad (19)$$

Distribution of poles for rank-2 gluing

Theorem

The eigenvalues of $G_2(T, p, \gamma)$ that are not eigenvalues of T are zeros of the rational function

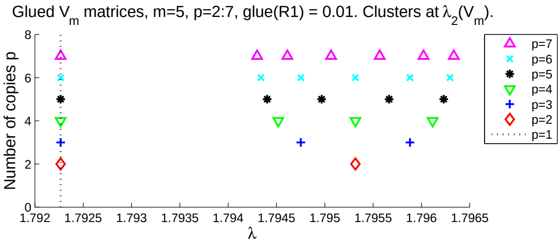
$$\Gamma_2(T, p, \gamma) = \det\left[Toep_{p-1}\{\gamma\rho, 1 - \gamma^2(\rho_{ff}\rho_{ll} - \rho^2), \gamma\rho\} - (\gamma\rho)^2 e_1 e_1^t \right]. \tag{20}$$

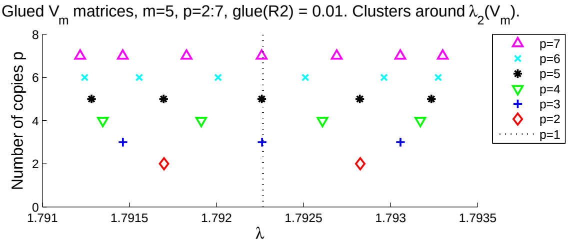
Insights from rank-1 updating theory:

- In this case, the poles have a Chebyshev distribution.
- Rank-1 update shifts eigenvalues away from the poles.

Technical details are complicated, see SIMAX paper.
No simple nice expressions like for rank-1 gluing.

Chebyshev distributions of zeros & poles





Summary: rank-1 and rank-2 gluing

- Good test matrices with very tight eigenvalue clusters.
- Determinants of Toeplitz matrices of rational functions govern spectral distributions.

More in these papers:

- I. Dhillon, B. Parlett, and C. V. *Glued matrices and the MRRR algorithm*, *SIAM J. Sci. Comp.*, 27(2):496-510, 2005.
- B. Parlett and C. V. *The spectrum of a glued matrix* To appear in *SIAM J. Matrix Anal. Appl.*

Thank you.