Structure preserving treatment of PCP-palindromic eigenvalue problems

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Joint work with H. Fassbender (TU Braunschweig),
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Introduction

PCP linearization (briefly)

PCP Schur form

Application, Numerical Experiments

PCP Palindromic Eigenvalue problems

- Consider a regular polynomial eigenvalue problem

$$Q(\lambda)x = (A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^k A_k)x = 0.$$

with $A_i \in \mathbb{C}^{n \times n}$ given, $x \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$ wanted

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- Let P be a real, square, and involutory matrix, i.e., $P^2 = I$.
- $Q(\lambda)$ is PCP palindromic, iff $A_i = P\overline{A_{k-i}}P$
($\overline{A}$ is the complex conjugate of A)

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- This talk is a summary of a paper $\Rightarrow$ [PCP]
- reminicent of $*$-palindromic problems, $A_i = A_{k-i}^*$, see [MMMM2]
- Application: Stability analysis of time delay equations (later)

Eigenvalue pairing

Let (λ, x) be an eigenpair of $Q(\lambda)$. Then

$$\begin{aligned}
(A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^k A_k)x &= 0 \\
P(A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^k A_k)PPx &= 0 \\
(\overline{A_k} + \lambda \overline{A_{k-1}} + \lambda^2 \overline{A_{k-2}} + \cdots + \lambda^k \overline{A_0})Px &= 0 \\
(A_k + \overline{\lambda} A_{k-1} + \overline{\lambda}^2 A_{k-2} + \cdots + \overline{\lambda}^k A_0)\overline{Px} &= 0 \\
\overline{\lambda}^k Q(1/\overline{\lambda})(P\overline{x}) &= 0
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so, $(1/\overline{\lambda}, P\overline{x})$ is also an eigenpair.

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Theorem:([PCP])*Let* $Q(\lambda) = \sum_{i=0}^k \lambda^i A_i,\ A_k \neq 0$ *be a regular PCP matrix polynomial. Then the spectrum of* $Q(\lambda)$ *has the pairing* $(\lambda, 1/\overline{\lambda})$*. Moreover, algebraic, gemetric and partial multiplicities of the eigenvalues in each pair are equal. Here,* $\lambda = 0$ *is allowed and is paired with the eigenvalue* ∞*.*

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Same pairing, as $*$-palindromic polynomials [MMMM2]

A method, a problem, and a remedy

- Eigenvalues of PCP polynomials either
  - come in pairs $(\lambda, 1/\bar{\lambda})$,
  - or they are on the unit circle, i.e., $|\lambda| = 1$ (those are the interesting ones for TDSs)
- Standard method for polynomial EVPs: Companion form

$$\lambda \begin{bmatrix} A_k & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} + \begin{bmatrix} A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I & 0 \end{bmatrix}$$

and QZ algorithm

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  and QZ algorithm
- Problem: neither companion form nor QZ algorithm care about PCP structure $\Rightarrow$ eigenvalue pairing will only be approximate (rounding errors)
- Remedy: structure preserving linearization and structure preserving Schur form

$\mathbb{L}_1$,$\mathbb{L}_2$, and $\mathbb{DL}$

Consider the pencil spaces [MMMM1,MMMM2] $(\Lambda = [\lambda^{k-1}, \ldots, \lambda, 1]^T)$

$$\begin{aligned} \mathbb{L}_1(Q) &:= \{L(\lambda) = \lambda X + Y : L(\lambda)\cdot(\Lambda \otimes I_n) = v \otimes Q(\lambda), v \in \mathbb{C}^k\} \\ \mathbb{L}_2(Q) &:= \{L(\lambda) = \lambda X + Y : (\Lambda^T \otimes I_n)\cdot L(\lambda) = w^T \otimes Q(\lambda), w \in \mathbb{C}^k\} \\ \mathbb{DL}(Q) &:= \mathbb{L}_1(Q) \cap \mathbb{L}_2(Q), \quad \text{with } v = w \end{aligned}$$

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- source for structured pencils:
  Theorem:([PCP]) (Existence/Uniqueness of PCP Pencils in $\mathbb{DL}(Q)$)
  Suppose $Q(\lambda)$ is a PCP-polynomial with respect to the involution P. Let F be the flip matrix and let $v \in \mathbb{C}^k$ be any vector such that $Fv = \overline{v}$, and let $L(\lambda)$ be the unique pencil in $\mathbb{DL}(Q)$ with ansatz vector v. Then $L(\lambda)$ is a PCP-pencil with respect to the involution $\tilde{P} = F \otimes P$.

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- Eigenvalue exclusion [MMMM1]: $L(\lambda)$ is a linearization of $Q(\lambda)$ iff no root of the polynomial $v_1 x^{k-1} + v_2 x^{k-2} + \ldots + \overline{v}_2 x + \overline{v}_1$ is an eigenvalue of $Q(\lambda)$.

Example: quadratic case

$$Q(\lambda)x = \lambda^2 A_2 + \lambda A_1 + P\overline{A}_2 P x = 0, \quad \text{with } A_1 = P\overline{A}_1 P, P^2 = I$$

- chose $v = [\alpha, \overline{\alpha}]^T$ where $-\overline{\alpha}/\alpha$ is not an eigenvalue of $Q(\lambda)$

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- $\mathbb{DL}(Q)$-linearization is (rows resamble $Q(\lambda)$)

$$\left(\lambda \begin{bmatrix} \alpha A_2 & \overline{\alpha} A_2 \\ \overline{\alpha} A_2 & \overline{\alpha} A_1 - \alpha P \overline{A_2} P \end{bmatrix} + \begin{bmatrix} \alpha A_1 - \overline{\alpha} A_2 & \alpha P \overline{A}_2 P \\ \alpha P \overline{A}_2 P & \overline{\alpha} P \overline{A}_2 P \end{bmatrix}\right) \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- it is PCP, because

$$\begin{bmatrix} \alpha A_2 & \overline{\alpha} A_2 \\ \overline{\alpha} A_2 & \overline{\alpha} A_1 - \alpha P \overline{A_2} P \end{bmatrix} = \begin{bmatrix} & P \\ P & \end{bmatrix} \overline{\begin{bmatrix} \alpha A_1 - \overline{\alpha} A_2 & \alpha P \overline{A}_2 P \\ \alpha P \overline{A}_2 P & \overline{\alpha} P \overline{A}_2 P \end{bmatrix}} \begin{bmatrix} & P \\ P & \end{bmatrix}$$

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- method to determine $L(\lambda)$ for higher order PCP polynomials: [PCP]

PCP Schur form

$\lambda A + P\overline{A}P$, where $A \in \mathbb{C}^{m\times m}$, $P \in \mathbb{R}^{m\times m}$ with $P^2 = I, P = P^T$

- generalized Schur form $\lambda S + T = Q(\lambda A + P\overline{A}P)Z$ with S, T upper triangular and Q, Z unitary is not structure preserving

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- we will make it structure preserving
- 1st step: determine Schur decomposition of P:

$$P = WDW^T, \quad \text{with } W \in \mathbb{R}^{m\times m} \text{ orthogonal}, \quad D = \begin{bmatrix} I_p & \\ & -I_{m-p} \end{bmatrix}$$

- A change of basis results in the PCP pencil

$$\begin{aligned} W^T(\lambda A + P\overline{A}P)WW^T x &= 0 \\ (\lambda \tilde{A} + D\overline{\tilde{A}}D)\tilde{x} &= 0 \end{aligned}$$

with $\tilde{A} = W^T A W$, $\tilde{x} = W^T x$

PCP Schur form II

$$(\lambda\tilde{A} + D\overline{\tilde{A}}D)\tilde{x} = 0$$

- rewrite problem (Cayley transform)

$$\left(\frac{\lambda - 1}{\lambda + 1}(\tilde{A} - D\overline{\tilde{A}}D)/2 + (\tilde{A} + D\overline{\tilde{A}}D)/2\right)\tilde{x} = 0$$

$$(\mu N + M)\tilde{x} = 0$$

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- Eigenvalue pairs $(\lambda, 1/\overline{\lambda})$ are mapped to $(\mu, -\overline{\mu})$, symmetry w.r.t. imaginary axis
- Structure is called PCP even [PCP] (similar properties as $*$-even problems [MMMM2])

PCP Schur form III

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- is this any better than before?

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$$= 2\begin{bmatrix} i\mathrm{Im}(A_{11}) & \mathrm{Re}(A_{12}) \\ \mathrm{Re}(A_{21}) & i\mathrm{Im}(A_{22}) \end{bmatrix} \Rightarrow \tilde{D}N\tilde{D} = -i\begin{bmatrix} -\mathrm{Im}(A_{11}) & \mathrm{Re}(A_{12}) \\ \mathrm{Re}(A_{21}) & \mathrm{Im}(A_{22}) \end{bmatrix}$$

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- With $\tilde{D} = \begin{bmatrix} I & \\ & -iI \end{bmatrix}$ we get

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- With $\tilde{D} = \begin{bmatrix} I & \\ & -iI \end{bmatrix}$ we get the *real* EVP

$$(\nu\tilde{N} + \tilde{M})\hat{x} := \big((-i\mu)(i\tilde{D}N\tilde{D}) + \tilde{D}M\tilde{D}\big)(\tilde{D}^{-1}\tilde{x}) = 0$$

PCP Schur form IV

$$(\nu\tilde{N} + \tilde{M})\hat{x} = 0 \quad \text{(real)}$$

- Let $\nu\tilde{S} + \tilde{T} = \tilde{Q}(\nu\tilde{N} + \tilde{M})\tilde{Z}$ be a real generalized Schur form
  $\Rightarrow Q, Z$ orthogonal, T upper triangular, S quasi upper triangular

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- Let $\nu\tilde{S} + \tilde{T} = \tilde{Q}(\nu\tilde{N} + \tilde{M})\tilde{Z}$ be a real generalized Schur form $\Rightarrow Q, Z$ orthogonal, T upper triangular, S quasi upper triangular
- 2×2 diagonal blocks in $\tilde{S}$ correspond to complex conjugate eigenvalue pairs $(\nu, \overline{\nu})$ of $(\nu\tilde{N} + \tilde{M})$, which correspond to reciprocal pairs $(\lambda, 1/\overline{\lambda})$ of $\lambda A + P\overline{A}P$.
- 1×1 diagonal blocks in $\tilde{S}$ correspond to a real eigenvalue ν of $(\nu\tilde{N} + \tilde{M})$ which corresponds to a unit-circle eigenvalue λ of $\lambda A + P\overline{A}P$.

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- 1×1 diagonal blocks in $\tilde{S}$ correspond to a real eigenvalue ν of $(\nu\tilde{N} + \tilde{M})$ which corresponds to a unit-circle eigenvalue λ of $\lambda A + P\overline{A}P$.
- Putting everything together:

$$(\underbrace{\tilde{Q}\tilde{D}W^T}_{Q})(\lambda A + P\overline{A}P)(\underbrace{W\tilde{D}\tilde{Z}}_{Z}) = \lambda(\underbrace{\tilde{T} - i\tilde{S}}_{S}) + (\underbrace{\tilde{T} + i\tilde{S}}_{\overline{S}})$$

S is complex, quasi upper triangular. $\lambda S + \overline{S}$ is PCP Schur form

Algorithm

Input: $A \in \mathbb{C}^{m\times m}$ and $P \in \mathbb{R}^{m\times m}$ with $P^2 = I$ and $P^T = P$.
Output: Unitary $Q, Z \in \mathbb{C}^{m\times m}$ and quasi upper triangular $S \in \mathbb{C}^{m\times m}$ such that $QAZ = S$ and $QP\overline{A}PZ = \overline{S}$;

1: $P \rightarrow WDW^T$ with $D = \mathrm{diag}(I_p, -I_{m-p})$ %real symmetric Schur form
2: $\tilde{A} \leftarrow W^TAW$

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2: $\tilde{A} \leftarrow W^TAW$
3: $\tilde{N} \leftarrow \begin{bmatrix} -\mathrm{Im}(A_{11}) & \mathrm{Re}(A_{12}) \\ \mathrm{Re}(A_{21}) & \mathrm{Im}(A_{22}) \end{bmatrix}$ where $\tilde{A}_{11} \in \mathbb{C}^{p\times p}$
4: $\tilde{M} \leftarrow \begin{bmatrix} \mathrm{Re}(A_{11}) & \mathrm{Im}(A_{12}) \\ \mathrm{Im}(A_{21}) & -\mathrm{Re}(A_{22}) \end{bmatrix}$ where $\tilde{A}_{11} \in \mathbb{C}^{p\times p}$
5: $(\tilde{N}, \tilde{M}) \rightarrow (\tilde{Q}^T\tilde{S}\tilde{Z}^T, \tilde{Q}^T\tilde{T}\tilde{Z}^T)$ %real generalized Schur form

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Input: $A \in \mathbb{C}^{m\times m}$ and $P \in \mathbb{R}^{m\times m}$ with $P^2 = I$ and $P^T = P$.
Output: Unitary $Q, Z \in \mathbb{C}^{m\times m}$ and quasi upper triangular $S \in \mathbb{C}^{m\times m}$ such that $QAZ = S$ and $QP\overline{A}PZ = \overline{S}$;

1: $P \to WDW^T$ with $D = \operatorname{diag}(I_p, -I_{m-p})$ %real symmetric Schur form
2: $\tilde{A} \leftarrow W^T AW$
3: $\tilde{N} \leftarrow \begin{bmatrix} -\mathrm{Im}(A_{11}) & \mathrm{Re}(A_{12}) \\ \mathrm{Re}(A_{21}) & \mathrm{Im}(A_{22}) \end{bmatrix}$ where $\tilde{A}_{11} \in \mathbb{C}^{p\times p}$
4: $\tilde{M} \leftarrow \begin{bmatrix} \mathrm{Re}(A_{11}) & \mathrm{Im}(A_{12}) \\ \mathrm{Im}(A_{21}) & -\mathrm{Re}(A_{22}) \end{bmatrix}$ where $\tilde{A}_{11} \in \mathbb{C}^{p\times p}$
5: $(\tilde{N}, \tilde{M}) \to (\tilde{Q}^T\tilde{S}\tilde{Z}^T, \tilde{Q}^T\tilde{T}\tilde{Z}^T)$ %real generalized Schur form
6: $Q \leftarrow \tilde{Q}\operatorname{diag}(I_p, -iI_{m-p})W^T$, $Z \leftarrow W\operatorname{diag}(I_p, -iI_{m-p})\tilde{Z}$
7: $S \leftarrow \tilde{T} - i\tilde{S}$

- faster than general complex QZ algorithm, because main work is a real QZ algorithm.
- yields structured results

Application: Stability of time delay systems

Neutral linear time-delay system (TDS) with m constant delays

$$\mathcal{S} = \begin{cases} \sum_{k=0}^{m} D_k \dot{x}(t-h_k) = \sum_{k=0}^{m} B_k x(t-h_k)\,, & t \geq 0 \\ x(t) = \varphi(t)\,, & t \in [-h_{\max}, 0) \end{cases}$$

- is stable for $(h_1, \ldots, h_k)$, if all eigenvalues s of $(-s\sum_{k=0}^{m} D_k e^{-h_k s} + \sum_{k=0}^{m} B_k e^{-h_k s})x = 0$ have negative real part.

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- eigenvalues are continuous wrt. delays $\Rightarrow$ approach: fix $h_1, \ldots, h_{m-1}$, find those h_m such that there are purely imaginary eigenvalues.
- Results in quadratic EVP [PCP] $(A_0 + zA_1 + z^2 A_2)v = 0$ where

$$A_0 = B_m \otimes D_S + D_m \otimes B_S, \quad A_1 = \ldots$$
$$A_2 = \overline{D_S \otimes B_m} + \overline{B_S \otimes D_m}$$

where D_S, B_S can be computed from $D_k, B_k, h_k, k = 0, \ldots, m-1$.

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  where D_S, B_S can be computed from $D_k, B_k, h_k, k = 0, \dots, m-1$.
- Has PCP structure wrt. the (involutory) permutation P such that $B_m \otimes D_S = P(D_S \otimes B_m)P \Rightarrow A_0 = P\overline{A_2}P$.

Numerical example

- TDS: partial TDS discetized in space with various stepsizes
- Algorithms: `polyeig`: companion form, `QZ`: PCP linearization+QZ, `PCP`: structures linearization+PCP Schur form
- n: size of TDS, $2n^2$: size of linearized EVP, $err = \max_{\lambda_i} \min_{\lambda_j} \frac{|\lambda_i - (1/\overline{\lambda_j})|}{|\lambda_i|}$
- #: number of found unit-circle eigenvalues (for unstructured methods: those with $||\lambda| - 1| < 10^{-14}$)

n	$2n^2$	t_{polyeig}	t_{QZ}	t_{PCP}	err_{polyeig}	err_{QZ}	$\#_{\text{polyeig}}$	$\#_{\text{QZ}}$	$\#_{\text{PCP}}$
5	50	0.02	0.02	0.01	5.5e-15	3.7e-15	4	4	4
10	200	0.50	0.55	0.28	6.5e-14	1.2e-13	4	4	4
15	450	5.5	6.3	3.0	4.4e-13	2.6e-13	4	3	4
20	800	33	36	20	1.3e-12	4.8e-13	3	0	4
25	1250	131	137	72	3.1e-12	6.6e-13	3	0	4
30	1800	413	435	227	1.1e-11	7.5e-13	0	0	4

Structured Algorithm is faster and finds all unit-circle eigenvalues.

Conclusion

- new variant of palindromic
- structure preserving linearization and Schur form
- is important in application

References

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[MMMM2] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Structured Polynomial Eigenvalue Problems: Good Vibrations from Good Linearizations*, SIAM Journal on Matrix Analysis and Applications, 28(4), pp 1029–1051, 2006.

Thanks for your attention!