CHAPTER 5: NONLINEAR EIGENVALUE PROBLEMS

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Examples
Nonlinear eigenvalue problem

The term nonlinear eigenvalue problem is not used in a unique way in the literature.

On the one hand: Parameter dependent nonlinear (with respect to the state variable) operator equations

\[ T(\lambda, u) = 0 \]

are discussed concerning

- positivity of solutions
- multiplicity of solution
- dependence of solutions on the parameter; bifurcation
- (change of) stability of solutions

In this presentation
For \( \lambda \in D \subset \mathbb{C} \) let \( T(\lambda) \) be a linear self-adjoint and bounded operator on a Hilbert space \( \mathcal{H} \).

Find \( \lambda \in D \) and \( x \neq 0 \) such that

\[ T(\lambda)x = 0. \tag{1} \]

Then \( \lambda \) is called an eigenvalue of \( T(\cdot) \), and \( x \) a corresponding eigenvector.

Nonlinear eigenproblems arise in

- dynamic/stability analysis of structures and in fluid mechanics
- regularization of total least squares problems
- vibration of sandwich plates
- electronic behavior of semiconductor hetero-structures
- vibration of fluid-solid structures
- accelerator design
- vibro-acoustics of piezoelectric/poroelastic structures
- nonlinear integrated optics

Example 1: Vibration of structures

Equations of motion arising in dynamic analysis of structures (with a finite number of degrees of freedom) are

\[ M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t) \]

where \( q \) are the Lagrangean coordinates, \( M \) is the mass matrix, \( K \) the stiffness matrix, \( C \) the viscous damping matrix, and \( f \) an external force vector.

Suppose that the system is exited by a time harmonic force \( f(t) = f_0 e^{i\omega_0 t} \).

If \( C = \alpha K + \beta M \) (modal damping) and \( (x_j, \mu_j), j = 1, \ldots, n \) denotes an orthonormal eigensystem of \( Kx = \mu Mx \) (i.e. \( x_j^T M x_j = \delta_{jk} \)) then the periodic response of the system is

\[ q(t) = e^{i\omega_0 t} \sum_{j=1}^{n} \frac{x_j^T f_0}{\mu_j - \omega_0^2 + i\omega_0 (\alpha \mu_j + \beta)} x_j. \]

Usually, one gets good approximations taking into account only a small number of eigenvalues in the vicinity of \( i\omega_0 \) (truncated mode superposition).
**Example 2: Dynamic element method**
Discretizing a linear eigenproblem

\[ Lu(x) = \lambda Mu(x), \ x \in \Omega. \quad Bu(x) = 0, \ x \in \partial \Omega \]

by finite elements which depend on the eigenparameter yields a nonlinear eigenproblem; cf. Przemieniecki (1968).

If the ansatz functions \( \phi_j(x) + \lambda \psi_j(x) \) depend linearly on the eigenparameter then one obtains a cubic eigenvalue problem (cf. V. 1987)

\[
\sum_{j=0}^{3} \lambda^3 A_j x = 0
\]

where

\[
A_3 = \left( \int_{\Omega} \psi_j M \psi_k \ dx \right)_{jk}, \quad A_2 = \left( \int_{\Omega} \left( \psi_j M \phi_k + \psi_j M \psi_k - \psi_j L \psi_k \right) dx \right)_{jk}
\]

\[
A_1 = \left( \int_{\Omega} \left( \psi_j M \phi_k - \psi_j L \psi_k - \psi_j L \phi_k \right) dx \right)_{jk}, \quad A_0 = - \left( \int_{\Omega} \phi_j L \phi_k dx \right)_{jk}
\]

**Example 4: Controlled systems with delayed feedback**

The governing equation of a mechanical system with delayed feedback is (Elsgolts & Norkin 1974, Hale 1977)

\[ M\ddot{u} + C\dot{u} + Ku = G_u u(t - \tau) + G_p \dot{u}(t - \tau). \]

This delay normally originates from physical limitations like finite switching times in controllers or unavailability of the current state of the system.

An ansatz \( u(t) = e^{\lambda t} x \) yields the eigenproblem

\[
\lambda^3 Mx + \lambda Cx + Kx - e^{-\lambda \tau} G_u x - \lambda e^{-\lambda \tau} G_p x = 0
\]

The system is stable if all eigenvalues have negative real part.

**Example 3: Conservative gyroscopic systems**

Simulation of acoustic behavior of rotating structures (rotating wheels, e.g.) results in

\[ M\ddot{q} + G\dot{q} + Kq = 0 \]

where \( M \) is positive definite, \( K \) is positive (semi-)definite, \( G \) skew-symmetric, i.e. \( G^T = -G \).

The corresponding eigenvalue problem

\[ \omega^2 Mx + \omega Gx + Kx = 0 \]

has purely imaginary eigenvalues.

Wanted in simulation of rotating tires is a large number of (not necessarily extreme) eigenvalues, for instance for rolling wheels all eigenfrequencies between 500 Hz and 2000 Hz, an important interval of perception of the human ear (cf. Nackenhorst 2004).

**Example 5: Viscoelastic model of damping**

Using a viscoelastic constitutive relation to describe the material behavior in the equations of motion yields a rational eigenvalue problem in the case of free vibrations.

Discretizing by finite elements yields (cf. Hager & Wiberg 2000)

\[
T(\omega)x := \left( \omega^2 M + K - \sum_{j=1}^{K} \frac{1}{1 + b_j^2 \omega^2} \Delta K_j \right)x = 0
\]

where \( M \) is the consistent mass matrix, \( K \) is the stiffness matrix with the instantaneous elastic material parameters used in Hooke’s law, and \( \Delta K_j \) collects the contributions of damping from elements with relaxation parameter \( b_j \).
Example 6: Fluid–solid vibrations

can be modelled in terms of solid displacement and fluid pressure and one obtains the classical form of an eigenproblem

\[
\text{div } [\sigma(u)] + \omega^2 \rho_s u = 0 \text{ in } \Omega_s,
\]
\[
\Delta p + \omega^2 \rho_f p = 0 \text{ in } \Omega_f,
\]
\[
\sigma(u) \cdot n - pn = 0 \text{ on } \Gamma_I,
\]
\[
\nabla p \cdot n + \omega^2 \rho_f u \cdot n = 0 \text{ on } \Gamma_I,
\]
\[
u p \cdot n = 0 \text{ on } \Gamma_N,
\]

where

- \( u \): solid displacement
- \( p \): fluid pressure
- \( \lambda = \omega^2 \): eigenparameter
- \( \sigma(u) \): linearized stress tensor
- \( \rho_s, \rho_f \): densities of solid and fluid

Variational and operator form

Find \( \lambda = \omega^2 \in \mathbb{C} \) and \( (u,p) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f) \) such that

\[
a_s(v,u) + c(v,p) = \lambda b_s(v,u) \quad \text{and} \quad a_t(q,p) = \lambda (-c(u,q) + b(q,p)).
\]

for every \( (v,q) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f) \).

which (using the Lax-Milgram Lemma) can be transformed into a linear (but not self-adjoint) eigenvalue problem

\[
K_s u + C p = \lambda M_s u \quad \text{(2a)}
\]
\[
K_f p = \lambda (-C' u + M_f p) \quad \text{(2b)}
\]

where \( K_s : H^1_{\Gamma_D}(\Omega_s)^3 \to H^1_{\Gamma_D}(\Omega_s)^3 \) is self-adjoint, elliptic, bounded, ...

Rational form of fluid-solid eigenproblem

Let \( 0 < \sigma_1 \leq \sigma_2 \leq \ldots \) denote the eigenvalues of the decoupled eigenproblem

\[
K_s u = \sigma M_s u
\]

and denote by \( u_1, u_2, \ldots \) corresponding orthonormal eigenfunctions. Then the spectral theorem yields

\[
(K_s - \lambda M_s)^{-1} u = \sum_{n=1}^{\infty} \frac{1}{\sigma_n - \lambda} \langle u_n, u \rangle u_n.
\]

If \( \lambda \) is not contained in the spectrum of the decoupled solid eigenproblem, then \( \lambda \) is an eigenvalue of the coupled fluid-solid problem if and only if it is an eigenvalue of the rational eigenvalue problem

\[
T(\lambda) p := -K_f p + \lambda M_f p + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} C_n p, \quad C_n p := \langle u_n, C p \rangle C' u_n.
\]

\[
T(\lambda) : H^1(\Omega_f) \to H^1(\Omega_f) \text{ is self-adjoint and bounded.}
\]

Quadratic form of fluid-solid eigenproblem

Another self-adjoint form of the fluid-solid eigenproblem is obtained if the second equation in (2) is multiplied by \( \omega \) and \( p \) is substituted by \( p =: \omega w \).

Then problem (2) is equivalent to the quadratic eigenvalue problem

\[
\left( \begin{array}{cc}
K_s & O \\
O & K_f
\end{array} \right) + \omega \left( \begin{array}{cc}
O & C' \\
C & O
\end{array} \right) - \omega^2 \left( \begin{array}{cc}
M_s & O' \\
O & M_f
\end{array} \right) \left( \begin{array}{c}
u \\
w
\end{array} \right) = 0.
\]
Examples

Example 7: Electronic structure of quantum dots

Semiconductor nanostructures have attracted tremendous interest in the past few years because of their special physical properties and their potential for applications in micro- and optoelectronic devices.

In such nanostructures, the free carriers are confined to a small region of space by potential barriers, and if the size of this region is less than the electron wavelength, the electronic states become quantized at discrete energy levels.

The ultimate limit of low dimensional structures is the quantum dot, in which the carriers are confined in all three directions, thus reducing the degrees of freedom to zero. Therefore, a quantum dot can be thought of as an artificial atom.

Electronic structure of quantum dots ct.

Problem: Determine relevant energy states (i.e. eigenvalues) and corresponding wave functions (i.e. eigenfunctions) of a three-dimensional quantum dot embedded in a matrix.

Governing equation: Schrödinger equation

$$-\nabla \cdot \left( \frac{\hbar^2}{2m(x, \lambda)} \nabla u \right) + V(x)u = \lambda u, \quad x \in \Omega_q \cup \Omega_m$$

where $\hbar$ is the reduced Planck constant, $m(x, \lambda)$ is the electron effective mass, and $V(x)$ is the confinement potential.

$m$ and $V$ are discontinuous across the heterojunction.

Boundary and interface conditions

$$u = 0 \quad \text{on outer boundary of matrix } \Omega_m$$

BenDaniel–Duke condition

$$\left. \frac{1}{m_m} \frac{\partial u}{\partial n} \right|_{\partial \Omega_m} = \left. \frac{1}{m_q} \frac{\partial u}{\partial n} \right|_{\partial \Omega_q} \quad \text{on interface}$$
Variational characterization of eigenvalues

The variational characterization of eigenvalues is a powerful tool to analyze selfadjoint operators in Hilbert spaces. In this subsection we discuss generalizations to operators depending nonlinearly on the eigenparameter. Although we are mainly interested in numerical methods (and therefore on the finite dimensional case) we consider the infinite dimensional case.

For linear selfadjoint eigenvalue problems the following characterizations of eigenvalues are well known (cf. Rektorys 1984):

**Theorem 1**
Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a selfadjoint and completely continuous operator on a real Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot , \cdot \rangle$, and denote by $R_A(x):=\langle Ax, x \rangle / \langle x, x \rangle$ the Rayleigh quotient of $A$ at $x \neq 0$.

Let $\lambda_1 \geq \lambda_2 \geq \ldots$ be the positive eigenvalues of $A$ ordered by magnitude and regarding their multiplicities, and let $\{x^1, x^2, \ldots \}$ be a system of orthogonal eigenelements where $x^i$ corresponds to $\lambda_i$.

Assume that $A$ has at least $n$ positive eigenvalues. Then the following characterizations hold:

**Variational characterization**

Find $\lambda \in \mathbb{R}$ and $u \in H^1_0(\Omega)$, $u \neq 0$, $\Omega := \Omega_q \cup \Omega_m$, such that

$$a(u, v; \lambda) := \frac{\hbar^2}{2} \int_{\Omega_q} \frac{1}{m_q(x, \lambda)} \nabla u \cdot \nabla v \, dx + \frac{\hbar^2}{2} \int_{\Omega_m} \frac{1}{m_m(x, \lambda)} \nabla u \cdot \nabla v \, dx$$
$$+ \int_{\Omega_q} V_q(x) uv \, dx + \int_{\Omega_m} V_m(x) uv \, dx$$
$$= \lambda \int_{\Omega} uv \, dx =: \lambda b(u, v) \quad \text{for every } v \in H^1_0(\Omega)$$

**Example**

**Electron effective mass**

The dependence of $m(x, \lambda)$ on $\lambda$ can be derived from the eight-band $k \cdot p$ analysis and effective mass theory. Projecting the $8 \times 8$ Hamiltonian onto the conduction band results in the single Hamiltonian eigenvalue problem with

$$m(x, \lambda) = \begin{cases} m_q(\lambda), & x \in \Omega_q \\ m_m(\lambda), & x \in \Omega_m \end{cases}$$

$$\frac{1}{m_j(\lambda)} = \frac{P_j^2}{\hbar^2} \left( \frac{2}{\lambda + g_j - V_j} + \frac{1}{\lambda + g_j - V_j + \delta_j} \right), j \in \{q, m\}$$

where $m_j$ is the electron effective mass, $V_j$ the confinement potential, $P_j$ the momentum, $g_j$ the main energy gap, and $\delta_j$ the spin-orbit splitting in the $j$th region.

Other types of effective mass (taking into account the effect of strain, e.g.) appear in the literature. They are all rational functions of $\lambda$ where $1/m(x, \lambda)$ is monotonically decreasing with respect to $\lambda$, and that's all we need.

**Variational characterization ct.**

(i) **Rayleigh’s principle**

$$\lambda_n = \max \{ R_A(x) : \langle x, x \rangle = 0, \; i = 1, \ldots, n - 1 \}.$$

(ii) **Maxmin characterization of Poincaré**

$$\lambda_n = \max_{\dim V=n} \min_{x \in V, x \neq 0} R_A(x).$$

(iii) **Minmax characterization of Courant, Fischer, Weyl**

$$\lambda_n = \min_{\dim V=n-1} \max_{x \in V, x \neq 0} R_A(x).$$

Analogously the negative eigenvalues $\lambda_{-1} \leq \lambda_{-2} \leq \ldots$ of $A$ can be characterized by the three principles above if we replace min by max and vice versa.
Variational form

Quite often the eigenvalue problem is given in variational form:

Let

$$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \quad \text{and} \quad b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$

be symmetric (i.e. $a(u, v) = a(v, u)$, $b(u, v) = b(v, u)$ for every $u, v \in \mathcal{H}$) and bounded quadratic forms, i.e. there exist $K_a, K_b > 0$ such that

$$|a(u, v)| \leq K_a \|u\| \cdot \|v\| \quad \text{and} \quad |b(u, v)| \leq K_b \|u\| \cdot \|v\| \quad \text{for every} \ u, v \in \mathcal{H}.$$ 

Let a be $\mathcal{H}$-elliptic (i.e. $a(u, u) \geq \alpha \|u\|^2$ for some $\alpha > 0$ for all $u \in \mathcal{H}$), and let $b$ be positive (i.e. $b(u, u) > 0$ for every $u \in \mathcal{H}$, $u \neq 0$) and completely continuous, i.e.

$$u_n \to u, \ v_n \to v \implies b(u_n, v_n) \to b(u, v),$$

where $u_n \to u$ denotes the weak convergence of $u_n$ to $u$.

Find $\lambda \in \mathbb{R}$ and $u \in \mathcal{H}$, $u \neq 0$ such that

$$a(u, v) = \lambda b(u, v) \quad \text{for every} \ v \in \mathcal{H}.$$ 

Variational form ct.

By the Lax-Milgram theorem there exists a self-adjoint and (by Rellich’s lemma) completely continuous operator

$$A : \mathcal{H} \to \mathcal{H} \quad \text{such that} \ b(u, v) = a(Au, v) \text{ for every} \ u, v \in \mathcal{H}. $$

Hence, the variational eigenvalue problem is equivalent to

$$a(u, v) = \lambda a(Au, v) \quad \forall v \in \mathcal{H} \quad \iff \quad u = \lambda Au,$$

and $\lambda$ is an eigenvalue of the variational eigenproblem if and only if $\lambda^{-1}$ is an eigenvalue of $A$.

Since $A$ is positive definite (note that $b(u, u) > 0$ for $u \neq 0$), the variational problems has positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots,$$

and $\lim_{k \to \infty} \lambda_k = \infty$, if $\dim \mathcal{H} = \infty$.

Nonlinear eigenvalue problem

Consider a family of bounded and selfadjoint operators

$$T(\lambda) : \mathcal{H} \to \mathcal{H}, \quad \lambda \in J,$$

where $J \subset \mathbb{R}$ is an open interval which may be unbounded.

Let

$$f : J \times \mathcal{H} \to \mathbb{R}, \ (\lambda, x) \mapsto \langle T(\lambda)x, x \rangle$$

be continuous, and assume that for every fixed $x \in \mathcal{H}$, $x \neq 0$, the real equation

$$f(\lambda, x) = 0 \quad (2)$$

has at most one solution in $J$.

Then equation (2) implicitly defines a functional $p$ on some subset $D$ of $H \setminus \{0\}$ which we call the Rayleigh functional, and which is exactly the Rayleigh quotient in case of a linear eigenproblem $T(\lambda) = \lambda I - A$. 
Nonlinear eigenvalue problem

We assume further that
\[(\lambda - p(x))f(\lambda; x) > 0 \quad \text{for every } x \in D(p) \text{ and } \lambda \neq p(x)\]

which generalizes the definiteness of the operator $B$ for the generalized linear eigenproblem $T(\lambda) := \lambda B - A$, and which is satisfied if $T$ is differentiable and
\[\frac{\partial}{\partial \lambda} f(\lambda, x)|_{\lambda=p(x)} > 0 \quad \text{for every } x \in D \quad (3)\]

The implicit function theorem gives that $D$ is an open set and $p$ is continuously differentiable on $D$.

Differentiating the governing equation
\[f(p(x)), x = \langle T(p(x))x, x \rangle \equiv 0\]

yields that the eigenvalues of problem $T(\cdot)$ are the stationary vectors of the Rayleigh functional $p$.

Overdamped problems

If the Rayleigh functional $p$ is defined on the entire space $\mathcal{H} \setminus \{0\}$ then the eigenproblem $T(\lambda)x = 0$ is called overdamped.

The notation is motivated by the finite dimensional quadratic eigenvalue problem
\[T(\lambda)x = \lambda^2 Mx + \lambda Cx + Kx = 0 \quad (4)\]

where $M$, $C$ and $K$ are symmetric and positive definite matrices.

- $\alpha = 0$ all eigenvalues on imaginary axis
- increase $\alpha$ eigenvalues go into left half plane as conjugate complex pairs
- increase $\alpha$ complex pairs reach real axis, run in opposite directions
- increase $\alpha$ all eigenvalues on the negative real axis
- increase $\alpha$ all eigenvalues going to the left are smaller than all eigenvalues going to the right

system is overdamped

Quadratic overdamped problems

For quadratic overdamped systems the two solutions
\[p_\pm(x) = \frac{1}{2\langle Mx, x \rangle} \left(-\alpha \langle Cx, x \rangle \pm \sqrt{\alpha^2 \langle Cx, x \rangle^2 - 4 \langle Mx, x \rangle \langle Kx, x \rangle}\right)\]

of the quadratic equation
\[\langle T(\lambda)x, x \rangle = \lambda^2 \langle Mx, x \rangle + \lambda \alpha \langle Cx, x \rangle + \langle Kx, x \rangle = 0 \quad (5)\]

are real, and they satisfy $\sup_{x \neq 0} p_-(x) < \inf_{x \neq 0} p_+(x)$.

Hence, equation (5) defines two Rayleigh functionals $p_-$ and $p_+$ corresponding to the intervals
\[J_- := (-\alpha, \inf_{x \neq 0} p_+(x)) \quad \text{and} \quad J_+ := (\sup_{x \neq 0} p_-(x), \infty).\]

Rayleigh’s principle

For general (not necessarily quadratic) overdamped problems Hadeler (1967) for the finite dimensional case, and 1968 for $\dim \mathcal{H} = \infty$ generalized Rayleigh’s principle proving that the eigenvectors are orthogonal with respect to the generalized scalar product
\[[x, y] := \begin{cases} \langle T(p(x))x, y \rangle, & \text{if } p(x) \neq p(y) \\ \frac{\langle T(p(x))x, y \rangle}{\langle T(p(x))x, x \rangle}, & \text{if } p(x) = p(y) \end{cases} \quad (6)\]

which is symmetric, definite and homogeneous, but in general is not bilinear.
Rayleigh’s principle

Theorem 2 (Haderel)
Let \( T(\lambda) : H \to H \), \( \lambda \in J \) be a family of selfadjoint and bounded operators. Assume that the problem \( T(\lambda)x = 0 \) is overdamped and that for every \( \lambda \in J \) there exists \( \nu(\lambda) > 0 \) such that \( T(\lambda) - \nu(\lambda)I \) is completely continuous.

Then problem \( T(\lambda)x = 0 \) has at most a countable set of eigenvalues in \( J \) which we assume to be ordered by magnitude \( \lambda_1 \geq \lambda_2 \geq \ldots \) regarding their multiplicities.

The corresponding eigenvectors \( x^1, x^2, \ldots \) can be chosen orthonormally with respect to the generalized scalar product (6), and the eigenvalues can be determined recursively by

\[
\lambda_n = \max \{ \rho(x) : [x, x_i] = 0, \ i = 1, \ldots, n - 1, \ x \neq 0 \}. \tag{7}
\]

minmax and maxmin for overdamped problems

Theorem 3 (Haderel 1968)
Let \( T(\lambda) : H \to H \), \( \lambda \in J \) be a family of selfadjoint and bounded operators. Assume that the problem \( T(\lambda)x = 0 \) is overdamped and that for every \( \lambda \in J \) there exists \( \nu(\lambda) > 0 \) such that \( T(\lambda) - \nu(\lambda)I \) is completely continuous.

Let the eigenvalues \( \lambda_n \) of \( T(\lambda)x = 0 \) be numbered in nonincreasing order according to multiplicities. Then they can be characterized by the following two variational principles

\[
\lambda_n = \max_{\dim V=n} \min_{x \in V, x \neq 0} \rho(x)
= \min_{\dim V=n-1} \max_{x \in V, x \neq 0} \rho(x).
\]

It is obvious that these principles can be transformed to minmax and maxmin principles for variational forms of nonlinear eigenvalue problems.
Variational characterization

Properties

- \( a(\cdot, \cdot, \lambda) \) bilinear, symmetric, bounded, \( H^0_0(\Omega) \)-elliptic for \( \lambda \geq 0 \)
- \( b(\cdot, \cdot) \) bilinear, positive definite, bounded, completely continuous

By the Lax–Milgram lemma the variational eigenproblem is equivalent to

\[ T(\lambda)u = 0 \]

where \( T(\lambda) : H^0_0(\Omega) \to H^0_0(\Omega), \quad \lambda \geq 0, \)

is a family of bounded operators such that . . .

Nonoverdamped systems

For nonoverdamped eigenproblems the natural ordering to call the largest eigenvalue the first one, the second largest the second one, etc., is not appropriate.

This is obvious if we make a linear eigenvalue

\[ T(\lambda)x := (\lambda I - A)x = 0 \]

nonlinear by restricting it to an interval \( J \) which does not contain the largest eigenvalue of \( A \).

Then all conditions are satisfied, \( p \) is the restriction of the Rayleigh quotient \( R_A \) to \( D := \{ x \neq 0 : R_A(x) \in J \} \), and sup\( x \in D \) \( p(x) \) will not be an eigenvalue.

Variational characterization

Properties ct.

- For

\[ f(\lambda; u) := \langle T(\lambda)u, u \rangle = \lambda b(u, u) - a(u, u; \lambda) \]

it holds

\[ f(0; u) < 0 < \lim_{\lambda \to \infty} f(\lambda; u) = \infty \quad \text{for every } u \neq 0 \]

- \( \frac{\partial}{\partial \lambda} f(\lambda; u) > 0 \) for every \( u \neq 0 \) and \( \lambda \geq 0 \)

- For fixed \( \lambda \geq 0 \) the eigenvalues of the linear eigenproblem \( T(\lambda)u = \mu u \)

satisfy a maxmin characterization.

Hence, the nonlinear Schrödinger equation has an infinite number of eigenvalues which can be characterized as minmax values of the Rayleigh functional.

Variational characterization

Enumeration of eigenvalues

\( \lambda \in J \) is an eigenvalue of \( T(\cdot) \) if and only if \( \mu = 0 \) is an eigenvalue of the linear problem \( T(\lambda)y = \mu y \). The key idea is to orientate the number of \( \lambda \) on the location on the eigenvalue \( \mu = 0 \) in the spectrum of the linear operator \( T(\lambda) \).

This to end we assume that for every \( \lambda \in J \) there exists \( \nu(\lambda) > 0 \) such that the linear operator \( S := T(\lambda) - \nu(\lambda)I \) is completely continuous.

If \( \lambda \in J \) is an eigenvalue of the nonlinear problem \( T(\lambda)x = 0 \) then \( \mu = 0 \) is an eigenvalue of \( T(\lambda) \), and \( -\nu(\lambda) \) is a negative eigenvalue of \( S \).

By Theorem 1 there exists \( n \in \mathbb{N} \) such that

\[ -\nu(\lambda) = \min_{\dim V=n} \max_{x \in V, x \neq 0} R_S(x), \quad \text{i.e.} \quad 0 = \min_{\dim V=n} \max_{x \in V, x \neq 0} R_{T(\lambda)}(x) \]

In this case we assign \( n \) to the eigenvalue \( \lambda \) of problem \( T(\lambda)x = 0 \) as its number.
maxmin for nonoverdamped problems

Theorem 4 (V. & Werner (1982), V. (2010))
Assume that for every \( x \in H \), \( x \neq 0 \) the real equation \( f(\lambda, x) = 0 \) has at most one solution \( \lambda =: p(x) \in J \), and that condition (3) holds, and suppose that for every \( \lambda \in J \) there exists \( \nu(\lambda) > 0 \) such that \( T(\lambda) - \nu(\lambda)I \) is completely continuous.

Then the nonlinear eigenvalue problem \( T(\lambda)x = 0 \) has at most a countable number of eigenvalues. Enumerating them as above the following characterizations hold:
If \( \lambda_n \in J \) is an \( n \)-th eigenvalue then
\[
\lambda_n = \max_{\dim V = n, \ V \cap D \neq \emptyset} \inf_{x \in D \cap V} p(x). \tag{8}
\]

If conversely
\[
\lambda_n = \sup_{\dim V = n, \ V \cap D \neq \emptyset} \inf_{x \in D \cap V} p(x) \in J
\]
then \( \lambda_n \) is an \( n \)-th eigenvalue of \( T(\lambda)x = 0 \) and (8) holds.

Sketch of proof

Step 1 (technical): Let \( \lambda \in J \), and assume that \( V \) is a finite dimensional subspace of \( \mathcal{H} \) such that \( V \cap D \neq \emptyset \). Then it holds that
\[
\lambda \left\{ \begin{array}{c}
\leq \sup_{x \in V \cap D(p)} p(x) \\
> \min_{x \in V} \langle T(\lambda)x, x \rangle
\end{array} \right\} \iff \min_{x \in V} \langle T(\lambda)x, x \rangle \leq 0 \tag{3}
\]

Step 2: If \( \lambda_n \) is an \( n \)-th eigenvalue, then \( \mu_n(\lambda_n) = 0 \), and
\[
\mu_n(\lambda_n) = \max_{\dim V = n \in V, \ ||x|| = 1} \min_{x \in V} \langle T(\lambda_n)x, x \rangle = \min_{x \in V} \langle T(\lambda_n)x, x \rangle.
\]

Hence, \( \min_{x \in V, \ ||x|| = 1} \langle T(\lambda_n)x, x \rangle \leq 0 \) for every \( V \) with \( \dim V = n \), and (3) implies
\[
\sup_{x \in V \cap D} p(x) \geq \lambda_n = \sup_{x \in V \cap D} p(x).
\]

Hence, \( \lambda_n \) is a minmax value of \( p \).

maxmin for nonoverdamped problems ct.

Theorem 5
Under the conditions of Theorem 4 let \( \sup_{V \in D} p(V) \in J \) and assume that there exists \( W \in \mathcal{H} \) such that
\[
W \cap D \neq \emptyset \quad \text{and} \quad \inf_{V \in W \cap D} p(V) \in J.
\]

Then for \( j = 1, \ldots, n \) every \( V \in \mathcal{H}_j \) with
\[
V \cap D \neq \emptyset \quad \text{and} \quad \lambda_j = \inf_{V \in D} p(V)
\]
is contained in \( D \), and the maxmin characterization of \( \lambda_j \) can be replaced by
\[
\lambda_j = \max_{V \in \mathcal{H}_j, \ V \setminus \{0\} \subseteq D} \min_{V \setminus \{0\}} p(V).
\]
Further literature

Variational characterizations for non-overdamped nonlinear eigenvalue problems were proved (independently from our work) by

- Barston (1974) for some extreme eigenvalues of finite dimensional quadratic eigenproblems

and for the infinite dimensional case by

- Griniv & Mel’nik (1996): $T(\lambda) = A(\lambda) - I$, $A(\lambda)$ compact
- Eschwé & M. Langer (2004) unbounded operators

Fluid-solid structure

The variational form of the eigenvalue problem governing vibrations of a structure immersed in a fluid is:

Find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega_0)$ such that for every $v \in H^1(\Omega_0)$

$$a(u, v) := \int_{\Omega_0} \nabla u \cdot \nabla v \, dx$$

$$= \lambda \int_{\Omega_0} uv \, dx + \sum_{j=1}^{\ell} \frac{\lambda_{j0}}{k_j - \lambda m_j} \int_{\Gamma_j} u n \, ds \cdot \int_{\Gamma_j} v n \, ds := b(u, v; \lambda).$$

Obviously, $f(\lambda, u) := b(u, u; \lambda) - a(u, u)$ is monotonically increasing for every fixed $u \neq 0$, and therefore $f(\lambda, u) = 0$ defines a Rayleigh functional in every real interval $J$ which does not contain a pole $k_j/m_j$.

For $J_0 := (0, \min\{k_j/m_j : j = 1, \ldots, \ell\}$ the infimum of the Rayleigh functional is contained in $J_0$, and Theorem 5 applies (i.e. the eigenvalues are enumerated in the natural way), for further intervals the enumeration according to their location in the spectrum of the linear eigenproblems is relevant in Theorem 4.

A priori bound for CMS

Theorem 5 can be used to derive an a priori bound for CMS (cf. Elssel & V. 2006).

We already obtained in Chapter 12 the Craig-Bampton form of an eigenvalue problem

$$\begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & \Omega_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} I & 0 & M_{i1} \\ 0 & I & M_{i2} \\ M_{ii1} & M_{ii2} & M_{ii} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and that this problem is reduced to

$$\begin{pmatrix} \Omega_2 & 0 \\ 0 & K_3 \end{pmatrix} y = \lambda \begin{pmatrix} I & M_{ii2} \\ M_{ii2} & M_{ii} \end{pmatrix} y$$

in the component mode synthesis method, where all diagonal elements of $\Omega_1$ are greater than the cut-off frequency $\omega_c$, and all eigenvalues in $\Omega_2$ are less than $\omega_c$.

A priori bound for CMS ct.

For $\lambda \in J := (0, \omega_c)$ the first equation of (1) yields

$$x_1 = \lambda(\Omega_1 - \lambda I)^{-1} M_{i1} x_3,$$

and $\lambda$ is an eigenvalue of $Kx = \lambda Mx$ if and only if it is an eigenvalue of the rational eigenproblem

$$T(\lambda)y = 0$$

where

$$T(\lambda) = -\begin{pmatrix} \Omega_2 & 0 \\ 0 & K_3 \end{pmatrix} + \lambda I \begin{pmatrix} M_{ii2} & M_{ii} \\ M_{ii2} & M_{ii} \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & M_{i1} \\ M_{i1} & \Omega_1 - \lambda I \end{pmatrix}$$

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\ell$ denote the eigenvalues of problem (1) ordered by magnitude, and let $m \in \mathbb{N}$ such that $\lambda_m < \omega_c \leq \lambda_{m+1}$.

Then $\lambda_1, \ldots, \lambda_m \in J$ are the eigenvalues of the nonlinear eigenproblem (1) in $J$. 
A priori bound for CMS ct.

For \( f(\lambda; y) := y^T T(\lambda)y \) it follows from the positive definiteness of \( \left( \begin{array}{cc} I & M_{M/2} \\ M_{M/2} & M_3 \end{array} \right) \) that
\[
\frac{\partial}{\partial \lambda} f(\lambda; y) = y^T \left( I \ M_{M/2} \right) y + \sum_{\omega \geq \omega_0} \frac{(2\omega y_j - \lambda^2 \omega_j) a_j^2}{(\omega_j - \lambda)^2} > 0
\]
for every \( y \in \mathbb{R}^\nu \setminus \{0\} \), where \( \nu \) denotes the dimension of the reduced problem (2), and \( a := (0 \ M_{M/1}) \).

Hence, the monotonicity of \( f(\lambda; y) \) for every \( y \in \mathbb{R}^\nu \setminus \{0\} \) the real equation \( f(\lambda; y) = 0 \) has at most one solution \( p(y) \in J \), and the condition
\[
\frac{\partial}{\partial \lambda} f(\lambda; y) \bigg|_{\lambda=p(y)} > 0 \quad \text{for every } y \in D
\]
holds.

The eigenvalues
\[
\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_\nu
\]
of the reduced problem (2) are minmax values of the Rayleigh quotient \( p(x) \) corresponding to problem (2).

Comparing \( \rho \) and \( \rho \) on appropriate subspaces of \( \mathbb{R}^\nu \) we arrive at the following a priori bound for the relative errors of the CMS approximations \( \tilde{\lambda}_j \) to \( \lambda_j \).

Theorem

Let \( K, M \in \mathbb{R}^{n \times n} \) be symmetric and positive definite, and let \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) be the eigenvalues of problem (1), which we assume to be ordered by magnitude.

Denote by \( \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_\nu \), the eigenvalues of the CMS approximation (2) of problem (1) corresponding to some partition of the graph \( |K| + |M| \) and some cut-off threshold \( \omega \).

Assume that the interval \((0, \omega]\) contains \( m \) eigenvalues \( \lambda_1, \ldots, \lambda_m \) of (1).

Then it holds
\[
0 \leq \frac{\tilde{\lambda}_j - \lambda_j}{\lambda_j} \leq \frac{\lambda_j}{\omega - \lambda_j} \leq \frac{\lambda_j}{\omega - \tilde{\lambda}_j}, \quad j = 1, \ldots, m.
\]

Hence, for eigenvalues which are far away from the cut-off threshold we may expect accurate approximations by the CMS method.
Proof

The left inequality, i.e. \( \lambda_j \leq \tilde{\lambda}_j \), is trivial since CMS is a projection method. The right inequality follows from the monotonicity of the function \( \lambda \mapsto \lambda/(\omega - \lambda) \).

To prove the inequality in the middle denote by \( V \subseteq S_j \), \( V \setminus \{0\} \subseteq D \) the \( j \)-dimensional subspace of \( \mathbb{R}^\nu \) such that

\[
\lambda_j = \max_{y \in V, y \neq 0} \rho(y).
\]

Then \( \rho(y) \leq \lambda_j \) for every \( y \in V, y \neq 0 \), and therefore it follows from the monotonicity of the function \( f(\lambda; y) \) with respect to \( \lambda \)

\[
- y^T \begin{pmatrix} \Omega_2 & 0 \\ 0 & K_0 \end{pmatrix} y + \lambda_j y^T \begin{pmatrix} I & \tilde{M}_{l1} \\ \tilde{M}_{l1} & \tilde{M}_{l2} \end{pmatrix} y + \lambda_j^2 y^T \begin{pmatrix} I \\ \tilde{M}_{l1} \end{pmatrix} (\Omega_1 - \lambda_j I)^{-1} (0 \tilde{M}_{l1}) y \geq 0.
\]

Hence, for every \( y \in V, y \neq 0 \) one obtains

\[
\lambda_j \geq \max_{y \in V, y \neq 0} \rho(y) - \lambda_j^2 \max_{y \in \mathbb{R}^\nu, y \neq 0} \frac{y^T \begin{pmatrix} 0 \tilde{M}_{l1} \\ 0 \tilde{M}_{l2} \end{pmatrix} y}{y^T \begin{pmatrix} I & \tilde{M}_{l2} \\ \tilde{M}_{l2} & \tilde{M}_{l1} \end{pmatrix} y} \geq \tilde{\lambda}_j - \frac{\lambda_j^2}{\omega - \lambda_j}.
\]

which completes the proof.

Proof ct.

Consider the model of a container ship with 10 substructures which has 1960 DoFs on the interfaces.

We solved the eigenproblem by the CMS method using a cut-off bound of 20,000 (about 10 times the largest wanted eigenvalue \( \lambda_{50} \approx 2183 \)).

329 eigenvalues of the substructure problems were less than our threshold, and the dimension of the resulting projected problem was 2289.

The following figure shows the relative errors for the smallest 50 eigenvalues (lower crosses in blue) and the error bounds (upper crosses in red).
Example ct.

A priori bound for AMLS

AMLs can be understood as a sequence of $\ell$ consecutive CMS steps and a terminating spectral truncation. It is clear how to obtain an a priori bound for the general AMLS method.

Hence, if $\lambda_j^{(\nu)}$ denotes the eigenvalues of the reduced eigenvalue problem corresponding to the $\nu$—th level ordered by magnitude, then it holds

$$\lambda_j^{(\nu)} \leq \lambda_j^{(\nu-1)} \left( 1 + \frac{\lambda_j^{(\nu-1)}}{\omega_{\nu}} - \frac{\lambda_j^{(\nu-1)}}{\omega_{\nu+1}} \right), \quad \nu = 1, 2, \ldots, p + 1.$$

where on the $\nu$—th level eigenvalues exceeding $\omega_{\nu}$ are neglected.

Thus, it follows for all $\lambda_j \leq \min_{\nu=1, \ldots, p} \omega_{\nu}$

$$\lambda_j^{(\ell+1)} \leq \lambda_j \prod_{\nu=0}^{\ell} \left( 1 + \frac{\lambda_j^{(\nu)}}{\omega_{\nu+1}} - \frac{\lambda_j^{(\nu)}}{\omega_{\nu}} \right),$$

and the following theorem follows.

A priori bound

In the example the relative error is overestimated by two or three orders of magnitude.

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ m & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with $\omega > 1$ and $m \in (0, 1)$ demonstrates that the a priori bound cannot be improved without further assumptions.

If $x_1$ and $x_2$ are the local degrees of freedom, and $x_3$ is the interface variable, then with cut-off frequency $\omega$ the minimum eigenvalue of the reduced problem is $\tilde{\lambda}_1 = 1$.

Letting $m \to 1 - 0$ l'Hospital's rule yields

$$\lim_{m \to 1 - 0} \frac{\tilde{\lambda}_1 - \lambda_1}{\omega} = \frac{1}{\omega} = \lim_{m \to 1 - 0} \frac{\lambda_1}{\omega - \lambda_1}.$$

Theorem

Let $K$, $M$ and $\lambda_j, j = 1, \ldots, n$ be given as in the last theorem. Let the graph of $|K| + |M|$ be substructured with $\ell$ levels, and denote by $\tilde{\lambda}_1^{(\nu)} \leq \tilde{\lambda}_2^{(\nu)} \leq \cdots$ the eigenvalues obtained by AMLS with cut-off threshold $\omega_{\nu}$ on level $\nu$.

If $m \in \mathbb{N}$ such that

$$\lambda_m < \min_{\nu=0, \ldots, p} \omega_{\nu},$$

then it holds

$$\frac{\tilde{\lambda}_j - \lambda_j}{\lambda_j} \leq \prod_{\nu=0}^{\ell} \left( 1 + \frac{\lambda_j^{(\nu)}}{\omega_{\nu} - \lambda_j^{(\nu)}} \right) - 1, \quad j = 1, \ldots, m.$$

Since the final problem is a projection of each of the intermediate eigenproblems in the AMLS reduction, it follows from the minmax characterization that $\lambda_j^{(\nu)} \leq \lambda_j$ for $\nu = 0, \ldots, p$. Therefore the a priori bound can be replaced by the computable bound

$$\frac{\tilde{\lambda}_j - \lambda_j}{\lambda_j} \leq \prod_{\nu=0}^{\ell} \left( 1 + \frac{\tilde{\lambda}_j}{\omega_{\nu} - \lambda_j} \right) - 1, \quad j = 1, \ldots, m.$$
Example

We substructured the FE model of the container ship by Metis with 4 levels of substructuring.

Neglecting eigenvalues exceeding 20,000 and 40,000 on all levels AMLS produced a projected eigenvalue problem of dimension 451 and 911, respectively.

The relative errors and the bounds are shown in the following figure where the lower and upper crosses correspond to the threshold 40,000, and the lower and upper circles to 20,000.