Approximation & Stability

Course notes [pdf]

The video lectures

This course is designed for 13-14 weeks. Each horizontal bar is marking the end of your weekly portion.

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week 1

Welcome

1. Introduction [pdf]

- **1.1 What this course is about - and what it's not**
- **1.2 What problems?**
  - 1.) linear equations
    - existence and
    - uniqueness of solutions
  - 2.) least squares problems
  - Insert: algebraic approach to \(A^{-1}\) and \(A^+\)
  - 3.) eigenvalue vs. spectral problems
- **1.3 Outlook: Approximation methods**

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week 2

2. Crash course on Functional Analysis [pdf]

- **2.1 Metric spaces and convergence**
  - metric spaces
  - examples
  - convergent and Cauchy sequences
  - completeness
  - closedness, vs completeness
- **2.2 Normed spaces**
  - definition

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week 3

- **2.2 Normed spaces (continued)**
  - examples
  - finite-dimensional
  - infinite-dimensional
  - norm induces metric
- **2.3 Banach spaces**
  - definition and first examples
  - further examples
- **2.4 Hilbert spaces**
  - scalar product and examples
  - scalar product implies norm

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week 4

- **2.4 Hilbert spaces (continued)**
  - definitions: angles, Euclidean space, Hilbert space
  - The (PI) or: Which norms come from a scalar product?
  - examples
  - orthogonal projection and the approximation theorem
- **2.5 Linear operators**
  - definitions: bounded sets and bounded linear operators
  - the operator norm
  - for linear operators, boundedness = continuity
  - the set \(L(X,Y)\) as a normed space
  - *\(L(X,Y)\) is a Banach space if \(Y\) is* one

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week 5

- **2.5 Linear operators (continued)**
  - relatively compact sets
  - compact operators
  - the adjoint operator
    - definition
    - examples
- **2.6 Direct sums and projections**
  - direct sums
  - projections

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week 6

3. Approximation methods for \(Ax=b\) [pdf]

- Applicability
- **3.1 Uniform convergence**
  - (i) invertibility is infectious
  - (ii) inverses converge to the inverse
  - stability
- **3.2 Strong convergence**
  - motivation, example and definition
  - again: strong vs. uniform convergence

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week 7

- **3.2 Strong convergence (continued)**
  - \(A_n\) as an operator \(X_n\to Y_n\)
  - example: the finite section method (FSM)
    - definition and reformulation via \(P_n AP_n\)
    - three perspectives on \(A_n\) and \(X_n\)
  - example: forward shift
    - the FSM is not applicable
    - a small modification makes it applicable
    - preview: approximation of spectra
- **3.3 Projection methods**
  - construction of projection methods
  - properties of the strong limit
  - examples of projectors (and projection methods)
    - example 1 - 3

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week 8

- **3.3 Projection methods (continued)**
  - examples of projectors (continued)
    - example 4 - 6
    - example 7 - 9
- **3.4 Polski's theorem**
- **3.5 More examples of projection methods**
  - collocation method
  - Galerkin method
  - comparison: Galerkin vs. FSM

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week 9

- **3.6 A little insert on operator theory**

  (3.6 is a bit technical. Don't worry if you're not catching every little bit of the proofs. But make sure you remember the results.)

- **3.7 Positive definiteness and rectangular finite sections**
  - positive definiteness
  - positive definiteness is sufficient for applicability of the FSM
  - rectangular finite sections work well
    - a simple idea and a little setback...
    - ...lead to a positive result for arbitrary invertible operators \(A\)
- **3.8 Differential equations: The Laplace BVP**
  - the Laplace boundary value problem
  - fundamental solutions of the Laplace operator
  - the double layer potential ansatz

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week 10

4. Approximation methods for \(Ax\approx b\) [pdf]

- 4.1 Recall: The Moore-Penrose pseudoinverse \(A^+\)
- 4.2 Singular value decomposition (SVD)
  - simple illustration for 2-by-2
  - general case
  - constructing \(A^+\) via the SVD of \(A\)
- **4.3 Stability and the SVD**
- **4.4 The condition number**
- **4.5 Regularization: The TSVD**

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week 11

- **4.6 Regularization: Tichonov regularization**
- **4.7 Regularization: TSVD vs. Tichonov**
- **4.8 Regularization of approximation methods**
  - What if my projection method is not applicable? A new Polski theorem.
  - sketch of proof
  - discussion of conditions and statements
  - stability vs. M.P.-stability vs. stable regularizability

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week 12

5. Spectra and their approximation [pdf]

- **5.1 Properties of the spectrum**
  - eigenvalues, spectral values, examples
  - The spectrum is closed, bounded and non-empty.
- **5.2 Spectra and dynamical systems**
  - discrete time
  - continuous time
- **5.3 The spectral radius**
  - formula and proof
  - When is the spectral radius equal to the norm?

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week 13

- **5.4 Spectra of convolution operators**
  - definition and examples
  - invertibility and spectrum
  - back to the examples: invertibility and spectrum
  - outlook: periodic operators
  - Matlab: spectra and singular value splitting
- **5.5 Approximation of spectra**

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week 14

- **5.6 Pseudospectra**
  - definition and a little advertisement block
  - How large can the pseudospectrum be?
  - convergence of pseudospectra and approximation of spectra