ITERATIVE PROJECTION METHODS FOR SPARSE LINEAR SYSTEMS AND EIGENPROBLEMS
CHAPTER 3 : SEMI-ITERATIVE METHODS

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Motivation

Consider a splitting method

$$x^{k+1} := B^{-1}\Big((B-A)x^k + b\Big) =: Gx^k + \tilde{b}$$

for the solution of the linear system of equations

$$Ax = b, \quad (A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n \text{ given}).$$

Question: Is it possible to improve the approximations $x^0, x^1, x^2, \dots, x^k$ by a linear combination

$$y^k := \sum_{j=0}^{k} \alpha_{k,j} x^j?$$

Error

Let

$$q_k(t) := \sum_{j=0}^{k} \alpha_{k,j} t^j$$

be a given polynomial of degree k such that $q_k(1) = 1$, and let $y^k = \sum_{j=0}^{k} \alpha_{k,j} x^j$.

Then it holds for the error of the original sequence

$$x^k - x^* = Gx^{k-1} + \tilde{b} - (Gx^* + \tilde{b}) = G(x^{k-1} - x^*) = \cdots = G^k(x^0 - x^*)$$

from which we obtain for the error $e^k := y^k - x^*$ of the modified sequence

$$e^k = \sum_{j=0}^{k} \alpha_{k,j} x^j - \sum_{j=0}^{k} \alpha_{k,j} x^* = \sum_{j=0}^{k} \alpha_{k,j}(x^j - x^*) = q_k(G)e^0.$$

If $G = T^{-1}JT$ is the Jordan canonical form of G, then obviously, $G^j = T^{-1}J^jT$, and therefore,

$$q_k(G) = T^{-1}q_k(J)T.$$

$q_k(G)$ is similar to the upper triangular matrix $q_k(J)$ with diagonal elements $q_k(\lambda_j)$, where λ_j are the eigenvalues of G.

Hence, the spectral radius of $q_k(G)$ is given by

$$\rho(q_k(G)) = \max_{j=1,\dots,n} |q_k(\lambda_j)|.$$

To obtain the optimum polynomial method we have to solve the discrete Chebyshev approximation problem:
Find a polynomial q_k of maximum degree k such that

$$\max_{j=1,\dots,n} |q_k(\lambda_j)| = \min! \quad \text{and} \quad q_k(1) = 1. \qquad (*)$$

Usually, the spectrum of the iteration matrix is not known but only a set $\Omega \subset \mathbb{C}$ which contains spec (G). In this case problem (*) is replaced by the continuous approximation problem:
Find a polynomial q_k of maximum degree k such that

$$\max_{z\in\Omega} |q_k(z)| = \min! \quad \text{and} \quad q_k(1) = 1. \qquad (**)$$

For the general case of an arbitrary set $\Omega \subset \mathbb{C}$ including the complex spectrum of G problem (**) is very difficult to solve. For special sets like circles, ellipses, rectangles and intervals on the real or on the imaginary axis the solution of (**) can be given (Hageman, Young (1981) and the literature given therein). We restrict ourselves to the case that Ω is a real interval.

From now on we assume that the eigenvalues λ_j of the iteration matrix G of the base iteration are real and that bounds m_E and M_E are known

$$m_E \le m(G) := \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n =: M(G) \le M_E.$$

Moreover, we suppose that (observe condition $q_k(1) = 1$!)

$$M_E < 1.$$

These assumptions are satisfied, for example, if the system matrix A is symmetric and positive definite and G is the iteration matrix of the symmetric Gauß-Seidel method or of the SSOR method. In this case G is positive semidefinite, and $m_E = 0$ is a lower bound of the spectrum of G.

For $\Omega = [m_E, M_e]$ the solution of the approximation problem (**) can be given in terms of the Chebyshev polynomials.

Definition
The Chebyshev polynomials are defined as

$$C_k(t) := \cos(k \cdot \arccos t), \quad |t| \le 1, \quad k \in \mathbb{N} \cup \{0\}.$$

In the following lemma we collect some properties of the Chebyshev polynomials that are needed in the sequel.

Lemma 3.1

(i) The functions $C_k(t)$ satisfy the recurrence formula

$$C_0(t) \equiv 1, \quad C_1(t) = t, \quad C_{k+1}(t) = 2tC_k(t) - C_{k-1}(t),\ k \geq 1.$$

In particular, C_k is a polynomial of degree k.

(ii) For $|t| \geq 1$ the Chebyshev polynomials have the following representation

$$C_k(t) = \cosh(k \cdot \text{Arcosh}\ t), \quad t \in \mathbb{R},\ |t| \geq 1.$$

Proof: (i) follows from the properties of the cosine function, in particular from

$$\cos((k+1)t) + \cos((k-1)t) = 2\cos(kt) \cdot \cos t.$$

(ii) follows from the fact that the functions $\cosh(k \cdot \text{Arcosh}\ t)$ satisfy the same recurrence formula. $\square$

Theorem 3.2

Let $[m_E, M_E] \subset (-\infty, 1)$ be an interval. Then the approximation problem:
Find a polynomial p_k of maximum degree k such that $p_k(1) = 1$ and

$$\max_{m_E \le t \le M_E} |p_k(t)| = \min!$$

has the unique solution

$$\tilde{p}_k(t) = \frac{1}{c_k} C_k\left(\frac{2t - m_E - M_E}{M_E - m_E}\right), \quad c_k := C_k\left(\frac{2 - m_E - M_E}{M_E - m_E}\right),$$

where C_k denotes the k-th Chebyshev polynomial.

The minimizer $\tilde{p}_k$ is of degree k and the minimum value is

$$\max\{|\tilde{p}_k(t)| \,:\, m_E \le t \le M_E\} = \frac{1}{c_k}.$$

Proof

From $\hat{t} := (2 - m_E - M_E)/(M_E - m_E) \notin [-1, 1]$ it follows that $c_k \neq 0$. Hence, $\tilde{p}_k$ is defined, and obviously, $\tilde{p}_k$ is a polynomial of degree k, and $\tilde{p}_k(1) = 1$ holds.

$(2t - m_E - M_E)/(m_E + M_E) \in [-1, 1]$ for every $t \in [m_E, M_E]$.
Hence, $|\tilde{p}_k(t)| \leq 1/c_k$ and that maximum value of $|\tilde{p}_k|$ is $1/c_k$.

It remains to show, that $\tilde{p}_k$ is the unique solution of the minimum problem.

Assume that q_k is a polynomial of degree k such that $q_k(1) = 1$ and

$$\max_{m_E \leq t \leq M_E} |q_k(t)| \leq \frac{1}{c_k}.$$

Obviously, the Chebyshev polynomial $C_k(\tau)$ attains the values $+1$ and -1 alternately at the arguments $\tau_j := \cos(j\pi/k)$, $j = 0, \dots, k$. Hence,

$$\tilde{p}_k(t_j) = \frac{(-1)^j}{c_k} \quad \text{for } t_j := \frac{2\tau_j - m_E - M_E}{M_E - m_E}, \quad j = 0, \dots, k.$$

Proof ct.

Let $r_k := \tilde{p}_k - q_k$. Then from

$$|q_k(t_j)| \le \frac{1}{c_k} = |\tilde{p}_k(t_j)|, \quad j = 0, \dots, k,$$

one obtains that

$$r(t_j) \ge 0, \text{ if } j \text{ is even}, \quad r(t_j) \le 0, \text{ if } j \text{ is odd}.$$

From the continuity of r_k we get the existence of a root $\tilde{t}_j$ in every interval $[t_j, t_{j-1}]$, $j = 1, \dots, k$. Moreover, $r_k(1) = \tilde{p}_k(1) - q_k(1) = 0$.

Therefore, the polynomial r_k of degree k has at least $k + 1$ roots, and it follows that $r_k(t) \equiv 0$, i.e. $q_k = \tilde{p}_k$. □

By Theorem 3.2 the Chebyshev polynomials yield the optimum acceleration method for a base method $x^{k+1} := Gx^k + \tilde{b}$, if the spectrum of G is known to be real and one uses an interval $[m_E, M_E]$ as including set of the spectrum. The resulting semi-iteration is called Chebyshev iteration.

We briefly analyze the improvement of the convergence by the Chebyshev iteration. On account of the assumption that all eigenvalues of G are contained in the interval $[m_E, M_E]$,

$$\frac{1}{C_k(d)} = \max_{m_E \le t \le M_E} |\tilde{p}_k(t)|, \quad d := \frac{2 - M_E - m_E}{M_E - m_E},$$

is an upper bound of the spectral radius of $\tilde{p}_k(G)$.

If $\rho(\tilde{p}_k(G))$ denotes the error reduction factor of the transition from y^0 to y^k then in the first k steps the error is reduced on an average by the factor

$$v_k := \sqrt[k]{\rho(\tilde{p}_k(\ G))} \leq \frac{1}{\sqrt[k]{C_k(d)}}.$$

From $C_k(t) = \cosh(k \cdot \operatorname{Arcosh} t)$ one obtains (notice that $d > 1$)

$$\begin{aligned} C_k(d) &= \frac{1}{2}\Big(\exp(k \cdot \operatorname{Arcosh}(d)) + \exp(-k \cdot \operatorname{Arcosh}(d))\Big) \\ &= \frac{1}{2}\Big(R^{k/2} + R^{-k/2}\Big) \qquad (+), \end{aligned}$$

where $R := \exp(2 \cdot \operatorname{Arcosh}(d))$.

In particular, for $k = 2$ $(+)$ is the quadratic equation for R

$$\frac{1}{2}(R + R^{-1}) = C_2(d) = 2d^2 - 1,$$

from which we obtain

$$R = \frac{1 - \sqrt{1 - \sigma^2}}{1 + \sqrt{1 - \sigma^2}}, \quad \sigma := \frac{1}{d}.$$

Theorem 3.3

Suppose that the spectrum of the iteration matrix G of the base iteration is contained in the interval $[m_E, M_E]$.

Then the average error reduction factor v_k of the first k steps of the Chebyshev iteration satisfies

$$v_k \le \sqrt{R} \cdot \sqrt[k]{\frac{2}{1 + R^k}} =: V_k$$

where

$$R := \frac{1 - \sqrt{1 - \sigma^2}}{1 + \sqrt{1 - \sigma^2}} < 1 \quad \text{and} \quad \sigma := \frac{M_E - m_E}{2 - M_E - m_E}.$$

For $k \to \infty$ the upper bound V_k converges to the asymptotic error reduction factor $V_\infty = \sqrt{R}$.

Example

To obtain an impression of the convergence improvement by the Chebyshev acceleration we consider the Jacobi method for the difference approximation of the model problem for $h = 1/128$.

Then the spectrum of the base iteration is real, and

$$-\cos\frac{\pi}{128} \le \lambda_j \le \cos\frac{\pi}{128} \approx 0.999699, \quad j = 1, 2, \dots, 128.$$

Hence, we can choose $M_E = -m_E = \rho(G) = \cos(\pi/128)$.

To reduce the error by the base iteration by the factor 10^{-3} it takes about $\ln 0.001/\ln\rho \approx 22932$ iteration steps.

Example ct.

In this example $R \approx 0.952083$, from which we obtain the asymptotic error reduction factor $V_\infty \approx 0.975753$.

If we estimate the number of steps to reduce the error by 10^{-3} using the Chebyshev iteration by the requirement

$$V_\infty^k = 10^{-3},$$

then we get $k = 282$.

This is a little too optimistic. Actually

$$V_{282} \approx 0.9782 \quad \text{and} \quad V_{282}^{282} \approx 0.00197 > 10^{-3},$$

and the required error reduction is arrived in the 310-th step:

$$V_{310} \approx 0.9779 \quad \text{and} \quad V_{310}^{310} \approx 9.779 * 10^{-4} < 10^{-3}.$$

The following table contains for the first k steps the upper bound of the average error reduction factor V_k, the bound of the error reduction factor V_k^k, the estimate V_∞^k of the error reduction and the reduction factor for the Jacobi method, the base iteration. It is seen that one obtains a substantial improvement only if the number of iterations is not too small. □

Example ct.

k	V_k	V_k^k	V_∞^k	$\rho(G)^k$
1	9.9970e-001	9.9970e-001	9.7575e-001	9.9970e-001
2	9.9940e-001	9.9880e-001	9.5209e-001	9.9940e-001
3	9.9910e-001	9.9729e-001	9.2901e-001	9.9910e-001
4	9.9880e-001	9.9520e-001	9.0648e-001	9.9880e-001
5	9.9850e-001	9.9252e-001	8.8450e-001	9.9850e-001
10	9.9702e-001	9.7061e-001	7.8234e-001	9.9699e-001
20	9.9422e-001	8.9052e-001	6.1206e-001	9.9399e-001
30	9.9171e-001	7.7905e-001	4.7884e-001	9.9100e-001
40	9.8955e-001	6.5703e-001	3.7462e-001	9.8802e-001
50	9.8774e-001	5.3979e-001	2.9308e-001	9.8505e-001
100	9.8247e-001	1.7053e-001	8.5896e-002	9.7033e-001
150	9.8027e-001	5.0317e-002	2.5175e-002	9.5582e-001
200	9.7914e-001	1.4756e-002	7.3782e-003	9.4153e-001
250	9.7846e-001	4.3248e-003	2.1624e-003	9.2746e-001
300	9.7801e-001	1.2675e-003	6.3376e-004	9.1359e-001
310	9.7794e-001	9.9163e-004	4.9581e-004	9.1085e-001

We do not have to compute the iterates $x^0, x^1, \dots, x^k$ by the base method and to store them, and to improve the approximations by the polynomial method under considerations afterwards.

This would require high arithmetic cost and a large amount of storage.

Taking advantage of the recurrence relation of the Chebyshev polynomials we now describe a less costly way to obtain the improved approximations directly.

This simpler computational form is possible whenever the sequence of polynomials satisfies a three term recurrence relation of the following type. Therefore, we more generally consider a set of polynomials q_k satisfying

$$\left.\begin{array}{l} q_0(t) \equiv 1, \quad q_1(t) = \gamma_1 t - \gamma_1 + 1, \\ q_{k+1}(t) = \delta_{k+1}(\gamma_{k+1} t + 1 - \gamma_{k+1}) q_k(t) + (1 - \delta_{k+1}) q_{k-1}(t), \; k \geq 1, \end{array}\right\}$$

where γ_j and δ_j are given real constants.

Recurrence relations of this type appear for example if one considers regular, symmetric and indefinite systems, where $[m_E, M_E]$ is replaced by the union of two disjoint intervals, which contains the spectrum of A but not the origin.

Theorem 3.4

Suppose that the system of equations $x = Gx + \tilde{b}$ has a unique solution x^*. Let the sequence $\{x^k\}$ of vectors be generated by

$$x^{k+1} := Gx^k + \tilde{b}, \qquad (1)$$

and let

$$y^k := \sum_{j=0}^{k} \alpha_{k,j} x^j, \quad k = 0, 1, \dots . \qquad (2)$$

If the polynomials

$$q_k(t) := \sum_{j=0}^{k} \alpha_{k,j} t^j$$

satisfy the three term recurrence relation

$$\left. \begin{array}{l} q_0(t) \equiv 1, \quad q_1(t) = \gamma_1 t - \gamma_1 + 1, \\ q_{k+1}(t) = \delta_{k+1}(\gamma_{k+1} t + 1 - \gamma_{k+1}) q_k(t) + (1 - \delta_{k+1}) q_{k-1}(t), \; k \geq 1, \end{array} \right\} \qquad (3)$$

Theorem 3.4 ct.

then the sequence $\{y^k\}$ can be generated by

$$\left.\begin{array}{l} y^0 = x^0 \\ y^1 = \gamma_1(Gy^0 + \tilde{b}) + (1-\gamma_1)y^0 \\ y^{k+1} = \delta_{k+1}\Big(\gamma_{k+1}(Gy^k + \tilde{b}) + (1-\gamma_{k+1})y^k\Big) + (1-\delta_{k+1})y^{k-1}, \quad k \geq 1. \end{array}\right\}$$

If, conversely, the vectors y^k are generated by (4), then they satisfy (2), where the vectors x^j are given by (1).

Proof

Let $e^k := y^k - x^*$ be the error vector of y^k.
Then $e^k = q_k(G)e^0$, and we obtain from equation (3) for $k \geq 1$

$$e^{k+1} = \left[\delta_{k+1}\Big(\gamma_{k+1}\ G + (1-\gamma_{k+1})\ I\Big)q_k(\ G) + (1-\delta_{k+1})q_{k-1}(\ G)\right] e^0,$$

and therefore,

$$e^{k+1} = \delta_{k+1}\Big(\gamma_{k+1}\ G + (1-\gamma_{k+1})\ I\Big)e^k + (1-\delta_{k+1})e^{k-1}. \tag{5}$$

Adding x^* on both sides one gets from

$$\begin{aligned} &\delta_{k+1}\Big(\gamma_{k+1}\ G + (1-\gamma_{k+1})\ I\Big)x^* + (1-\delta_{k+1})x^* \\ = \quad &\delta_{k+1}\gamma_{k+1}Gx^* - \delta_{k+1}\gamma_{k+1}x^* + x^* = x^* - \delta_{k+1}\gamma_{k+1}\tilde{b} \end{aligned}$$

Proof ct.

the equation

$$y^{k+1} = \delta_{k+1}\Big(\gamma_{k+1}\; G + (1-\gamma_{k+1})\; I\Big)y^k + (1-\delta_{k+1})y^{k-1} + \delta_{k+1}\gamma_{k+1}\tilde{b},$$

which is the three term recurrence relation (3). The case for e^1 follows similarly.

Conversely, let the sequence of vectors $\{y^k\}$ be generated by (4), and denote by $\tilde{e}^k := y^k - x^*$ the error vector of y^k.

Repeating the steps above in reverse order, where e^k is replaced by $\tilde{e}^k$, we obtain $\tilde{e}^{k+1} = q_{k+1}(\; G)\tilde{e}^0$ with q_k defined by (2). Therefore, it follows that the iterative method (4) is equivalent to a polynomial method, where the polynomials are given by (1). □

Form the recurrence relation of the Chebyshev polynomials one obtains the following recurrence relation of the transformed Chebyshev polynomials $\tilde{p}_k$ on the interval $[m_E, M_E]$:

Let

$$\sigma := \frac{M_E - m_E}{2 - M_E - m_E}, \quad \tilde{\gamma} := \frac{2}{2 - M_E - m_E},$$

$$\tilde{\delta}_1 := 1, \quad \tilde{\delta}_2 := \left(1 - 0.5\sigma^2\right)^{-1}, \quad \tilde{\delta}_{k+1} := (1 - 0.25\sigma^2\tilde{\delta}_k)^{-1}, \ k \geq 2.$$

Then

$$\left.\begin{array}{l} \tilde{p}_0(t) \equiv 1, \quad \tilde{p}_1(t) = \tilde{\gamma}t - \tilde{\gamma} + 1, \\ \tilde{p}_{k+1}(t) = \tilde{\delta}_{k+1}(\tilde{\gamma}t - \tilde{\gamma} + 1)\tilde{p}_k(t) + (1 - \tilde{\delta}_{k+1})\tilde{p}_{k-1}(t), \quad k \geq 1. \end{array}\right\}$$

Chebyshev iteration

This recurrence relation is of the form (3). Hence, from Theorem 3.4 we get the Chebyshev iteration for the iterative solution of the linear system of equations $Ax = b$:

Let

$$x^{k+1} = B^{-1}\Big((B - A)x^k + b\Big) = Gx^k + \tilde{b}$$

be a base iteration such that the iteration matrix G has real eigenvalues which are contained in $(-\infty, 1)$.

Determine estimates $M_E < 1$ of the largest eigenvalue and $m_E < M_E$ of the smallest eigenvalue of G.

Chebyshev iteration ct.

Set

$$\sigma := \frac{M_E - m_E}{2 - M_E - m_E}, \quad \text{and} \quad \gamma := \frac{2}{2 - M_E - m_E},$$

and iterate as follows:

$$\begin{aligned}
k = 0 \; &: \quad y^0 \text{ given initial approximation} \\
k = 1 \; &: \quad y^1 := \gamma(Gy^0 + \tilde{b}) + (1-\gamma)y^0 \\
k = 2 \; &: \quad \delta_2 := (1 - 0.5\sigma^2)^{-1} \\
&\quad\;\; y^2 := \delta_2\Big(\gamma(Gy^1 + \tilde{b}) + (1-\gamma)y^1\Big) + (1-\delta_2)y^0 \\
k \geq 2 \; &: \quad \delta_{k+1} := (1 - 0.25\sigma^2\delta_k)^{-1} \\
&\quad\;\; y^{k+1} := \delta_{k+1}\Big(\gamma(Gy^k + \tilde{b}) + (1-\gamma)y^k\Big) + (1-\delta_{k+1})y^{k-1}.
\end{aligned}$$

Example

Consider the model problem with $h = 1/128$ where the unknowns are ordered lexicographically.

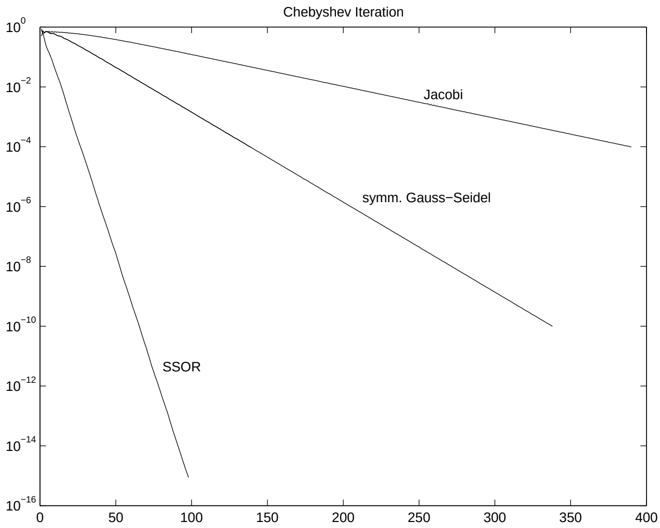
The following picture contains the convergence history for the Chebyshev iteration. The base iterations are

(i) the Jacobi method, where the minimum and maximum eigenvalue of the iteration are known to be $M_E = \cos(\pi/128)$ and $m_E = -\cos(\pi/128)$

(ii) the symmetric Gauss-Seidel method where the iteration matrix is similar to a symmetric and positive semidefinite matrix. Therefore, we choose $m_E = 0$. The spectral radius of G was determined experimentally $M_E = 0.9988$.

(iii) the SSOR method for the experimentally determined optimum parameter $\omega = 1.96$ where $m_E = 0$ and $M_E = 0.9682$.

Example ct.



Chebyshev Iteration
10^0
10^−2
10^−4
10^−6
10^−8
10^−10
10^−12
10^−14
10^−16
0
50
100
150
200
250
300
350
400
Jacobi
symm. Gauss−Seidel
SSOR

Choice of Parameters

In the Chebyshev acceleration we need estimates m_E of smallest eigenvalue $m(G)$ and M_E of the largest eigenvalue $M(G)$ of the iteration matrix of the base method. Convergence is only guaranteed, if $m_E \leq m(G)$ and $M(G) \leq M_E < 1$, but bounds of the eigenvalues are usually not at hand.

If the estimates m_E and M_E are not too bad (and if the error of the initial vector contains components of all eigenvectors of G), then the error of the iteration should decrease in that way that is predicted by Theorem 3.2.

If this is not the case, then we can try to improve the estimates by the comparison of the expected and the observed error reduction.

Choice of Parameters ct.

If new estimates $\tilde{m}_E$ and/or $\tilde{M}_E$ have been obtained by some adaptive method and are included into the computation, then all polynomials change, and hence, the acceleration algorithm has to be restarted.

Notice however, that the improvement of the base method by the Chebyshev acceleration becomes effective only if the number of iterations is not too small. Hence the method should not be restarted too often.

We will not discuss the algorithms for the improvement of the estimates in detail. Adaptive Chebyshev acceleration procedures are developed in Hageman & Young (1981).

Choice of Parameters ct.

In the following we develop some rules of thumb for the choice of the parameters m_E and M_E. We motivate them by the first acceleration step which yields y^1.

The error of y^1 is given by

$$e^1 = \tilde{p}_1(G)e^0.$$

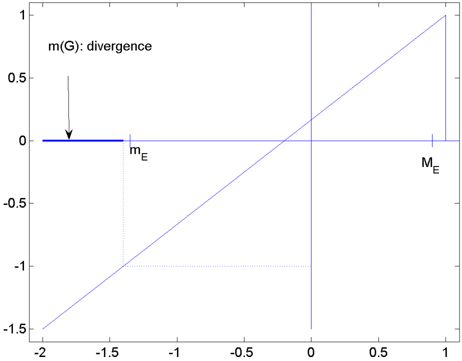
The reduction of e^1 is determined by the maximum size of $|\tilde{p}_1(t)|$ on the spectrum of G. The polynomial $\tilde{p}_1$ is linear, it satisfies $\tilde{p}_1(1) = 1$, and its Chebyshev norm is minimal in the class of all linear polynomials p_1 such that $p_1(1) = 1$.

It is obvious that $\tilde{p}_1$ is characterized by the fact that it attains the value 0 at the midpoint of the interval $[m_E, M_E]$. Hence, it is given by

$$\tilde{p}_1(t) = \frac{2}{2 - M_E - m_E}\Big(t - \frac{M_E + m_E}{2}\Big).$$

Choice of Parameters ct.

Because $M_E < 1$ is very close to 1 (otherwise it would not be necessary to accelerate the convergence of the base method) divergence can occur very easily if $m_E > m(G)$.



RULE 1: *m_E should not overestimate the smallest eigenvalue $m(G)$.*

Choice of Parameters ct.

Usually, it is not very difficult to find a safe lower bound of $m(G)$.

For the Jacobi iteration the spectrum of G is symmetric. Hence, from the convergence of the base method, one obtains the lower bound $m_E := -1 < m(G)$.

For the symmetric Gauß-Seidel method and the SSOR method the iteration matrices are similar to symmetric and positive semidefinite matrices. Hence, $m_E = 0$ is a safe choice.

The bound $m_E = 0$ can always be chosen, if we replace the base iteration by the corresponding 'double-step method' where we consider only those iterates x^{2k} with an even index:

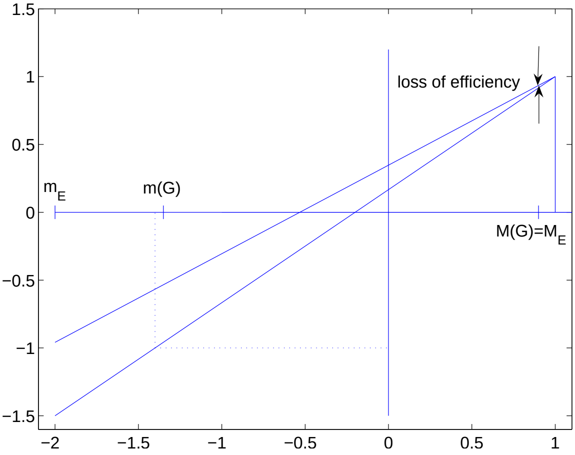
$$\hat{x}^{k+1} = G(G\hat{x}^k + \tilde{b}) + \tilde{b} =: \hat{G}\hat{x}^k + \hat{b}, \quad \hat{G} := G^2,\ \hat{b} := \tilde{b} + G\tilde{b}.$$

Obviously, the spectrum of the iteration matrix $\hat{G}$ is nonnegative.

Choice of Parameters ct.

RULE 2: *A moderate underestimate of $m(G)$ is not critical.*

Because usually M_E is very close to 1 the rate of convergence is not deteriorated very much if m_E is chosen too small.



Choice of Parameters ct.

RULE 3: *If the estimate is to be improved during the iteration, it is better to choose $M_E < M(G)$ than $M_E > M(G)$.*

If $M_E < M(G)$ then all components of the error corresponding to eigenvalues below M_E are reduced efficiently whereas the components corresponding to eigenvalues above M_E are even amplified. Hence, we can use the Rayleigh quotient of the generalized eigenvalue problem

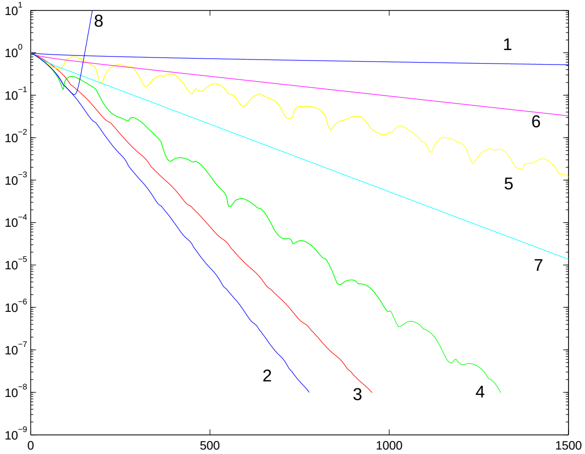
$$(B - A)z = \lambda Bz,$$

which has the same eigenvalues as $G := B^{-1}(B - A)$, to improve the estimate of G considerably.

Example

Consider the difference approximation of the model problem with stepsize $h = 1/128$.
The next figure contains in a semilogarithmic scale the errors of the Jacobi method (graph 1) and Chebyshev acceleration based on the Jacobi method for different choices of the parameters m_E and M_E, namely

graph 2	blue	$m_E = -\cos(\pi/128)$	$M_E = \cos(\pi/128)$
graph 3	red	$m_E = -2.0$	$M_E = \cos(\pi/128)$
graph 4	green	$m_E = -\cos(\pi/128)$	$M_E = 0.9999$
graph 5	yellow	$m_E = -\cos(\pi/128)$	$M_E = 0.99999$
graph 6	magenta	$m_E = -\cos(\pi/128)$	$M_E = 0.99$
graph 7	cyan	$m_E = -\cos(\pi/128)$	$M_E = 0.999$
graph 8	blue	$m_E = -0.99$	$M_E = \cos(\pi/128)$.


10^1
10^0
10^−1
10^−2
10^−3
10^−4
10^−5
10^−6
10^−7
10^−8
10^−9
0
500
1000
1500
1
2
3
4
5
6
7
8