Dynamic Behavior of Complex Structures
CHAPTER 5: NONLINEAR EIGENVALUE PROBLEMS

Heinrich Voss
voss@tu-harburg.de

Hamburg University of Technology
Institute of Numerical Simulation
Nonlinear eigenvalue problems

Let $D \subset \mathbb{C}$ be an open set (maybe unbounded), and let

$$T(\lambda) \in \mathbb{C}^{n \times n}, \quad \lambda \in D$$

be a family of matrices.

Find $\lambda \in D$ and $x \neq 0$ and/or $y \neq 0$ such that

$$T(\lambda)x = 0 \quad \text{and/or} \quad y^H T(\lambda) = 0.$$  

Then $\lambda$ is called an eigenvalue of $T(\cdot)$, and $x$ and $y$ are corresponding right and left eigenvectors.

Problems of this type arise in vibrations of conservative gyroscopic systems, damped vibrations of structures, problems with retarded argument, lateral buckling problems, fluid-solid vibrations, quantum dot heterostructures, and sandwich plates, e.g.
Example 1: Vibration of structures

Equations of motion arising in dynamic analysis of structures (with a finite number of degrees of freedom) are

$$M \ddot{q}(t) + C \dot{q}(t) + Kq(t) = f(t)$$

where $q$ are the Lagrangean coordinates, $M$ is the mass matrix, $K$ the stiffness matrix, $C$ the viscous damping matrix, and $f$ an external force vector.

Suppose that the system is exited by a time harmonic force $f(t) = f_0 e^{i\omega_0 t}$.

If $i\omega_0$ is not an eigenvalue of the quadratic eigenproblem

$$(\lambda^2 M + \lambda C + K)x = 0$$

and if all eigenvalues are distinct then the periodic response of the system is

$$q(t) = e^{i\omega_0 t} \sum_{j=1}^{2n} \frac{y_j^H f_0}{i\omega_0 - \lambda_j} x_j$$

where $x_j$ and $y_j$ denote the right- and left eigenvectors.

Usually, one gets good approximations taking into account only a small number of eigenvalues in the vicinity of $i\omega_0$ (truncated mode superposition).
Example 2: Dynamic element method

Discretizing a linear eigenproblem

\[ Lu(x) = \lambda Mu(x), \quad x \in \Omega, \quad Bu(x) = 0, \quad x \in \partial \Omega \]

by finite elements which depend on the eigenparameter yields a nonlinear eigenproblem; cf. Przemieniecki (1968).

If the ansatz functions \( \phi_j(x) + \lambda \psi_j(x) \) depend linearly on the eigenparameter then one obtains a cubic eigenvalue problem (cf. V. 1987)

\[
\sum_{j=0}^{3} \lambda^j A_j x = 0
\]

where

\[
A_3 = \left( \int_{\Omega} \psi_j M \psi_k \, dx \right)_{jk}, \quad A_2 = \left( \int_{\Omega} (\psi_j M \phi_k + \phi_j M \psi_k - \psi_j L \psi_k) \, dx \right)_{jk}
\]

\[
A_1 = \left( \int_{\Omega} (\phi_j M \phi_k - \phi_j L \psi_k - \psi_j L \phi_k) \, dx \right)_{jk}, \quad A_0 = -\left( \int_{\Omega} \phi_j L \phi_k \, dx \right)_{jk}
\]
Example 3: Conservative gyroscopic systems

Simulation of acoustic behavior of rotating structures (rotating wheels, e.g.) results in

\[ M\ddot{q} + G\dot{q} + Kq = 0 \]

where \( M \) is positive definite, \( K \) is positive (semi-)definite, \( G \) skew-symmetric, i.e. \( G^T = -G \).

The corresponding eigenvalue problem

\[ \omega^2 Mx + \omega Gx + Kx = 0 \]

has purely imaginary eigenvalues.

Wanted in simulation of rotating tires is a large number of (not necessarily extreme) eigenvalues, for instance for rolling wheels all eigenfrequencies between 500 Hz and 2000 Hz, an important interval of perception of the human ear (cf. Nackenhorst 2004).
Example 4: Controlled systems with delayed feedback

The governing equation of a mechanical system with delayed feedback is (Elsgolts & Norkin 1974, Hale 1977)

\[ M\ddot{u} + C\dot{u} + Ku = G_u u(t - \tau) + G\dot{u}\dot{u}(t - \tau). \]

This delay normally originates from physical limitations like finite switching times in controllers or unavailability of the current state of the system.

An ansatz \( u(t) = e^{\lambda t} x \) yields the eigenproblem

\[ \lambda^2 Mx + \lambda Cx + Kx - e^{-\lambda \tau} G_u x - \lambda e^{-\lambda \tau} G\dot{u} x = 0 \]

The system is stable if all eigenvalues have negative real part.
Example 5: Viscoelastic model of damping

Using a viscoelastic constitutive relation to describe the material behavior in the equations of motion yields a rational eigenvalue problem in the case of free vibrations.

Discretizing by finite elements yields (cf. Hager & Wiberg 2000)

\[ T(\omega)x := \left( \omega^2 M + K - \sum_{j=1}^{K} \frac{1}{1 + b_j \omega} \Delta K_j \right)x = 0 \]

where \( M \) is the consistent mass matrix, \( K \) is the stiffness matrix with the instantaneous elastic material parameters used in Hooke’s law, and \( \Delta K_j \) collects the contributions of damping from elements with relaxation parameter \( b_j \).
Example 6: Fluid–solid vibrations

Determine acoustic eigenfrequencies of a (very long) cavity containing a tube bundle.

Tubes are
- immersed in an inviscid compressible fluid
- rigid, assembled in parallel inside the fluid,
- elastically mounted such that they can vibrate transversally, but can not move in the direction perpendicular to their sections.

Due to these assumptions, three-dimensional effects are neglected, and the problem is studied in any transversal section of the cavity.
Fluid–solid vibrations ct.

$\Omega$ : section of cavity, $\Omega_j$ : section of tube $j$
The fluid-solid vibration problem was modeled by Conca, Planchard & Vanninathan (1989, 1995)

Find $\omega \in \mathbb{R}$ and $u \neq 0$ such that

$$c^2 \Delta u + \omega^2 u = 0 \quad \text{in } \Omega_0$$

$$\frac{\partial u}{\partial n} = \frac{\rho_0 \omega^2}{k_j - \omega^2 m_j} n \cdot \int_{\Gamma_j} u n \, ds \quad \text{on } \Gamma_j, \ j = 1, \ldots, \ell$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma$$

Here $u$ is the potential of the velocity field, $n$ is the outward normal on the boundary of $\Omega_0$, $c$ denotes the speed of sound in the fluid, $\rho_0$ is the specific density of the fluid, $k_j$ represents the stiffness constant of the spring system supporting the tube $j$ and $m_j$ is the mass per unit length of the tube $j$. 
Fluid–solid vibrations ct.

Rewriting the eigenproblem in its variational form the Lax–Milgram lemma yields that it is equivalent to the rational eigenvalue problem

Determine $\lambda := \omega^2$ and $u \in \mathcal{H} := \{ u \in H^1(\Omega_0) : \int_{\Omega_0} u \, dx = 0 \}$ such that

$$T(\lambda)u := (-c^2 I + \lambda A + \sum_{j=1}^{K} \frac{\rho_0 \lambda}{k_j - \lambda m_j} B_j)u = 0$$

where the linear symmetric operators $A$ and $B_j$ are defined by

$$\langle Au, v \rangle := \int_{\Omega_0} uv \, dx, \quad \langle B_j u, v \rangle := \int_{\Gamma_j} u n \, ds \cdot \int_{\Gamma_j} v n \, ds \quad \text{for every } u, v \in H.$$

Discretizing by finite elements yields a rational matrix eigenvalue problem

$$Kx = \lambda Mx + \sum_{j=1}^{\ell} \frac{\lambda}{\sigma_j - \lambda} C_j x, \quad \lambda = \omega^2,$$

where $K$ and $M$ are symmetric and positive (semi-)definite, and $C_j$ is symmetric and positive semidefinite of rank 2.
Example 7: Electronic structure of quantum dots

Semiconductor nanostructures have attracted tremendous interest in the past few years because of their special physical properties and their potential for applications in micro- and optoelectronic devices.

In such nanostructures, the free carriers are confined to a small region of space by potential barriers, and if the size of this region is less than the electron wavelength, the electronic states become quantized at discrete energy levels.

The ultimate limit of low dimensional structures is the quantum dot, in which the carriers are confined in all three directions, thus reducing the degrees of freedom to zero. Therefore, a quantum dot can be thought of as an artificial atom.
Example 7: Electronic structure of quantum dots

Electron Spectroscopy Group, Fritz-Haber-Institute, Berlin
Electronic structure of quantum dots ct.

**Problem:** Determine relevant energy states (i.e. eigenvalues) and corresponding wave functions (i.e. eigenfunctions) of a three-dimensional quantum dot embedded in a matrix.

![Pyramidal (InAs) quantum dot in (GaAs) matrix](image-url)
Electronic structure of quantum dots ct.

Governing equation: Schrödinger equation

\[-\nabla \cdot \left( \frac{\hbar^2}{2m(x, \lambda)} \nabla u \right) + V(x)u = \lambda u, \quad x \in \Omega_q \cup \Omega_m\]

where \( \hbar \) is the reduced Planck constant, \( m(x, \lambda) \) is the electron effective mass, and \( V(x) \) is the confinement potential.

\( m \) and \( V \) are discontinuous across the heterojunction.

Boundary and interface conditions

\[u = 0 \quad \text{on outer boundary of matrix } \Omega_m\]

BenDaniel–Duke condition

\[
\frac{1}{m_m} \frac{\partial u}{\partial n} \bigg|_{\partial \Omega_m} = \frac{1}{m_q} \frac{\partial u}{\partial n} \bigg|_{\partial \Omega_q} \quad \text{on interface}
\]
Variational form

Find \( \lambda \in \mathbb{R} \) and \( u \in H^1_0(\Omega) \), \( u \neq 0 \), \( \Omega := \overline{\Omega_q} \cup \Omega_m \), such that

\[
a(u, v; \lambda) := \frac{\hbar^2}{2} \int_{\Omega_q} \frac{1}{m_q(x, \lambda)} \nabla u \cdot \nabla v \, dx + \frac{\hbar^2}{2} \int_{\Omega_m} \frac{1}{m_m(x, \lambda)} \nabla u \cdot \nabla v \, dx
+ \int_{\Omega_q} V_q(x)uv \, dx + \int_{\Omega_m} V_m(x)uv \, dx
= \lambda \int_{\Omega} uv \, dx =: \lambda b(u, v) \quad \text{for every } v \in H^1_0(\Omega)
\]
Electron effective mass

The dependence of $m(x, \lambda)$ on $\lambda$ can be derived from the eight-band $k \cdot p$ analysis and effective mass theory. Projecting the $8 \times 8$ Hamiltonian onto the conduction band results in the single Hamiltonian eigenvalue problem with

$$m(x, \lambda) = \begin{cases} m_q(\lambda), & x \in \Omega_q \\ m_m(\lambda), & x \in \Omega_m \end{cases}$$

$$\frac{1}{m_j(\lambda)} = \frac{P_j^2}{\hbar^2} \left( \frac{2}{\lambda + g_j - V_j} + \frac{1}{\lambda + g_j - V_j + \delta_j} \right), \quad j \in \{m, q\}$$

where $m_j$ is the electron effective mass, $V_j$ the confinement potential, $P_j$ the momentum, $g_j$ the main energy gap, and $\delta_j$ the spin-orbit splitting in the $j$th region.

Other types of effective mass (taking into account the effect of strain, e.g.) appear in the literature. They are all rational functions of $\lambda$ where $1/m(x, \lambda)$ is monotonically decreasing with respect to $\lambda$, and that’s all we need.
The variational characterization of eigenvalues is a powerful tool to analyze selfadjoint operators in Hilbert spaces. In this subsection we discuss generalizations to operators depending nonlinearly on the eigenparameter. Although we are mainly interested in numerical methods (and therefore on the finite dimensional case) we consider the infinite dimensional case.

For linear selfadjoint eigenvalue problems the following characterizations of eigenvalues are well known (cf. Rektorys 1984):

**Theorem 1**
Let \( A : \mathcal{H} \to \mathcal{H} \) be a selfadjoint and completely continuous operator on a real Hilbert space \( \mathcal{H} \) with scalar product \( \langle \cdot , \cdot \rangle \), and denote by \( R_A(x) := \langle Ax, x \rangle / \langle x, x \rangle \) the Rayleigh quotient of \( A \) at \( x \neq 0 \).

Let \( \lambda_1 \geq \lambda_2 \geq \ldots \) be the positive eigenvalues of \( A \) ordered by magnitude and regarding their multiplicities, and let \( x^1, x^2, \ldots \) be a system of orthogonal eigenelements where \( x^j \) corresponds to \( \lambda_j \).

Assume that \( A \) has at least \( n \) positive eigenvalues. Then the following characterizations hold:
Variational characterization ct.

(i) Rayleigh’s principle

\[ \lambda_n = \max \{ R_A(x) : \langle x, x_i \rangle = 0, \ i = 1, \ldots, n-1 \} . \]

(ii) Maxmin characterization of Poincaré

\[ \lambda_n = \max_{\dim V=n} \min_{x \in V, x \neq 0} R_A(x) . \]

(iii) Minmax characterization of Courant, Fischer, Weyl

\[ \lambda_n = \min_{\dim V=n-1} \max_{x \in V^\perp, x \neq 0} R_A(x) . \]

Analogously the negative eigenvalues \( \lambda_{-1} \leq \lambda_{-2} \leq \ldots \) of \( A \) can be characterized by the three principles above if we replace min by max and vice versa.
Quite often the eigenvalue problem is given in variational form:

Let

\[ a : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \quad \text{and} \quad b : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \]

be symmetric (i.e. \( a(u, v) = a(v, u), b(u, v) = b(v, u) \) for every \( u, v \in \mathcal{H} \)) and bounded quadratic forms, i.e. there exist \( K_a, K_b > 0 \) such that

\[ |a(u, v)| \leq K_a \|u\| \cdot \|v\| \quad \text{and} \quad |b(u, v)| \leq K_b \|u\| \cdot \|v\| \quad \text{for every} \ u, v \in \mathcal{H}. \]

Let \( a \) be \( \mathcal{H} \)-elliptic (i.e. \( a(u, u) \geq \alpha \|u\|^2 \) for some \( \alpha > 0 \) for all \( u \in \mathcal{H} \)), and let \( b \) be positive (i.e. \( b(u, u) > 0 \) for every \( u \in \mathcal{H}, u \neq 0 \)) and completely continuous, i.e.

\[ u_n \rightharpoonup u, \ v_n \rightharpoonup v \quad \Rightarrow \quad b(u_n, v_n) \to b(u, v), \]

where \( u_n \rightharpoonup u \) denotes the weak convergence of \( u_n \) to \( u \).

Find \( \lambda \in \mathbb{R} \) and \( u \in \mathcal{H}, u \neq 0 \) such that

\[ a(u, v) = \lambda b(u, v) \quad \text{for every} \ v \in \mathcal{H}. \]
By the Lax-Milgram theorem there exists a selfadjoint and (by Rellich’s lemma) completely continuous operator

$$A : \mathcal{H} \to \mathcal{H}$$

such that $$b(u, v) = a(Au, v)$$ for every $$u, v \in \mathcal{H}$$.

Hence, the variational eigenvalue problem is equivalent to

$$a(u, v) = \lambda a(Au, v) \quad \forall v \in \mathcal{H} \iff u = \lambda Au,$$

and $$\lambda$$ is an eigenvalue of the variational eigenproblem if and only if $$\lambda^{-1}$$ is an eigenvalue of $$A$$.

Since $$A$$ is positive definite (note that $$b(u, u) > 0$$ for $$u \neq 0$$), the variational problems has positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots,$$

and $$\lim_{k \to \infty} \lambda_k = \infty$$, if $$\dim \mathcal{H} = \infty$$. 

Since the Rayleigh quotient is given by
\[ R_A(u) := \frac{a(Au, u)}{a(u, u)} = \frac{b(u, u)}{a(u, u)} \]
it follows from Theorem 1 (cf. Weinberger 1974):
Rayleigh’s principle
\[ \lambda_j = \min \left\{ \frac{a(u, u)}{b(u, u)} : u \in \mathcal{H} \setminus \{0\}, \ b(u, u^k) = 0 \text{ for } k = 1, \ldots, j - 1 \right\}, \]
where \( u^j, j \in \mathbb{N} \) denotes the eigenelement corresponding to \( \lambda_j \), and the eigenelements are chosen such that \( b(u^j, u^k) = \delta_{jk} \).
Minmax characterization; Poincaré
\[ \lambda_j = \min_{\dim V = j} \max_{u \in V, u \neq 0} \frac{a(u, u)}{b(u, u)}. \]
Maxmin characterization, Courant, Fischer
\[ \lambda_j = \max_{\dim V = j - 1} \min_{u \in V^\perp, u \neq 0} \frac{a(u, u)}{b(u, u)}. \]
Consider a family of bounded and selfadjoint operators

\[ T(\lambda) : \mathcal{H} \rightarrow \mathcal{H}, \quad \lambda \in J, \]

where \( J \subset \mathbb{R} \) is an open interval which may be unbounded.

Let

\[ f : J \times \mathcal{H} \rightarrow \mathbb{R}, \quad (\lambda, x) \mapsto \langle T(\lambda)x, x \rangle \]

be continuously differentiable, and assume that for every fixed \( x \in \mathcal{H}, x \neq 0 \), the real equation

\[ f(\lambda, x) = 0 \quad (2) \]

has at most one solution in \( J \).

Then equation (2) implicitly defines a functional \( p \) on some subset \( D \) of \( H \setminus \{0\} \) which we call the Rayleigh functional, and which is exactly the Rayleigh quotient in case of a linear eigenproblem \( T(\lambda) = \lambda I - A \).
We assume further that

\[ \frac{\partial}{\partial \lambda} f(\lambda, x) \bigg|_{\lambda=p(x)} > 0 \quad \text{for every } x \in D \quad (3) \]

which generalizes the definiteness of the operator \( B \) for the generalized linear eigenproblem \( T(\lambda) := \lambda B - A \).

The implicit function theorem gives that \( D \) is an open set and \( p \) is continuously differentiable on \( D \).

Differentiating the governing equation

\[ f(p(x), x) = \langle T(p(x)x, x) \equiv 0 \]

yields that the eigenelements of problem \( T(\cdot) \) are the stationary vectors of the Rayleigh functional \( p \).
Overdamped problems

If the Rayleigh functional $\psi$ is defined on the entire space $\mathcal{H} \setminus \{0\}$ then the eigenproblem $T(\lambda)x = 0$ is called overdamped.

The notation is motivated by the finite dimensional quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda \alpha Cx + Kx = 0 \quad (4)$$

where $M$, $C$ and $K$ are symmetric and positive definite matrices.

$\alpha = 0$ all eigenvalues on imaginary axis

increase $\alpha$ eigenvalues go into left half plane as conjugate complex pairs

increase $\alpha$ complex pairs reach real axis, run in opposite directions

increase $\alpha$ all eigenvalues on the negative real axis

increase $\alpha$ all eigenvalues going to the left are smaller than all eigenvalues going to the right

system is overdamped
For quadratic overdamped systems the two solutions

\[ p_{\pm}(x) = \frac{1}{2\langle Mx, x \rangle} \left( -\alpha \langle Cx, x \rangle \pm \sqrt{\alpha^2 \langle Cx, x \rangle^2 - 4\langle Mx, x \rangle \langle Kx, x \rangle} \right) . \]

of the quadratic equation

\[ \langle T(\lambda)x, x \rangle = \lambda^2 \langle Mx, x \rangle + \lambda \alpha \langle Cx, x \rangle + \langle Kx, x \rangle = 0 \quad (5) \]

are real, and they satisfy \( \sup_{x \neq 0} p_{-}(x) < \inf_{x \neq 0} p_{+}(x) \).

Hence, equation (5) defines two Rayleigh functionals \( p_{-} \) and \( p_{+} \) corresponding to the intervals

\[ J_{-} := (-\infty, \inf_{x \neq 0} p_{+}(x)) \quad \text{and} \quad J_{+} := (\sup_{x \neq 0} p_{-}(x), \infty). \]
For general (not necessarily quadratic) overdamped problems Hadeler (1967 for the finite dimensional case, and 1968 for \( \dim \mathcal{H} = \infty \)) generalized Rayleigh’s principle proving that the eigenvectors are orthogonal with respect to the generalized scalar product

\[
[x, y] := \begin{cases} 
\left\langle \frac{T(p(x)) - T(p(y))}{p(x) - p(y)} x, y \right\rangle, & \text{if } p(x) \neq p(y) \\
\left\langle T'(p(x)) x, y \right\rangle, & \text{if } p(x) = p(y)
\end{cases}
\]

which is symmetric, definite and homogeneous, but in general is not bilinear.
Theorem 2 (Hadeler)
Let $T(\lambda) : \mathcal{H} \rightarrow \mathcal{H}$, $\lambda \in J$ be a family of selfadjoint and bounded operators. Assume that the problem $T(\lambda)x = 0$ is overdamped and that for every $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) - \nu(\lambda)I$ is completely continuous.

Then problem $T(\lambda)x = 0$ has at most a countable set of eigenvalues in $J$ which we assume to be ordered by magnitude $\lambda_1 \geq \lambda_2 \geq \ldots$ regarding their multiplicities.

The corresponding eigenvectors $x^1, x^2, \ldots$ can be chosen orthonormally with respect to the generalized scalar product (6), and the eigenvalues can be determined recursively by

$$\lambda_n = \max\{\rho(x) : [x, x_i] = 0, \ i = 1, \ldots, n - 1, \ x \neq 0\}.$$  \hspace{1cm} (7)
Poincaré’s maxmin characterization was first generalized by Duffin (1955) to overdamped quadratic eigenproblems of finite dimension, and for more general overdamped problems of finite dimension it was proved by Rogers (1964).

Infinite dimensional eigenvalue problems were studied by Turner (1967), Langer (1968), and Weinberger (1969) who proved generalizations of both, the maxmin characterization of Poincaré and of the minmax characterization of Courant, Fischer and Weyl for quadratic (and by Turner (1968) for polynomial) overdamped problems.

The corresponding generalizations for general overdamped problems of infinite dimension were derived by Hadeler (1968). Similar results weakening the compactness requirements are contained in Rogers (1968), Werner (1971) and Hadeler (1975).
Theorem 3 (Hadjic) Let $T(\lambda) : H \to H$, $\lambda \in J$ be a family of selfadjoint and bounded operators. Assume that the problem $T(\lambda)x = 0$ is overdamped and that for every $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) - \nu(\lambda)I$ is completely continuous.

Let the eigenvalues $\lambda_n$ of $T(\lambda)x = 0$ be numbered in nonincreasing order according to multiplicities. Then they can be characterized by the following two variational principles

$$\lambda_n = \max_{\dim V = n} \min_{x \in V, x \neq 0} p(x)$$
$$= \min_{\dim V = n - 1} \max_{x \in V^\perp, x \neq 0} p(x).$$

It is obvious that these principles can be transformed to minmax and maxmin principles for variational forms of nonlinear eigenvalue problems.
Variational form of QD problem

Multiplying the Schrödinger equation of the quantum dot problem by $\nu \in H_0^1(\Omega)$ one obtains the weak form

Find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$, $u \neq 0$, $\Omega := \overline{\Omega}_q \cup \Omega_m$, such that

$$a(u, v; \lambda) := \frac{\hbar^2}{2} \int_{\Omega_q} \frac{1}{m_q(x, \lambda)} \nabla u \cdot \nabla v \, dx + \frac{\hbar^2}{2} \int_{\Omega_m} \frac{1}{m_m(x, \lambda)} \nabla u \cdot \nabla v \, dx$$

$$+ \int_{\Omega_q} V_q(x) uv \, dx + \int_{\Omega_m} V_m(x) uv \, dx$$

$$= \lambda \int_{\Omega} uv \, dx =: \lambda b(u, v) \quad \text{for every } v \in H_0^1(\Omega)$$
Properties

- $a(\cdot, \cdot, \lambda)$ bilinear, symmetric, bounded, $H^1_0(\Omega)$--elliptic for $\lambda \geq 0$
- $b(\cdot, \cdot)$ bilinear, positive definite, bounded, completely continuous

By the Lax–Milgram lemma the variational eigenproblem is equivalent to

$$T(\lambda)u = 0$$

where

$$T(\lambda) : H^1_0(\Omega) \to H^1_0(\Omega), \quad \lambda \geq 0,$$

is a family of bounded operators such that . . .
For

\[ f(\lambda; u) := \langle T(\lambda)u, u \rangle = \lambda b(u, u) - a(u, u; \lambda) \]

it holds

\[ f(0; u) < 0 < \lim_{\lambda \to \infty} f(\lambda; u) = \infty \quad \text{for every } u \neq 0 \]

\[ \frac{\partial}{\partial \lambda} f(\lambda; u) > 0 \quad \text{for every } u \neq 0 \text{ and } \lambda \geq 0 \]

For fixed \( \lambda \geq 0 \) the eigenvalues of the linear eigenproblem \( T(\lambda)u = \mu u \) satisfy a maxmin characterization.

Hence, the nonlinear Schrödinger equation has an infinite number of eigenvalues which can be characterized as minmax values of the Rayleigh functional.
For nonoverdamped eigenproblems the natural ordering to call the largest eigenvalue the first one, the second largest the second one, etc., is not appropriate.

This is obvious if we make a linear eigenvalue

\[ T(\lambda)x := (\lambda I - A)x = 0 \]

nonlinear by restricting it to an interval \( J \) which does not contain the largest eigenvalue of \( A \).

Then all conditions are satisfied, \( p \) is the restriction of the Rayleigh quotient \( R_A \) to \( D := \{x \neq 0 : R_A(x) \in J\} \), and \( \sup_{x \in D} p(x) \) will not be an eigenvalue.
Enumeration of eigenvalues

\( \lambda \in J \) is an eigenvalue of \( T(\cdot) \) if and only if \( \mu = 0 \) is an eigenvalue of the linear problem \( T(\lambda)y = \mu y \). The key idea is to orientate the number of \( \lambda \) on the location on the eigenvalue \( \mu = 0 \) in the spectrum of the linear operator \( T(\lambda) \).

To this end we assume that for every \( \lambda \in J \) there exists \( \nu(\lambda) > 0 \) such that the linear operator \( S := T(\lambda) - \nu(\lambda)I \) is completely continuous.

If \( \lambda \in J \) is an eigenvalue of the nonlinear problem \( T(\lambda)x = 0 \) then \( \mu = 0 \) is an eigenvalue of \( T(\lambda) \), and \( -\nu(\lambda) \) is a negative eigenvalue of \( S \).

By Theorem 1 there exists \( n \in \mathbb{N} \) such that

\[
-\nu(\lambda) = \min_{\dim V = n} \max_{x \in V, x \neq 0} R_S(x), \quad \text{i.e.} \quad 0 = \min_{\dim V = n} \max_{x \in V, x \neq 0} R_{T(\lambda)}(x).
\]

In this case we assign \( n \) to the eigenvalue \( \lambda \) of problem \( T(\lambda)x = 0 \) as its number.
Theorem 4 (V. & Werner (1982))
Assume that for every $x \in H$, $x \neq 0$ the real equation $f(\lambda, x) = 0$ has at most one solution $\lambda =: p(x) \in J$, and that condition (3) holds, and suppose that for every $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) - \nu(\lambda)I$ is completely continuous.

Then the nonlinear eigenvalue problem $T(\lambda)x = 0$ has at most a countable number of eigenvalues. Enumerating them as above the following characterizations hold:

If $\lambda_n \in J$ is an $n$-th eigenvalue then

$$
\lambda_n = \max_{\dim V=n, V \cap D \neq \emptyset} \inf_{x \in D \cap V} p(x). \quad (8)
$$

If conversely

$$
\lambda_n = \sup_{\dim V=n, V \cap D \neq \emptyset} \inf_{x \in D \cap V} p(x) \in J
$$

then $\lambda_n$ is an $n$-th eigenvalue of $T(\lambda)x = 0$ and (8) holds.
Theorem 5
Under the conditions of Theorem 4 let \( \sup_{v \in D} p(v) \in J \) and assume that there exists \( W \in \mathcal{H}_n \) such that

\[
W \cap D \neq \emptyset \quad \text{and} \quad \inf_{v \in W \cap D} p(v) \in J.
\]

Then for \( j = 1, \ldots, n \) every \( V \in \mathcal{H}_j \) with

\[
V \cap D \neq \emptyset \quad \text{and} \quad \lambda_j = \inf_{v \in V \cap D} p(v)
\]

is contained in \( D \), and the maxmin characterization of \( \lambda_j \) can be replaced by

\[
\lambda_j = \max_{V \in \mathcal{H}_j, V \backslash \{0\} \subset D} \min_{v \in V \backslash \{0\}} p(v).
\]
Theorem 6 (V. (2003))
Assume that for every \( x \in H, x \neq 0 \) the real equation \( f(\lambda, x) = 0 \) has at most one solution \( \lambda =: p(x) \in J \), and that condition (3) holds, and suppose that for every \( \lambda \in J \) there exists \( \nu(\lambda) > 0 \) such that \( T(\lambda) - \nu(\lambda)I \) is completely continuous.

Then the nonlinear eigenvalue problem \( T(\lambda)x = 0 \) has at most a countable number of eigenvalues. Enumerating them as above the following characterizations hold:

If problem \( T(\lambda)x = 0 \) has an \( n \)-th eigenvalue \( \lambda_n \) then

\[
\lambda_n = \min_{\dim V = n-1, \ V \perp D \neq \emptyset} \sup_{x \in D \cap V \perp} p(x).
\]
Fluid-solid structure

The variational form of the eigenvalue problem governing vibrations of a structure immersed in a fluid is:

Find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega_0)$ such that for every $v \in H^1(\Omega_0)$

$$
a(u, v) := c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \, dx \\
= \lambda \int_{\Omega_0} uv \, dx + \sum_{j=1}^{\ell} \frac{\lambda \rho_0}{k_j - \lambda m_j} \int_{\Gamma_j} un \, ds \cdot \int_{\Gamma_j} vn \, ds =: b(u, v; \lambda).
$$

Obviously, $f(\lambda, u) := b(u, u; \lambda) - a(u, u)$ is monotonically increasing for every fixed $u \neq 0$, and therefore $f(\lambda, u) = 0$ defines a Rayleigh functional in every real interval $J$ which does not contain a pole $k_j/m_j$.

For $J_0 := (0, \min\{k_j/m_j : j = 1, \ldots, \ell\}$ the infimum of the Rayleigh functional is contained in $J_0$, and Theorem 5 applies (i.e. the eigenvalues can be enumerated in the natural way), for further intervals the enumeration according to their location in the spectrum of the linear eigenproblems is relevant in Theorem 4.
A priori bound for CMS

Theorem 5 can be used to derive an a priori bound for CMS (cf. Elssel & V. 2006).

We already obtained in Chapter 12 the Craig-Bampton form of an eigenvalue problem

\[
\begin{pmatrix}
\Omega_1 & 0 & 0 \\
0 & \Omega_2 & 0 \\
0 & 0 & \tilde{K}_{ii}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \lambda
\begin{pmatrix}
I & 0 & \tilde{M}_{i1} \\
0 & I & \tilde{M}_{i2} \\
\tilde{M}_{i1} & \tilde{M}_{i2} & \tilde{M}_{ii}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\] (1)

and that this problem is reduced to

\[
\begin{pmatrix}
\Omega_2 & 0 \\
0 & \tilde{K}_{ii}
\end{pmatrix}
y = \lambda
\begin{pmatrix}
I & \tilde{M}_{i2} \\
\tilde{M}_{i2} & \tilde{M}_{ii}
\end{pmatrix}y
\] (2)

in the component mode synthesis method, where all diagonal elements of \( \Omega_1 \) are greater than the cut-off frequency \( \omega \), and all eigenvalues in \( \Omega_2 \) are less than \( \omega \).
A priori bound for CMS ct.

For $\lambda \in J := (0, \omega)$ the first equation of (1) yields

$$x_1 = \lambda (\Omega_1 - \lambda I)^{-1} \tilde{M}_{\ell i_1} x_3,$$

and $\lambda$ is an eigenvalue of $Kx = \lambda Mx$ if and only if it is an eigenvalue of the rational eigenproblem

$$T(\lambda) y = 0$$

where

$$T(\lambda) = - \begin{pmatrix} \Omega_2 & 0 \\ 0 & \tilde{K}_{ii} \end{pmatrix} + \lambda \begin{pmatrix} I & \tilde{M}_{i \ell 2} \\ \tilde{M}_{i i 2} & \tilde{M}_{ii} \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 \\ \tilde{M}_{i \ell 1} \end{pmatrix} (\Omega_1 - \lambda I)^{-1} \begin{pmatrix} 0 & \tilde{M}_{\ell i_1} \end{pmatrix}. \quad (3)$$

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of problem (1) ordered by magnitude, and let $m \in \mathbb{N}$ such that $\lambda_m < \omega \leq \lambda_{m+1}$.

Then $\lambda_1, \ldots, \lambda_m \in J$ are the eigenvalues of the nonlinear eigenproblem (1) in $J$. 
A priori bound for CMS ct.

For $f(\lambda; y) := y^T T(\lambda)y$ it follows from the positive definiteness of

\[
\begin{pmatrix}
I & \tilde{M}_{i\ell 2} \\
\tilde{M}_{i\ell 2} & \tilde{M}_{ii}
\end{pmatrix}
\]

that

\[
\frac{\partial}{\partial \lambda} f(\lambda; y) = y^T \begin{pmatrix}
I & \tilde{M}_{i\ell 2} \\
\tilde{M}_{i\ell 2} & \tilde{M}_{ii}
\end{pmatrix} y + \sum_{\omega_j \geq \omega} \frac{(2\omega_j - \lambda^2) a_j^2}{(\omega_j - \lambda)^2} > 0
\]

for every $y \in \mathbb{R}^\nu \setminus \{0\}$, where $\nu$ denotes the dimension of the reduced problem (2), and $a := (0 \  \tilde{M}_{i\ell 1}) y$.

Hence, due to the monotonicity of $f(\lambda; y)$ for every $y \in \mathbb{R}^\nu \setminus \{0\}$ the real equation $f(\lambda; y) = 0$ has at most one solution $p(y) \in J$, and the condition

\[
\frac{\partial}{\partial \lambda} f(\lambda, y) \bigg|_{\lambda=p(y)} > 0 \quad \text{for every } y \in D
\]

holds.
A priori bound for CMS ct.

If \( y := \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^\nu \) then it easily seen that for \( x_1 := \lambda(\Omega_1 - \lambda I)^{-1}\tilde{M}_{i\ell_1}x_3 \) it holds that

\[
(x_2^T, x_3^T)T(\lambda)\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = (x_1^T, x_2^T, x_3^T)\begin{pmatrix} \lambda I - \Omega_1 & O & \lambda \tilde{M}_{i\ell_1} \\ O & \lambda I - \Omega_2 & \lambda \tilde{M}_{i\ell_2} \\ \lambda \tilde{M}_{i\ell_1} & \lambda \tilde{M}_{i\ell_2} & \lambda \tilde{M}_{ii} - \tilde{K}_{ii} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

Hence, \( \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \in D \) if and only if the Rayleigh quotient \( R \) of the linear eigenproblem (1) at \( x := (x_1^T, x_2^T, x_3^T)^T \) is contained in \( J \), and \( p(y) = R(x) \).

In particular

\[
\inf_{y \in D} p(y) = \inf_{x \in \mathbb{R}^n, x \neq 0} R(x) = \lambda_1 \in J,
\]

and the eigenvalues \( \lambda_1, \ldots, \lambda_m \) of problem (3) satisfy the minmax characterization.
The eigenvalues

\[ \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_\nu \]

of the reduced problem (2) are minmax values of the Rayleigh quotient \( \rho(x) \) corresponding to problem (2).

Comparing \( p \) and \( \rho \) on appropriate subspaces of \( \mathbb{R}^\nu \) we arrive at the following a priori bound for the relative errors of the CMS approximations \( \tilde{\lambda}_j \) to \( \lambda_j \).
Theorem

Let $K, M \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of problem (1), which we assume to be ordered by magnitude.

Denote by $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_\nu$ the eigenvalues of the CMS approximation (2) of problem (1) corresponding to some partition of the graph $|K| + |M|$ and some cut-off threshold $\omega$.

Assume that the interval $(0, \omega)$ contains $m$ eigenvalues $\lambda_1, \ldots, \lambda_m$ of (1).

Then it holds

$$0 \leq \frac{\tilde{\lambda}_j - \lambda_j}{\lambda_j} \leq \frac{\lambda_j}{\omega - \lambda_j} \leq \frac{\tilde{\lambda}_j}{\omega - \tilde{\lambda}_j}, \quad j = 1, \ldots, m.$$ 

Hence, for eigenvalues which are far away from the cut-off threshold we may expect accurate approximations by the CMS method.
Proof

The left inequality, i.e. $\lambda_j \leq \tilde{\lambda}_j$, is trivial since CMS is a projection method. The right inequality follows from the monotonicity of the function $\lambda \mapsto \lambda/(\omega - \lambda)$.

To prove the inequality in the middle denote by $V \in S_j, V \setminus \{0\} \subset D$ the $j$ dimensional subspace of $\mathbb{R}^\nu$ such that

$$
\lambda_j = \max_{y \in V, \ y \neq 0} p(y).
$$

Then $p(y) \leq \lambda_j$ for every $y \in V, y \neq 0$, and therefore it follows from the monotonicity of the function $f(\lambda; y)$ with respect to $\lambda$

$$
-y^T \left( \begin{array}{cc}
\Omega_2 & 0 \\
0 & \tilde{K}_{ii}
\end{array} \right) y + \lambda_j y^T \left( \begin{array}{ccc}
I & \tilde{M}_{i\ell 2} \\
\tilde{M}_{i\ell 2} & \tilde{M}_{ii}
\end{array} \right) y \\
+ \lambda_j^2 y^T \left( \begin{array}{c}
0 \\
\tilde{M}_{i\ell 1}
\end{array} \right) (\Omega_1 - \lambda_j I)^{-1} \left( \begin{array}{c}
0 \\
\tilde{M}_{i\ell 1}
\end{array} \right) y \geq 0.
$$
Proof ct.

Hence, for every \( y \in V, y \neq 0 \) one obtains

\[
\lambda_j \geq \frac{y^T \begin{pmatrix} \Omega_2 & 0 \\ 0 & \tilde{K}_{ii} \end{pmatrix} y}{y^T \begin{pmatrix} I & \tilde{M}_{i\ell_2} \\ \tilde{M}_{i\ell_2} & \tilde{M}_{ii} \end{pmatrix} y} - \lambda_j^2 \frac{y^T \begin{pmatrix} 0 \\ \tilde{M}_{i\ell_1} \end{pmatrix} \Omega_1 - \lambda_j I \begin{pmatrix} 0 & \tilde{M}_{i\ell_1} \end{pmatrix} y}{y^T \begin{pmatrix} I & \tilde{M}_{i\ell_2} \\ \tilde{M}_{i\ell_2} & \tilde{M}_{ii} \end{pmatrix} y}.
\]

In particular for \( \hat{y} \in V \) such that \( \rho(\hat{y}) = \max_{y \in V, y \neq 0} \rho(y) \) we have

\[
\lambda_j \geq \max_{y \in V, y \neq 0} \rho(y) - \lambda_j^2 \frac{\hat{y}^T \begin{pmatrix} 0 \\ \tilde{M}_{i\ell_1} \end{pmatrix} \Omega_1 - \lambda_j I \begin{pmatrix} 0 & \tilde{M}_{i\ell_1} \end{pmatrix} \hat{y}}{\hat{y}^T \begin{pmatrix} I & \tilde{M}_{i\ell_2} \\ \tilde{M}_{i\ell_2} & \tilde{M}_{ii} \end{pmatrix} \hat{y}}.
\]

\[
\lambda_j \geq \tilde{\lambda}_j - \frac{\lambda_j^2}{\omega - \lambda_j} \max_{y \in \mathbb{R}^n, y \neq 0} \frac{y^T \begin{pmatrix} 0 \\ \tilde{M}_{i\ell_1} \end{pmatrix} \Omega_1 - \lambda_j I \begin{pmatrix} 0 & \tilde{M}_{i\ell_1} \end{pmatrix} y}{y^T \begin{pmatrix} I & \tilde{M}_{i\ell_2} \\ \tilde{M}_{i\ell_2} & \tilde{M}_{ii} \end{pmatrix} y}.
\]
Proof ct.

From the positive definiteness of the transformed mass matrix

\[
\begin{pmatrix}
  I & 0 & \tilde{M}_{ei1} \\
  0 & I & \tilde{M}_{ei2} \\
  \tilde{M}_{ie1} & \tilde{M}_{ie2} & \tilde{M}_{ii}
\end{pmatrix}
\]

it follows that the Schur complement

\[
\begin{pmatrix}
  I & \tilde{M}_{ei2} \\
  \tilde{M}_{ie2} & \tilde{M}_{ii}
\end{pmatrix}
- \begin{pmatrix}
  0 \\
  \tilde{M}_{ie1}
\end{pmatrix}
\begin{pmatrix}
  0 & \tilde{M}_{ei1}
\end{pmatrix}
\]

is positive definite as well. Thus,

\[
\max_{y \in \mathbb{R}^n, y \neq 0} \frac{y^T \begin{pmatrix}
  0 \\
  \tilde{M}_{ie1}
\end{pmatrix} \begin{pmatrix}
  0 & \tilde{M}_{ei1}
\end{pmatrix} y}{y^T \begin{pmatrix}
  I & \tilde{M}_{ei2} \\
  \tilde{M}_{ie2} & \tilde{M}_{ii}
\end{pmatrix} y} \leq 1,
\]

and inequality (4) yields

\[
\lambda_j \geq \tilde{\lambda}_j - \frac{\chi_j^2}{\omega - \lambda_j}
\]

which completes the proof.
Consider the model of a container ship with 10 substructures which has 1960 DoFs on the interfaces.

We solved the eigenproblem by the CMS method using a cut-off bound of 20,000 (about 10 times the largest wanted eigenvalue $\lambda_{50} \approx 2183$).

329 eigenvalues of the substructure problems were less than our threshold, and the dimension of the resulting projected problem was 2289.

The following figure shows the relative errors for the smallest 50 eigenvalues (lower crosses in blue) and the error bounds (upper crosses in red).
Example ct.

![Graph showing the relationship between the number of eigenvalues and their relative error or bound.](image)
In the example the relative error is overestimated by two or three orders of magnitude.

\[
\begin{pmatrix}
\omega & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \lambda
\begin{pmatrix}
1 & 0 & m \\
0 & 1 & 0 \\
m & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

with \( \omega > 1 \) and \( m \in (0, 1) \) demonstrates that the a priori bound can not be improved without further assumptions.

If \( x_1 \) and \( x_2 \) are the local degrees of freedom, and \( x_3 \) is the interface variable, then with cut-off frequency \( \omega \) the minimum eigenvalue of the reduced problem is \( \tilde{\lambda}_1 = 1 \).

Letting \( m \to 1 - 0 \) l’Hospital’s rule yields

\[
\lim_{m \to 1-0} \frac{\tilde{\lambda}_1 - \lambda_1}{\lambda_1} = \frac{1}{\omega} = \lim_{m \to 1-0} \frac{\lambda_1}{\omega - \lambda_1}.
\]
A priori bound for AMLS

AMLS can be understood as a sequence of $\ell$ consecutive CMS steps and a terminating spectral truncation. It is clear how to obtain an a priori bound for the general AMLS method.

Hence, if $\lambda^{(\nu)}_j$ denotes the eigenvalues of the reduced eigenvalue problem corresponding to the $\nu$–th level ordered by magnitude, then it holds

$$\lambda^{(\nu)}_j \leq \lambda^{(\nu-1)}_j \left( 1 + \frac{\lambda^{(\nu-1)}_j}{\omega_\nu - \lambda^{(\nu-1)}_j} \right), \quad \nu = 1, 2, \ldots, p + 1.$$  

where on the $\nu$-th level eigenvalues exceeding $\omega_\nu$ are neglected.

Thus, it follows for all $\lambda_j \leq \min_{\nu=1,\ldots,p} \omega_\nu$

$$\lambda^{(\ell+1)}_j \leq \lambda_j \prod_{\nu=0}^{p} \left( 1 + \frac{\lambda^{(\nu)}_j}{\omega_{\nu+1} - \lambda^{(\nu)}_j} \right),$$  

and the following theorem follows.
Variational characterization

**Theorem**

Let $K$, $M$ and $\lambda_j$, $j = 1, \ldots, n$ be given as in the last theorem. Let the graph of $|K| + |M|$ be substructured with $\ell$ levels, and denote by $\tilde{\lambda}_1^{(\nu)} \leq \tilde{\lambda}_2^{(\nu)} \leq \ldots$ the eigenvalues obtained by AMLS with cut-off threshold $\omega_\nu$ on level $\nu$.

If $m \in \mathbb{N}$ such that

$$\lambda_m < \min_{\nu=0,\ldots,p} \omega_\nu \leq \lambda_{m+1}$$

then it holds

$$\frac{\tilde{\lambda}_j - \lambda_j}{\lambda_j} \leq \prod_{\nu=0}^{p} \left( 1 + \frac{\lambda_j^{(\nu)}}{\omega_\nu - \tilde{\lambda}_j^{(\nu)}} \right) - 1, \quad j = 1, \ldots, m.$$ 

Since the final problem is a projection of each of the intermediate eigenproblems in the AMLS reduction, it follows from the minmax characterization that $\lambda_j^{(\nu)} \leq \tilde{\lambda}_j$ for $\nu = 0, \ldots, p$. Therefore the a priori bound can be replaced by the computable bound

$$\frac{\tilde{\lambda}_j - \lambda_j}{\lambda_j} \leq \prod_{\nu=0}^{p} \left( 1 + \frac{\tilde{\lambda}_j}{\omega_\nu - \tilde{\lambda}_j} \right) - 1, \quad j = 1, \ldots, m.$$
We substructured the FE model of the container ship by Metis with 4 levels of substructuring.

Neglecting eigenvalues exceeding 20,000 and 40,000 on all levels AMLS produced a projected eigenvalue problem of dimension 451 and 911, respectively.

The relative errors and the bounds are shown in the following figure where the lower and upper crosses correspond to the threshold 40,000, and the lower and upper circles to 20,000.
Example ct.