Technische Universität Hamburg **SoSe 2019**
Institute of Mathematics
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Exam Stochastics

2^{nd} of September, 2019

You have 120 minutes to complete the exam.

Please fill in your surname, forename, matriculation number, subject and semester in **capital letters**:

Surname:															
Forename:															
Matr.-Nr.:		Subj.:				Sem.	

Notebook Nr.:				

User-Login: XXXXXX Passwort: XXXXXX

The exam consists of 9 problems. You must solve problems 1 to 9. The total number of points is 40.

Problem	Points	Corr.
1		
2		
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7		
8		
9		
$\sum$		

To your attention:

- Do not use a pencil or red ink.
- You are allowed to use only the provided notebook and cheat sheet. No other aids or electronic devices are allowed!

Vorbehalt:
I was informed that my exam will only be evaluated if the Examination Office of the TUHH confirms my official approval.

(date, signature)

Problem 1. (3 points)

Consider the random experiment of throwing a coin three times in a row and observing the face of the coin.

a) State a corresponding sample space Ω.

b) Now assume that you are only interested in the event that you have thrown three heads. State the smallest σ-algebra that contains this event and a corresponding probability measure.

Problem 2. (4 points)

Let A and B be two events such that

$$\mathbb{P}(A \mid B) = \frac{1}{2}, \quad \mathbb{P}(B \mid \complement A) = \frac{1}{4}, \quad \mathbb{P}(B) = \frac{1}{4}.$$

Calculate the probability of the event A. Furthermore, check whether A and B are independent.

Problem 3. (5 points)

For $h \in \mathbb{R}$, consider the function

$$f_h(x) = \begin{cases} h - (2+2h)x + 6x^2 & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

and assume that $f_h^{-1}((-\infty, a]) \in \mathcal{B}(\mathbb{R})$ for all $a, h \in \mathbb{R}$.

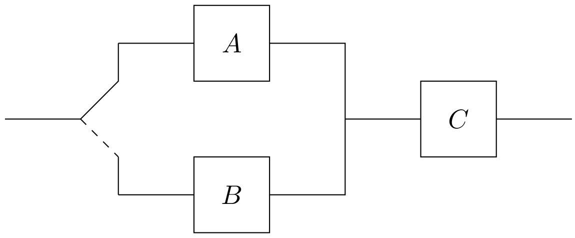
a) Determine **one** value for h such that $f_h(x)$ becomes a probability density function.

b) Determine the probability of the event $[-1/2, 1/2]$ for all suitable h.

Problem 4. (6 points)

Consider a system consisting of 3 components A, B, C.

A switch controls whether component A or B is being used. Component C is used all the time.



The lifetime of a single component is described by the independent random variables $T_k, k \in \{A, B, C\}$. Let $T = \max(T_A + T_B, T_C)$ be the random variable describing describing the lifetime of the system as a whole. Each of the random variables T_k is exponentially distributed with parameter $\lambda > 0$.

a) Determine the probability density function of the random variable $T_A + T_B$.

b) The cumulative distribution function of T is given as

$$F_T(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 - (\lambda t + 1)\exp(-2\lambda t) & \text{for } t \geq 0. \end{cases}$$

Calculate the expected lifetime of the system.

Problem 5. (5 points)

Consider the following region

$$A = \left\{(s,t) \in \mathbb{R}^2;\ s \geq 0,\ t \leq 0,\ s^2 + t^2 \leq 1\right\}\ .$$

and let $X = (X_1, X_2)$ be a random vector that is uniformly distributed on A.

a) Draw the region A.

b) Calculate the marginal density f_{X_2}.

c) Determine the expectation of X_2.

d) Determine $\mathbb{E}(X_2 \mid X_1 = s)$ for $0 \leq s \leq 1$.

Problem 6. (4 points)

Consider a stochastic process $(W_t)_{t\geq 0}$ with

- $\mathbb{P}(W_0 = 0) = 1$
- $W_t - W_s \sim \textsc{Normal}(0, t-s)$ for all $0 \leq s < t$.

a) Simulate the above process for $t \in [0,1]$ at the discrete points in time $t_n = \frac{n}{1000}$ for $n = 0, 1, \ldots, 1000$.

b) Let $T > 0$ be a fixed point in time and consider a new process

$$B_t = (T-t) \cdot W_t + t \cdot V_{T-t}, \quad 0 \leq t \leq T\,,$$

where $V_t \sim W_t$ for all $t \geq 0$. Simulate 2 paths of $(B_t)_{t\geq 0}$ with $T = 1$ on the interval $[0,1]$ at the discrete points in time $t_n = n/1000$ for $n = 0, 1, \ldots, 1000$ and plot your results.

c) Why is $(B_t)_{t\geq 0}$ called *Brownian Bridge*? To this end, calculate $\mathbb{P}(B_0 = 0)$ and $\mathbb{P}(B_1 = 0)$.

Problem 7. (4 points)

A local apple supplier advertises that with probability $p = 0.10$ a delivered apple is rotten.

a) A supermarket owner orders 1000 apples from this supplier. Use an appropriate limit theorem to approximately calculate the probability of the supermarket order containing at most 80 rotten apples.

b) The supermarket needs 1000 apples that are **not** rotten. How many apples does the supermarket owner have to order if she wants to assure with at least 99% probability that there are at least 1000 apples that are **not** rotten?

c) Simulate 5000 realizations of an order of 1000 apples using 1000 Bernoulli distributed random variables. For each realization calculate the number of rotten apples. Visualize the distribution of the number of rotten apples in a histogram.

Problem 8. (5 points)

Read in the dataset `'data.csv'` and answer the following questions.

a) Suppose that the data follows a normal distribution with variance $\sigma^2 = 1$. State the confidence interval for the mean μ with confidence level $1 - \alpha = 0.95$.

b) Carry out the statistical hypothesis testing scheme in order to decide with significance level $\alpha = 0.05$ if $\mu \geq 2$.

Hint: You may use the table below appropriately.

H_0	H_1	test T	critical region K
NORMAL(ϑ, σ^2), σ^2 known			
$\vartheta = \mu_0$	$\vartheta \neq \mu_0$		$K = \complement[-z_{1-\frac{\alpha}{2}}, z_{1-\frac{\alpha}{2}}]$
$\vartheta \leq \mu_0$	$\vartheta > \mu_0$	$T(x) = \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}$	$K = (z_{1-\alpha}, \infty)$
$\vartheta \geq \mu_0$	$\vartheta < \mu_0$		$K = (-\infty, -z_{1-\alpha})$
NORMAL(ϑ, σ^2), σ^2 unknown			
$\vartheta = \mu_0$	$\vartheta \neq \mu_0$		$K = \complement[-t_{n-1,1-\frac{\alpha}{2}}, t_{n-1,1-\frac{\alpha}{2}}]$
$\vartheta \leq \mu_0$	$\vartheta > \mu_0$	$T(x) = \frac{\bar{x}-\mu_0}{\frac{\sqrt{s_x^2}}{\sqrt{n}}}$	$K = (t_{n-1,1-\alpha}, \infty)$
$\vartheta \geq \mu_0$	$\vartheta < \mu_0$		$K = (-\infty, -t_{n-1,1-\alpha})$
NORMAL(μ, ϑ^2), μ unknown			
$\vartheta^2 = \sigma_0^2$	$\vartheta^2 \neq \sigma_0^2$		$K = \complement[-\chi^2_{n-1,\frac{\alpha}{2}}, \chi^2_{n-1,1-\frac{\alpha}{2}}]$
$\vartheta^2 \leq \sigma_0^2$	$\vartheta^2 > \sigma_0^2$	$T(x) = (n-1)\frac{s_x^2}{\sigma_0^2}$	$K = (\chi^2_{n-1,1-\alpha}, \infty)$
$\vartheta^2 \geq \sigma_0^2$	$\vartheta^2 < \sigma_0^2$		$K = (-\infty, \chi^2_{n-1,\alpha})$

Problem 9. (4 points)

Consider the following random experiment: An apparatus generates uniformly distributed random numbers in an interval $[0, \vartheta]$, where $0 < \vartheta < 5$ is unknown. We take samples of size $n \in \mathbb{N}$.

a) State an appropriate statistical model.

b) Now consider for all $n \in \mathbb{N}$ the point estimator

$$T_n(x) = 2x_n$$

where $x = (x_1, \ldots, x_n)$. Show that T_n is an unbiased point estimator for ϑ.

c) Formulate the likelihood function for the model from part a) and calculate its derivative with respect to ϑ.

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