On Sylvester's Law of Inertia for Nonlinear Eigenvalue Problems

Heinrich Voss
voss@tuhh.de

TU Hamburg–Harburg

Sylvester's law of inertia

Let $A \in \mathbb{C}^{n \times n}$ be Hermitian. Then the inertia (π, ν, ζ) is invariant under congruence transformations $B = S^H AS$, S nonsingular.

Sylvester's law of inertia

Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Then the inertia (π, ν, ζ) is invariant under congruence transformations $B = S^H AS$, S nonsingular.

Corollary: If $A = LDL^H$ is the LDL-factorization of A. Then π and ν equals the number of positive and negative entries of D, respectively.

Sylvester's law of inertia

Let $A \in \mathbb{C}^{n \times n}$ be Hermitian. Then the inertia (π, ν, ζ) is invariant under congruence transformations $B = S^H AS$, S nonsingular.

Corollary: If $A = LDL^H$ is the LDL-factorization of A. Then π and ν equals the number of positive and negative entries of D, respectively.

Corollary: Suppose that $B > 0$ and $A - \sigma B = LDL^H$. If D has ν negative entries, then $Ax = \lambda Bx$ has ν eigenvalues smaller than σ.

Sylvester's law of inertia

Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. Then the inertia (π, ν, ζ) is invariant under congruence transformations $B = S^H AS$, S nonsingular.

Corollary: If $A = LDL^H$ is the LDL-factorization of A. Then π and ν equals the number of positive and negative entries of D, respectively.

Corollary: Suppose that $B > 0$ and $A - \sigma B = LDL^H$. If D has ν negative entries, then $Ax = \lambda Bx$ has ν eigenvalues smaller than σ.

This leads to

- Localization of eigenvalues
- Bisection method and secant method for eigenvalues
- Determine initial approximation for faster eigensolvers
- Test whether eigenvalues in a given interval are missing

Outline

1. Minmax Characterization
2. Sylvester's law for nonlinear eigenproblems
3. Hyperbolic eigenvalue problems
4. Nonoverdamped quadratic eigenproblems
5. Gyroscopically stabilized systems

Outline

Nonlinear Eigenvalue Problem

Let $J \subset \mathbb{R}$ be an open interval (which may be unbounded), and $T(\lambda)$, $\lambda \in J$ be a family of Hermitian matrices.

Nonlinear Eigenvalue Problem

Let $J \subset \mathbb{R}$ be an open interval (which may be unbounded), and $T(\lambda)$, $\lambda \in J$ be a family of Hermitian matrices.

Nonlinear eigenvalue Problem:
Find $\lambda \in J$ and $x \neq 0$ such that

$$T(\lambda)x = 0.$$

Then λ is called an eigenvalue of $T(\cdot)$, and x a corresponding eigenvector.

Nonlinear Eigenvalue Problem

Let $J \subset \mathbb{R}$ be an open interval (which may be unbounded), and $T(\lambda)$, $\lambda \in J$ be a family of Hermitian matrices.

Nonlinear eigenvalue Problem:
Find $\lambda \in J$ and $x \neq 0$ such that

$$T(\lambda)x = 0.$$

Then λ is called an eigenvalue of $T(\cdot)$, and x a corresponding eigenvector.

Problems of this type arise in damped vibrations of structures, conservative gyroscopic systems, lateral buckling problems, problems with retarded arguments, fluid-solid vibrations, and quantum dot heterostructures, e.g.

Nonlinear minmax theory

Assume that for fixed $x \in \mathbb{C}^n$, $x \neq 0$ the real equation

$$f(\lambda, x) := x^H T(\lambda) x = 0$$

has at most one solution $\lambda =: p(x)$ in J.

Nonlinear minmax theory

Assume that for fixed $x \in \mathbb{C}^n$, $x \neq 0$ the real equation

$$f(\lambda, x) := x^H T(\lambda) x = 0$$

has at most one solution $\lambda =: p(x)$ in J.

Then equation $f(\lambda, x) = 0$ implicitly defines a functional p on some subset D of $\mathbb{C}^n$ which we call the Rayleigh functional.

Nonlinear minmax theory

Assume that for fixed $x \in \mathbb{C}^n$, $x \neq 0$ the real equation

$$f(\lambda, x) := x^H T(\lambda) x = 0$$

has at most one solution $\lambda =: p(x)$ in J.

Then equation $f(\lambda, x) = 0$ implicitly defines a functional p on some subset D of $\mathbb{C}^n$ which we call the Rayleigh functional.

Let

$$(\lambda - p(x)) f(\lambda, x) > 0 \quad \text{for every } \lambda \neq p(x) \text{ and every } x \in D.$$

Overdamped problems

If p is defined on $D = \mathbb{C}^n \setminus \{0\}$ then the problem $T(\lambda)x = 0$ is called overdamped.

Overdamped problems

If p is defined on $D = \mathbb{C}^n \setminus \{0\}$ then the problem $T(\lambda)x = 0$ is called overdamped.

Notation is motivated by the finite dimensional quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda\alpha Cx + Kx = 0$$

where M, C and K are Hermitian and positive definite matrices.

Overdamped problems

If p is defined on $D = \mathbb{C}^n \setminus \{0\}$ then the problem $T(\lambda)x = 0$ is called overdamped.

Notation is motivated by the finite dimensional quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda\alpha Cx + Kx = 0$$

where M, C and K are Hermitian and positive definite matrices.

Theorem (Duffin 1955, Rogers 1964)
Under the conditions above an overdamped problem has exactly n eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ which can be characterized by

$$\lambda_j = \min_{\dim V = j} \max_{x \in V \setminus \{0\}} p(x).$$

Nonoverdamped problems

For nonoverdamped eigenproblems the natural ordering to call the smallest eigenvalue the first one, the second smallest the second one, etc., is not appropriate.

Nonoverdamped problems

For nonoverdamped eigenproblems the natural ordering to call the smallest eigenvalue the first one, the second smallest the second one, etc., is not appropriate.

This is obvious if we make a linear eigenvalue

$$T(\lambda)x := (\lambda I - A)x = 0$$

nonlinear by restricting it to an interval J which does not contain the smallest eigenvalue of A.

Nonoverdamped problems

For nonoverdamped eigenproblems the natural ordering to call the smallest eigenvalue the first one, the second smallest the second one, etc., is not appropriate.

This is obvious if we make a linear eigenvalue

$$T(\lambda)x := (\lambda I - A)x = 0$$

nonlinear by restricting it to an interval J which does not contain the smallest eigenvalue of A.

Then all conditions are satisfied, p is the restriction of the Rayleigh quotient R_A to

$$D := \{x \neq 0 \,:\, R_A(x) \in J\},$$

and $\inf_{x \in D} p(x)$ will in general not be an eigenvalue.

Enumeration of eigenvalues

If $\lambda \in J$ is an eigenvalue of $T(\cdot)$ then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $\ell \in \mathbb{N}$ such that

$$0 = \max_{V \in H_\ell} \min_{v \in V \setminus \{0\}} \frac{v^H T(\lambda) v}{\|v\|^2}$$

where H_ℓ denotes the set of all ℓ–dimensional subspaces of $\mathbb{C}^n$.

Enumeration of eigenvalues

If $\lambda \in J$ is an eigenvalue of $T(\cdot)$ then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $\ell \in \mathbb{N}$ such that

$$0 = \max_{V \in H_\ell} \min_{v \in V \setminus \{0\}} \frac{v^H T(\lambda) v}{\|v\|^2}$$

where H_ℓ denotes the set of all ℓ–dimensional subspaces of $\mathbb{C}^n$.

In this case λ is called an ℓ-th eigenvalue of $T(\cdot)$.

Minmax characterization (V., Werner 1982, V. 2009)

Under the conditions given above it holds:

(i) For every $\ell \in \mathbb{N}$ there is at most one ℓ-th eigenvalue of $T(\cdot)$ which can be characterized by

$$\lambda_\ell = \min_{V \in H_\ell,\ V \cap D \neq \emptyset} \ \sup_{v \in V \cap D} p(v). \qquad (*)$$

The set of eigenvalues of $T(\cdot)$ in J is at most countable.

Minmax characterization (V., Werner 1982, V. 2009)

Under the conditions given above it holds:

(i) For every $\ell \in \mathbb{N}$ there is at most one ℓ-th eigenvalue of $T(\cdot)$ which can be characterized by

$$\lambda_\ell = \min_{V \in H_\ell,\ V \cap D \neq \emptyset} \ \sup_{v \in V \cap D} \ p(v). \qquad (*)$$

The set of eigenvalues of $T(\cdot)$ in J is at most countable.

(ii) If

$$\lambda_\ell := \inf_{V \in H_\ell,\ V \cap D \neq \emptyset} \ \sup_{v \in V \cap D} \ p(v) \in J$$

for some $\ell \in \mathbb{N}$ then λ_ℓ is the ℓ-th eigenvalue of $T(\cdot)$ in J, and (*) holds.

Minmax characterization (V., Werner 1982, V. 2009)

Under the conditions given above it holds:

(i) For every $\ell \in \mathbb{N}$ there is at most one ℓ-th eigenvalue of $T(\cdot)$ which can be characterized by

$$\lambda_\ell = \min_{V \in H_\ell,\ V \cap D \neq \emptyset} \sup_{v \in V \cap D} p(v). \qquad (*)$$

The set of eigenvalues of $T(\cdot)$ in J is at most countable.

(ii) If

$$\lambda_\ell := \inf_{V \in H_\ell,\ V \cap D \neq \emptyset} \sup_{v \in V \cap D} p(v) \in J$$

for some $\ell \in \mathbb{N}$ then λ_ℓ is the ℓ-th eigenvalue of $T(\cdot)$ in J, and (*) holds.

(iii) $\tilde{\lambda}$ is an ℓ-th eigenvalue if and only if $\mu = 0$ is the ℓ largest eigenvalue of the linear eigenproblem $T(\tilde{\lambda})x = \mu x$.

Minmax characterization (V., Werner 1982, V. 2009)

Under the conditions given above it holds:

(i) For every $\ell \in \mathbb{N}$ there is at most one ℓ-th eigenvalue of $T(\cdot)$ which can be characterized by

$$\lambda_\ell = \min_{V \in H_\ell,\ V \cap D \neq \emptyset} \sup_{v \in V \cap D} p(v). \qquad (*)$$

The set of eigenvalues of $T(\cdot)$ in J is at most countable.

(ii) If

$$\lambda_\ell := \inf_{V \in H_\ell,\ V \cap D \neq \emptyset} \sup_{v \in V \cap D} p(v) \in J$$

for some $\ell \in \mathbb{N}$ then λ_ℓ is the ℓ-th eigenvalue of $T(\cdot)$ in J, and (*) holds.

(iii) $\tilde{\lambda}$ is an ℓ-th eigenvalue if and only if $\mu = 0$ is the ℓ largest eigenvalue of the linear eigenproblem $T(\tilde{\lambda})x = \mu x$.

(iv) The minimum in (*) is attained for the invariant subspace of $T(\lambda_\ell)$ corresponding to its ℓ largest eigenvalues.

Outline

Sylvester's law for overdamped problems

Assume that $T : J \to \mathbb{C}^{n\times n}$ is overdamped and satisfies the conditions of the minmax characterization.

Sylvester's law for overdamped problems

Assume that $T : J \to \mathbb{C}^{n\times n}$ is overdamped and satisfies the conditions of the minmax characterization.

For $\sigma \in J$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has π eigenvalues that are smaller than σ, ν eigenvalues that exceed σ, and σ is an eigenvalue of multiplicity ζ.

Sylvester's law for overdamped problems

Assume that $T : J \to \mathbb{C}^{n\times n}$ is overdamped and satisfies the conditions of the minmax characterization.

For $\sigma \in J$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has π eigenvalues that are smaller than σ, ν eigenvalues that exceed σ, and σ is an eigenvalue of multiplicity ζ.

Sketch of proof: Show that

(i) If W denotes the invariant subspace of $T(\sigma)$ corresponding to its positive eigenvalues, then $f(\sigma, x) = x^H T(\sigma)x > 0$ for $x \in W$, $x \neq 0$. Hence, $p(x) < \sigma$, and therefore

$$\lambda_\pi = \min_{\dim V=\pi} \max_{x\in V} p(x) \leq \max_{x\in W} p(x) < \sigma.$$

Sylvester's law for overdamped problems

Assume that $T : J \to \mathbb{C}^{n\times n}$ is overdamped and satisfies the conditions of the minmax characterization.

For $\sigma \in J$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has π eigenvalues that are smaller than σ, ν eigenvalues that exceed σ, and σ is an eigenvalue of multiplicity ζ.

Sketch of proof: Show that

(i) If W denotes the invariant subspace of $T(\sigma)$ corresponding to its positive eigenvalues, then $f(\sigma, x) = x^H T(\sigma)x > 0$ for $x \in W$, $x \neq 0$. Hence, $p(x) < \sigma$, and therefore

$$\lambda_\pi = \min_{\dim V = \pi} \max_{x \in V} p(x) \leq \max_{x \in W} p(x) < \sigma.$$

(ii) For every subspace V with $\dim V = \pi + \zeta + 1$ there exists $x \in V$ with $p(x) > \sigma$, and thus

$$\lambda_{\pi+\zeta+1} = \min_{\dim V = \pi+\zeta+1} \max_{x \in V} p(x) > \sigma.$$

Sylvester's law for non-overdamped problems

Extreme eigenvalues:

Assume that $T : J \to \mathbb{C}^{n\times n}$ satisfies the conditions of the minmax characterization, and assume that there exists $\mu \in J$ such that $T(\mu)$ is negative definite.

Sylvester's law for non-overdamped problems

Extreme eigenvalues:

Assume that $T : J \to \mathbb{C}^{n\times n}$ satisfies the conditions of the minmax characterization, and assume that there exists $\mu \in J$ such that $T(\mu)$ is negative definite.

For $\sigma \in J$, $\sigma > \mu$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has exactly π eigenvalues in J that are smaller than σ.

Sylvester's law for non-overdamped problems

Extreme eigenvalues:
Assume that $T : J \to \mathbb{C}^{n\times n}$ satisfies the conditions of the minmax characterization, and assume that there exists $\mu \in J$ such that $T(\mu)$ is negative definite.

For $\sigma \in J$, $\sigma > \mu$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has exactly π eigenvalues in J that are smaller than σ.

General case:
Assume that for every ρ dimensional subspace V with $V \cap D \neq \emptyset$ there exists $x \in V \cap D$ with $p(x) > \mu$, $\mu \in J$.

Sylvester's law for non-overdamped problems

Extreme eigenvalues:
Assume that $T : J \to \mathbb{C}^{n\times n}$ satisfies the conditions of the minmax characterization, and assume that there exists $\mu \in J$ such that $T(\mu)$ is negative definite.

For $\sigma \in J$, $\sigma > \mu$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has exactly π eigenvalues in J that are smaller than σ.

General case:
Assume that for every ρ dimensional subspace V with $V \cap D \neq \emptyset$ there exists $x \in V \cap D$ with $p(x) > \mu$, $\mu \in J$.

For $\sigma \in J$, $\sigma > \mu$ let (π, ν, ζ) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has eigenvalues $\lambda_\rho \leq \cdots \leq \lambda_\pi < \sigma$.

Outline

Hyperbolic quadratic pencils

A quadratic matrix polynomial

$$Q(\lambda) := \lambda^2 A + \lambda B + C, \quad A = A^H > 0,\ B = B^H,\ C = C^H$$

is hyperbolic if for every $x \in \mathbb{C}^n$, $x \neq 0$ the quadratic polynomial

$$\lambda^2 x^H A x + \lambda x^H B x + x^H C x = 0$$

has two distinct real roots

Hyperbolic quadratic pencils

A quadratic matrix polynomial

$$Q(\lambda) := \lambda^2 A + \lambda B + C, \quad A = A^H > 0, \; B = B^H, \; C = C^H$$

is hyperbolic if for every $x \in \mathbb{C}^n$, $x \neq 0$ the quadratic polynomial

$$\lambda^2 x^H Ax + \lambda x^H Bx + x^H Cx = 0$$

has two distinct real roots

$$p_{\pm}(x) = -\frac{x^H Bx}{2x^H Ax} \pm \sqrt{\left(\frac{x^H Bx}{2x^H Ax}\right)^2 - \frac{x^H Cx}{x^H Ax}}.$$

Hyperbolic quadratic pencils

A quadratic matrix polynomial

$$Q(\lambda) := \lambda^2 A + \lambda B + C, \quad A = A^H > 0,\ B = B^H,\ C = C^H$$

is hyperbolic if for every $x \in \mathbb{C}^n$, $x \neq 0$ the quadratic polynomial

$$\lambda^2 x^H A x + \lambda x^H B x + x^H C x = 0$$

has two distinct real roots

$$p_\pm(x) = -\frac{x^H B x}{2x^H A x} \pm \sqrt{\left(\frac{x^H B x}{2x^H A x}\right)^2 - \frac{x^H C x}{x^H A x}}.$$

The ranges $J_\pm := p_\pm(\mathbb{C}^n \setminus \{0\})$ are disjoint real intervals with $\max J_- < \min J_+$

Hyperbolic quadratic pencils ct.

$Q(\lambda)$ is positive definite for $\lambda < \min J_-$ and $\lambda > \max J_+$, and it is negative definite for $\lambda \in (\max J_-, \min J_+)$.

Hyperbolic quadratic pencils ct.

$Q(\lambda)$ is positive definite for $\lambda < \min J_-$ and $\lambda > \max J_+$, and it is negative definite for $\lambda \in (\max J_-, \min J_+)$.

(Q, J_+) and $(-Q, J_-)$ satisfy the conditions of the variational characterization of eigenvalues, i.e. there exist $2n$ eigenvalues (cf. Duffin 1955)

$$\lambda_1 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots \leq \lambda_{2n}$$

Hyperbolic quadratic pencils ct.

$Q(\lambda)$ is positive definite for $\lambda < \min J_-$ and $\lambda > \max J_+$, and it is negative definite for $\lambda \in (\max J_-, \min J_+)$.

(Q, J_+) and $(-Q, J_-)$ satisfy the conditions of the variational characterization of eigenvalues, i.e. there exist $2n$ eigenvalues (cf. Duffin 1955)

$$\lambda_1 \le \cdots \le \lambda_n < \lambda_{n+1} \le \cdots \le \lambda_{2n}$$

and

$$\lambda_j = \min_{\dim V = j} \max_{x \in V, x \neq 0} p_-(x) \quad \text{and} \quad \lambda_{n+j} = \min_{\dim V = j} \max_{x \in V, x \neq 0} p_+(x), \quad j = 1, \dots, n.$$

Hyperbolic quadratic pencils ct.

$Q(\lambda)$ is positive definite for $\lambda < \min J_-$ and $\lambda > \max J_+$, and it is negative definite for $\lambda \in (\max J_-, \min J_+)$.

(Q, J_+) and $(-Q, J_-)$ satisfy the conditions of the variational characterization of eigenvalues, i.e. there exist $2n$ eigenvalues (cf. Duffin 1955)

$$\lambda_1 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots \leq \lambda_{2n}$$

and

$$\lambda_j = \min_{\dim V = j} \max_{x \in V, x \neq 0} p_-(x) \quad \text{and} \quad \lambda_{n+j} = \min_{\dim V = j} \max_{x \in V, x \neq 0} p_+(x), \quad j = 1, \dots, n.$$

For $\sigma \in \mathbb{R}$ and $x \neq 0$ with $f(\sigma, x) := x^H Q(\sigma) x > 0$ and $2\sigma x^H A x + x^H B x < 0$ it follows that $p_-(x) > \sigma$, and therefore $\sigma \leq \lambda_n$. Similarly, $f(\sigma, x) := x^H Q(\sigma) x > 0$ and $2\sigma x^H A x + x^H B x > 0$ implies $\sigma > \lambda_{n+1}$.

Sylvester's law for hyperbolic quadratic pencils

Let $Q(\lambda)$ be hyperbolic, and let (π, ν, ζ) be the inertia of $Q(\sigma)$ for $\sigma \in \mathbb{R}$.

Sylvester's law for hyperbolic quadratic pencils

Let $Q(\lambda)$ be hyperbolic, and let (π, ν, ζ) be the inertia of $Q(\sigma)$ for $\sigma \in \mathbb{R}$.

(1) If $\pi = \zeta = 0$ then the quadratic eigenvalue problem $Q(\lambda)x = 0$ has n eigenvalues in $(-\infty, \sigma)$ and n eigenvalues in (σ, ∞).

Sylvester's law for hyperbolic quadratic pencils

Let $Q(\lambda)$ be hyperbolic, and let (π, ν, ζ) be the inertia of $Q(\sigma)$ for $\sigma \in \mathbb{R}$.

(1) If $\pi = \zeta = 0$ then the quadratic eigenvalue problem $Q(\lambda)x = 0$ has n eigenvalues in $(-\infty, \sigma)$ and n eigenvalues in (σ, ∞).

(2) For $\pi = 0$ and $\zeta > 0$ let $x \neq 0$ be an element of the null space of $Q(\sigma)$. If $2\sigma x^H Mx + x^H Cx < 0$ then $Q(\lambda)x = 0$ has $n - \zeta$ eigenvalues in $(-\infty, \sigma)$, n eigenvalues in (σ, ∞) and $\sigma = \lambda_n$ with multiplicity ζ, and if $2\sigma x^H Mx + x^H Cx > 0$ then $Q(\lambda)x = 0$ has n eigenvalues in $(-\infty, \sigma)$, $n - \zeta$ eigenvalues in (σ, ∞) and $\sigma = \lambda_{n+1}$ with multiplicity ζ.

Sylvester's law for hyperbolic quadratic pencils

Let $Q(\lambda)$ be hyperbolic, and let (π, ν, ζ) be the inertia of $Q(\sigma)$ for $\sigma \in \mathbb{R}$.

(1) If $\pi = \zeta = 0$ then the quadratic eigenvalue problem $Q(\lambda)x = 0$ has n eigenvalues in $(-\infty, \sigma)$ and n eigenvalues in (σ, ∞).

(2) For $\pi = 0$ and $\zeta > 0$ let $x \neq 0$ be an element of the null space of $Q(\sigma)$. If $2\sigma x^H Mx + x^H Cx < 0$ then $Q(\lambda)x = 0$ has $n - \zeta$ eigenvalues in $(-\infty, \sigma)$, n eigenvalues in (σ, ∞) and $\sigma = \lambda_n$ with multiplicity ζ, and if $2\sigma x^H Mx + x^H Cx > 0$ then $Q(\lambda)x = 0$ has n eigenvalues in $(-\infty, \sigma)$, $n - \zeta$ eigenvalues in (σ, ∞) and $\sigma = \lambda_{n+1}$ with multiplicity ζ.

(3) For $\pi > 0$ and $\zeta = 0$ let $x \neq 0$ such that $f(\sigma; x) > 0$. If $2\sigma x^H Mx + x^H Cx < 0$ then $Q(\lambda)x = 0$ has $n - \pi$ eigenvalues in $(-\infty, \sigma)$ and $n + \pi$ eigenvalues in (σ, ∞), and if $2\sigma x^H Mx + x^H Cx > 0$ then $Q(\lambda)x = 0$ has $n + \pi$ eigenvalues in $(-\infty, \sigma)$ and $n - \pi$ eigenvalues in (σ, ∞).

Sylvester's law for hyperbolic quadratic pencils

Let $Q(\lambda)$ be hyperbolic, and let (π, ν, ζ) be the inertia of $Q(\sigma)$ for $\sigma \in \mathbb{R}$.

(1) If $\pi = \zeta = 0$ then the quadratic eigenvalue problem $Q(\lambda)x = 0$ has n eigenvalues in $(-\infty, \sigma)$ and n eigenvalues in (σ, ∞).

(2) For $\pi = 0$ and $\zeta > 0$ let $x \neq 0$ be an element of the null space of $Q(\sigma)$. If $2\sigma x^H Mx + x^H Cx < 0$ then $Q(\lambda)x = 0$ has $n - \zeta$ eigenvalues in $(-\infty, \sigma)$, n eigenvalues in (σ, ∞) and $\sigma = \lambda_n$ with multiplicity ζ, and if $2\sigma x^H Mx + x^H Cx > 0$ then $Q(\lambda)x = 0$ has n eigenvalues in $(-\infty, \sigma)$, $n - \zeta$ eigenvalues in (σ, ∞) and $\sigma = \lambda_{n+1}$ with multiplicity ζ.

(3) For $\pi > 0$ and $\zeta = 0$ let $x \neq 0$ such that $f(\sigma; x) > 0$. If $2\sigma x^H Mx + x^H Cx < 0$ then $Q(\lambda)x = 0$ has $n - \pi$ eigenvalues in $(-\infty, \sigma)$ and $n + \pi$ eigenvalues in (σ, ∞), and if $2\sigma x^H Mx + x^H Cx > 0$ then $Q(\lambda)x = 0$ has $n + \pi$ eigenvalues in $(-\infty, \sigma)$ and $n - \pi$ eigenvalues in (σ, ∞).

(4) For $\pi > 0$ and $\zeta > 0$ let $x \neq 0$ such that $f(\sigma; x) > 0$. If $2\sigma x^H Mx + x^H Cx < 0$ then $Q(\lambda)x = 0$ has $n - \pi - \zeta$ eigenvalues in $(-\infty, \sigma)$ and $n + \pi$ eigenvalues in (σ, ∞), and if $2\sigma x^H Mx + x^H Cx > 0$ then $Q(\lambda)x = 0$ has $n + \pi$ eigenvalues in $(-\infty, \sigma)$ and $n - \pi - \zeta$ eigenvalues in (σ, ∞). In either case σ is an eigenvalue with multiplicity ζ.

hyperbolic pencils

Let

$$P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j, \quad A_j = A_j^H, \; j = 0, \dots, k, \quad A_k > 0$$

be hyperbolic, i.e. $f(\lambda; x) := x^H P(\lambda) x = 0$ has exactly k real roots for $x \neq 0$

hyperbolic pencils

Let

$$P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j, \quad A_j = A_j^H,\ j = 0, \dots, k, \quad A_k > 0$$

be hyperbolic, i.e. $f(\lambda; x) := x^H P(\lambda) x = 0$ has exactly k real roots for $x \neq 0$

Then there exist k disjoint open intervals $\Delta_j \subset \mathbb{R}$, $j = 1, \dots, k$ such that $P(\lambda)x = 0$ has exactly n eigenvalues in each Δ_j which allow for a minmax characterization. To fix the numeration let $\sup \Delta_j < \inf \Delta_{j-1}$.

hyperbolic pencils

Let

$$P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j, \quad A_j = A_j^H, \; j = 0, \dots, k, \quad A_k > 0$$

be hyperbolic, i.e. $f(\lambda; x) := x^H P(\lambda) x = 0$ has exactly k real roots for $x \neq 0$

Then there exist k disjoint open intervals $\Delta_j \subset \mathbb{R}$, $j = 1, \dots, k$ such that $P(\lambda)x = 0$ has exactly n eigenvalues in each Δ_j which allow for a minmax characterization. To fix the numeration let $\sup \Delta_j < \inf \Delta_{j-1}$.

For $\sigma \in \mathbb{R}$ let (π, ν, ζ) be the inertia of $T(\sigma)$, and let $x \in \mathbb{C}^n$ such that $x^H T(\sigma) x > 0$.

hyperbolic pencils

Let

$$P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j, \quad A_j = A_j^H, \; j = 0, \dots, k, \quad A_k > 0$$

be hyperbolic, i.e. $f(\lambda; x) := x^H P(\lambda)x = 0$ has exactly k real roots for $x \neq 0$

Then there exist k disjoint open intervals $\Delta_j \subset \mathbb{R}$, $j = 1, \dots, k$ such that $P(\lambda)x = 0$ has exactly n eigenvalues in each Δ_j which allow for a minmax characterization. To fix the numeration let $\sup \Delta_j < \inf \Delta_{j-1}$.

For $\sigma \in \mathbb{R}$ let (π, ν, ζ) be the inertia of $T(\sigma)$, and let $x \in \mathbb{C}^n$ such that $x^H T(\sigma)x > 0$.

If $f(\cdot; x)$ has exactly j roots which exceed σ then it holds that

$$\sigma \in \Delta_{j+1} \quad \text{or} \quad \sigma \in [\sup \Delta_{j+1}, \inf \Delta_j] \quad \text{or} \quad \sigma \in \Delta_j.$$

hyperbolic pencils

Let

$$P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j, \quad A_j = A_j^H, \; j = 0, \dots, k, \quad A_k > 0$$

be hyperbolic, i.e. $f(\lambda; x) := x^H P(\lambda) x = 0$ has exactly k real roots for $x \neq 0$

Then there exist k disjoint open intervals $\Delta_j \subset \mathbb{R}$, $j = 1, \dots, k$ such that $P(\lambda)x = 0$ has exactly n eigenvalues in each Δ_j which allow for a minmax characterization. To fix the numeration let $\sup \Delta_j < \inf \Delta_{j-1}$.

For $\sigma \in \mathbb{R}$ let (π, ν, ζ) be the inertia of $T(\sigma)$, and let $x \in \mathbb{C}^n$ such that $x^H T(\sigma) x > 0$.

If $f(\cdot; x)$ has exactly j roots which exceed σ then it holds that

$$\sigma \in \Delta_{j+1} \quad \text{or} \quad \sigma \in [\sup \Delta_{j+1}, \inf \Delta_j] \quad \text{or} \quad \sigma \in \Delta_j.$$

The inertia and the derivative $\frac{\partial}{\partial \lambda} f(\sigma; x)$ yield the localization of σ within the spectrum of $T(\lambda)x = 0$.

Outline

Nonoverdamped quadratic eigenproblems

$$Q(\lambda) = \lambda^2 A + \lambda B + C, \quad A > 0,\ B > 0,\ C > 0$$

Nonoverdamped quadratic eigenproblems

$$Q(\lambda) = \lambda^2 A + \lambda B + C, \quad A > 0,\ B > 0,\ C > 0$$

Let

$$\delta_- := \sup\{p_-(x) \ : \ p_-(x) \in \mathbb{R}\}, \quad \delta_+ := \inf\{p_+(x) \ : \ p_+(x) \in \mathbb{R}\}$$

and $J_- := (-\infty, \delta_+)$, $J_+ = (\delta_-, \infty)$.

Nonoverdamped quadratic eigenproblems

$$Q(\lambda) = \lambda^2 A + \lambda B + C, \quad A > 0,\ B > 0,\ C > 0$$

Let

$$\delta_- := \sup\{p_-(x)\ :\ p_-(x) \in \mathbb{R}\}, \quad \delta_+ := \inf\{p_+(x)\ :\ p_+(x) \in \mathbb{R}\}$$

and $J_- := (-\infty, \delta_+)$, $J_+ = (\delta_-, \infty)$.

Then all eigenvalues in J_- are min sup values of p_-, and all eigenvalues in J_+ are max inf values of p_+.

Nonoverdamped quadratic eigenproblems

$$Q(\lambda) = \lambda^2 A + \lambda B + C, \quad A > 0,\ B > 0,\ C > 0$$

Let

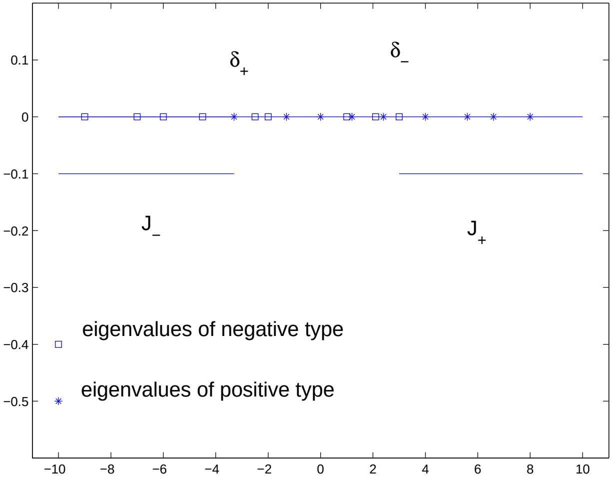
$$\delta_- := \sup\{p_-(x) \ : \ p_-(x) \in \mathbb{R}\}, \quad \delta_+ := \inf\{p_+(x) \ : \ p_+(x) \in \mathbb{R}\}$$

and $J_- := (-\infty, \delta_+)$, $J_+ = (\delta_-, \infty)$.

Then all eigenvalues in J_- are min sup values of p_-, and all eigenvalues in J_+ are max inf values of p_+.

Hence, for $\sigma < \delta_+$ with $\text{in}(T(\sigma)) = (\pi, \nu, \zeta)$ there are ν eigenvalues of $T(\lambda)x = 0$ in $(-\infty, \delta_+)$, and for $\sigma \in (0, \delta_-)$ with $\text{in}(T(\sigma)) = (\pi, \nu, \zeta)$ there are ν eigenvalues of $T(\lambda)x = 0$ in $(\delta_-, 0)$

Nonoverdamped quadratic eigenproblems



Nonoverdamped quadratic eigenproblems

Theorem

Let $A, B, C > 0$. Then it holds that

$$\tilde{\delta}_+ := -\sqrt{\max_{x \neq 0} \frac{x^H C x}{x^H A x}} \leq \delta_+ = \inf\{p_+(x) \; : \; p_+(x) \in \mathbb{R}\}$$

and

$$\delta_- = \sup\{p_-(x) \; : \; p_-(x) \in \mathbb{R}\} \leq -\sqrt{\min_{x \neq 0} \frac{x^H C x}{x^H A x}} =: \tilde{\delta}_-$$

Nonoverdamped quadratic eigenproblems

Theorem

Let $A, B, C > 0$. Then it holds that

$$\tilde{\delta}_+ := -\sqrt{\max_{x \neq 0} \frac{x^H C x}{x^H A x}} \leq \delta_+ = \inf\{p_+(x) \; : \; p_+(x) \in \mathbb{R}\}$$

and

$$\delta_- = \sup\{p_-(x) \; : \; p_-(x) \in \mathbb{R}\} \leq -\sqrt{\min_{x \neq 0} \frac{x^H C x}{x^H A x}} =: \tilde{\delta}_-$$

Hence, for $\sigma < \tilde{\delta}_+$ with $\text{in}(T(\sigma)) = (\pi, \nu, \zeta)$ there are ν eigenvalues of $T(\lambda)x = 0$ in $(-\infty, \tilde{\delta}_+)$, and for $\sigma \in (0, \tilde{\delta}_-)$ with $\text{in}(T(\sigma)) = (\pi, \nu, \zeta)$ there are ν eigenvalues of $T(\lambda)x = 0$ in $(\tilde{\delta}_-, 0)$

Nonoverdamped quadratic eigenproblems

Proof

$$f(p_+(x);x) = x^H Q(p_+(x))x = 0 \quad \Longleftrightarrow \quad x^H Bx = -p_+(x)x^H Ax - \frac{1}{p_+(x)} x^H Cx.$$

Nonoverdamped quadratic eigenproblems

Proof

$$f(p_+(x);x) = x^H Q(p_+(x))x = 0 \iff x^H Bx = -p_+(x)x^H Ax - \frac{1}{p_+(x)} x^H Cx.$$

Hence,

$$\frac{\partial}{\partial \lambda} f(p_+(x);x) = 2p_+(x)x^H Ax + x^H Bx = p_+(x)x^H Ax - \frac{1}{p_+(x)} x^H Cx \geq 0$$

Nonoverdamped quadratic eigenproblems

Proof

$$f(p_+(x);x) = x^H Q(p_+(x))x = 0 \quad \Longleftrightarrow \quad x^H Bx = -p_+(x)x^H Ax - \frac{1}{p_+(x)} x^H Cx.$$

Hence,

$$\frac{\partial}{\partial\lambda} f(p_+(x);x) = 2p_+(x)x^H Ax + x^H Bx = p_+(x)x^H Ax - \frac{1}{p_+(x)} x^H Cx \geq 0$$

$$\Longleftrightarrow \quad p_+(x)^2 \leq \frac{x^H Cx}{x^H Ax}, \quad \text{i.e.} \quad \delta_+ \geq -\sqrt{\lambda_{\max}(C,A)} = \tilde{\delta}_+.$$

And analogously

$$\delta_- \leq -\sqrt{\lambda_{\min}(C,A)} = \tilde{\delta}_-.$$

Nonoverdamped quadratic eigenproblems

Proof

$$f(p_+(x);x) = x^HQ(p_+(x))x = 0 \quad \Longleftrightarrow \quad x^HBx = -p_+(x)x^HAx - \frac{1}{p_+(x)}x^HCx.$$

Hence,

$$\frac{\partial}{\partial\lambda}f(p_+(x);x) = 2p_+(x)x^HAx + x^HBx = p_+(x)x^HAx - \frac{1}{p_+(x)}x^HCx \geq 0$$

$$\Longleftrightarrow \quad p_+(x)^2 \leq \frac{x^HCx}{x^HAx}, \quad \text{i.e.} \quad \delta_+ \geq -\sqrt{\lambda_{\max}(C,A)} = \tilde{\delta}_+.$$

And analogously

$$\delta_- \leq -\sqrt{\lambda_{\min}(C,A)} = \tilde{\delta}_-.$$

Eliminating x^HAx one gets $\quad -2\lambda_{\max}(C,B) \leq \delta_+ \leq \delta_- \leq -2\lambda_{min}(C,B).$

Example

$$A = \text{eye}(20); \quad B = \text{randn}(20); B = B' * B; \quad C = \text{randn}(20); C = C' * C;$$

Example

$$A = \text{eye}(20); \quad B = \text{randn}(20); B = B' * B; \quad C = \text{randn}(20); C = C' * C;$$

Then $Q(\lambda)x = 0$ has 26 real eigenvalues, 13 of either type, and the maximum of the eigenvalues of negative type is less than the minimum of the eigenvalue of positive type. So, Sylvester's theorem can be applied to all of them.

Example

$$A = \text{eye}(20); \quad B = \text{randn}(20); B = B' * B; \quad C = \text{randn}(20); C = C' * C;$$

Then $Q(\lambda)x = 0$ has 26 real eigenvalues, 13 of either type, and the maximum of the eigenvalues of negative type is less than the minimum of the eigenvalue of positive type. So, Sylvester's theorem can be applied to all of them.

12 eigenvalues of negative type are less than $-\sqrt{\max(\lambda(C, A))}$ and 6 eigenvalues of positive type exceed $-\sqrt{\min(\lambda(C, A))}$.

Outline

Gyroscopically stabilized pencils

A quadratic matrix polynomial

$$Q(\lambda) := \lambda^2 I + \lambda B + C, \quad B = B^H, \ \det B \neq 0, \ C = C^H > 0$$

is gyroscopically stabilized if for some $k > 0$ it holds that

$$|B| > kI + k^{-1}C$$

where $|B|$ denotes the positive square root of B^2

Gyroscopically stabilized pencils

A quadratic matrix polynomial

$$Q(\lambda) := \lambda^2 I + \lambda B + C, \quad B = B^H, \ \det B \neq 0, \ C = C^H > 0$$

is gyroscopically stabilized if for some $k > 0$ it holds that

$$|B| > kI + k^{-1} C$$

where $|B|$ denotes the positive square root of B^2

Motivation: For $G^H = -G$ the free motions of a conservative, time-invariant linear system oscillating about an unstable equilibrium under action of a gyroscopic force are governed by

$$\ddot{u}(t) + G\dot{u}(t) - Cu(t) = 0.$$

Gyroscopically stabilized pencils

A quadratic matrix polynomial

$$Q(\lambda) := \lambda^2 I + \lambda B + C, \quad B = B^H, \ \det B \neq 0, \ C = C^H > 0$$

is gyroscopically stabilized if for some $k > 0$ it holds that

$$|B| > kI + k^{-1} C$$

where $|B|$ denotes the positive square root of B^2

Motivation: For $G^H = -G$ the free motions of a conservative, time-invariant linear system oscillating about an unstable equilibrium under action of a gyroscopic force are governed by

$$\ddot{u}(t) + G\dot{u}(t) - Cu(t) = 0.$$

Making the substitution $u(t) = x \exp(\mu t)$ with x independent of t, and then the rotation of the parameter $\lambda = -i\mu$ leads to the eigenvalue problem $Q(\lambda)x = 0$ where $B = iG$ is clearly indefinite.

Gyroscopically stabilized pencils ct.

Theorem (Barkwell, Lancaster, Markus 1992)

- The spectrum of a gyroscopic stabilized pencil is real, i.e. Q is quasihyperbolic.

Gyroscopically stabilized pencils ct.

Theorem (Barkwell, Lancaster, Markus 1992)

- The spectrum of a gyroscopic stabilized pencil is real, i.e. Q is quasihyperbolic.
- All eigenvalues are either of positive type (i.e. $x^H Q'(\lambda)x > 0$) or of negative type (i.e. $x^H Q'(\lambda)x < 0$).

Gyroscopically stabilized pencils ct.

Theorem (Barkwell, Lancaster, Markus 1992)

- The spectrum of a gyroscopic stabilized pencil is real, i.e. Q is quasihyperbolic.
- All eigenvalues are either of positive type (i.e. $x^H Q'(\lambda)x > 0$) or of negative type (i.e. $x^H Q'(\lambda)x < 0$).
- If $(\pi, \nu, 0)$ is the inertia of B, then $Q(\lambda)x = 0$ has 2π negative and 2ν positive eigenvalues.

Gyroscopically stabilized pencils ct.

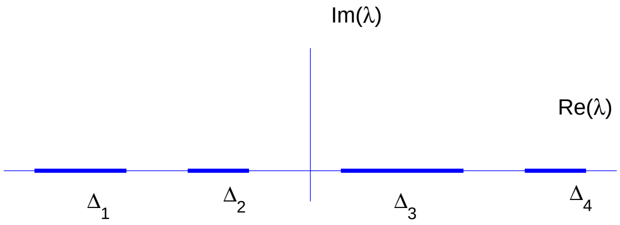
Theorem (Barkwell, Lancaster, Markus 1992)

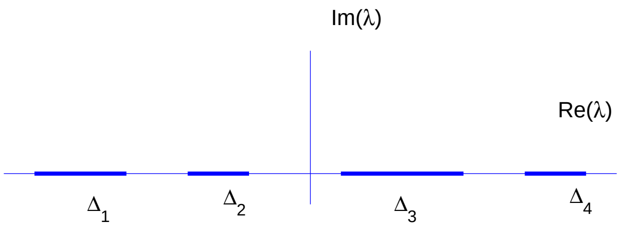
- The spectrum of a gyroscopic stabilized pencil is real, i.e. Q is quasihyperbolic.
- All eigenvalues are either of positive type (i.e. $x^H Q'(\lambda)x > 0$) or of negative type (i.e. $x^H Q'(\lambda)x < 0$).
- If $(\pi, \nu, 0)$ is the inertia of B, then $Q(\lambda)x = 0$ has 2π negative and 2ν positive eigenvalues.
- The 2π negative eigenvalues lie in two disjoint intervals, π eigenvalues in each; the ones in the left interval are of negative type, the ones in the right interval are of positive type.

Gyroscopically stabilized pencils ct.

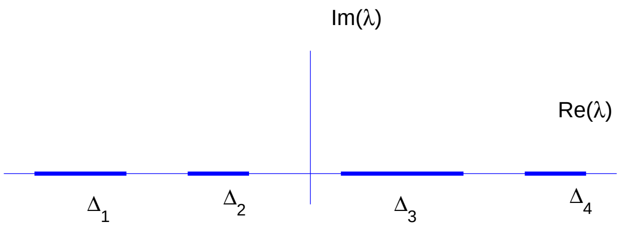
Theorem (Barkwell, Lancaster, Markus 1992)

- The spectrum of a gyroscopic stabilized pencil is real, i.e. Q is quasihyperbolic.
- All eigenvalues are either of positive type (i.e. $x^H Q'(\lambda)x > 0$) or of negative type (i.e. $x^H Q'(\lambda)x < 0$).
- If $(\pi, \nu, 0)$ is the inertia of B, then $Q(\lambda)x = 0$ has 2π negative and 2ν positive eigenvalues.
- The 2π negative eigenvalues lie in two disjoint intervals, π eigenvalues in each; the ones in the left interval are of negative type, the ones in the right interval are of positive type.
- The 2ν positive eigenvalues lie in two disjoint intervals, ν eigenvalues in each; the ones in the left interval are of negative type, the ones in the right interval are of positive type.


Im(λ)
Re(λ)
Δ1
Δ2
Δ3
Δ4



- in Δ_1 there are π eigenvalues of negative type
- in Δ_2 there are π eigenvalues of positive type
- in Δ_3 there are $n - \pi$ eigenvalues of negative type
- in Δ_4 there are $n - \pi$ eigenvalues of positive type



- in Δ_1 there are π eigenvalues of negative type
- in Δ_2 there are π eigenvalues of positive type
- in Δ_3 there are $n - \pi$ eigenvalues of negative type
- in Δ_4 there are $n - \pi$ eigenvalues of positive type

There is a basis of eigenvectors corresponding to eigenvalues in $\Delta_1 \cup \Delta_3$, and there is basis of eigenvectors corresponding to eigenvalues in $\Delta_2 \cup \Delta_4$.

Gyroscopically stabilized pencils ct.

Theorem (Lancaster, Markus, Zhou 2003)

- There exists a π dimensional subspace V of $\mathbb{C}^{n\times n}$ such that $x^H Q(-k)x < 0$ for every $x \in V \setminus \{0\}$.
- There exists a $n-\pi$ dimensional subspace W of $\mathbb{C}^{n\times n}$ such that $x^H Q(k)x < 0$ for every $x \in W \setminus \{0\}$.

Gyroscopically stabilized pencils ct.

Theorem (Lancaster, Markus, Zhou 2003)

- There exists a π dimensional subspace V of $\mathbb{C}^{n\times n}$ such that $x^H Q(-k)x < 0$ for every $x \in V \setminus \{0\}$.
- There exists a $n-\pi$ dimensional subspace W of $\mathbb{C}^{n\times n}$ such that $x^H Q(k)x < 0$ for every $x \in W \setminus \{0\}$.

Since $\lambda \mapsto x^H Q(\lambda)x$ is a parabola and $x^H Q(0)x > 0$ it follows at once $V \cap W = \{0\}$, i.e.

$$\dim(V+W) = n$$

Minmax characterization

Let

$$p_-^+(x) := \begin{cases} p_-(x) & \text{if } p_-(x) > 0 \\ \infty & \text{else} \end{cases} \qquad p_+^+(x) := \begin{cases} p_+(x) & \text{if } p_+(x) > 0 \\ 0 & \text{else} \end{cases}$$

Minmax characterization

Let

$$p_-^+(x) := \begin{cases} p_-(x) & \text{if } p_-(x) > 0 \\ \infty & \text{else} \end{cases} \qquad p_+^+(x) := \begin{cases} p_+(x) & \text{if } p_+(x) > 0 \\ 0 & \text{else} \end{cases}$$

If

$$\max\{p_-^+(x)\} < \min\{p_+^+(x)\}$$

then all eigenvalues in Δ_3 and Δ_4 are minmax and maxmin values of p_-^+ and p_+^+, respectively.

Minmax characterization

Let

$$p_-^+(x) := \begin{cases} p_-(x) & \text{if } p_-(x) > 0 \\ \infty & \text{else} \end{cases} \qquad p_+^+(x) := \begin{cases} p_+(x) & \text{if } p_+(x) > 0 \\ 0 & \text{else} \end{cases}$$

If

$$\max\{p_-^+(x)\} < \min\{p_+^+(x)\}$$

then all eigenvalues in Δ_3 and Δ_4 are minmax and maxmin values of p_-^+ and p_+^+, respectively.

If

$$\max\{p_-^-(x)\} < \min\{p_+^-(x)\}$$

then all eigenvalues in Δ_1 and Δ_2 are minmax and maxmin values of p_-^- and p_+^-, respectively.

From these variational characterizations of eigenvalues one obtains a Sylvester Theorem for gyroscopically stabilized quadratic eigenvalue problems in an obvious way.

Minmax for Gyroscopically Stabilized Pencils

Lemma

For $x \neq 0$ let $f(\lambda; x) := x^H Q(\lambda) x = 0$ and $\lambda \geq \sqrt{\lambda_{\max}(C)}$. Then it holds that $x^H Q'(\lambda) x \geq 0$.

Minmax for Gyroscopically Stabilized Pencils

Lemma
For $x \neq 0$ let $f(\lambda; x) := x^H Q(\lambda) x = 0$ and $\lambda \geq \sqrt{\lambda_{\max}(C)}$. Then it holds that $x^H Q'(\lambda) x \geq 0$.

Proof Similar to the nonoverdamped case.

Minmax for Gyroscopically Stabilized Pencils

Lemma
For $x \neq 0$ let $f(\lambda; x) := x^H Q(\lambda) x = 0$ and $\lambda \geq \sqrt{\lambda_{\max}(C)}$. Then it holds that $x^H Q'(\lambda) x \geq 0$.

Proof Similar to the nonoverdamped case.

In $(\sqrt{\lambda_{\max}(C)}, \infty)$ the conditions of the minmax characterization with Rayleigh functional p_+^+ are satisfied. Hence there are at most $n - \pi$ eigenvalues which are maxmin values of p_+^+. Hence, if $\sigma \geq \sqrt{\lambda_{\max}(C)}$ and in$Q(\sigma) = (\pi, \nu, 0)$, then there are exactly ν eigenvalues of $Q(\lambda)x = 0$ in (σ, ∞).

Minmax for Gyroscopically Stabilized Pencils

Lemma

For $x \neq 0$ let $f(\lambda; x) := x^H Q(\lambda)x = 0$ and $\lambda \geq \sqrt{\lambda_{\max}(C)}$. Then it holds that $x^H Q'(\lambda)x \geq 0$.

Proof Similar to the nonoverdamped case.

In $(\sqrt{\lambda_{\max}(C)}, \infty)$ the conditions of the minmax characterization with Rayleigh functional p_+^+ are satisfied. Hence there are at most $n - \pi$ eigenvalues which are maxmin values of p_+^+. Hence, if $\sigma \geq \sqrt{\lambda_{\max}(C)}$ and in$Q(\sigma) = (\pi, \nu, 0)$, then there are exactly ν eigenvalues of $Q(\lambda)x = 0$ in (σ, ∞).

Similar results hold for the intervals $(-\infty, -\sqrt{\lambda_{\max}(C)})$, $(-\sqrt{\lambda_{\min}(C)}, 0)$, and $(0, \sqrt{\lambda_{\min}(C)})$, and localization of eigenvalues via a Sylvester law is possible in these intervals as well.