Improving Eigenpairs from AMLS with Subspace Iteration

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Outline

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Automated Multi-Level Substructring

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Assuming that the interior degrees of freedom of substructures depend quasistatically on the interface degrees of freedom, and modeling the deviation from quasistatic dependence in terms of a small number of selected substructure eigenmodes the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest.

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Recent studies in vibro-acoustic analysis of passenger car bodies where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed have shown that AMLS is considerably faster than Lanczos type approaches.

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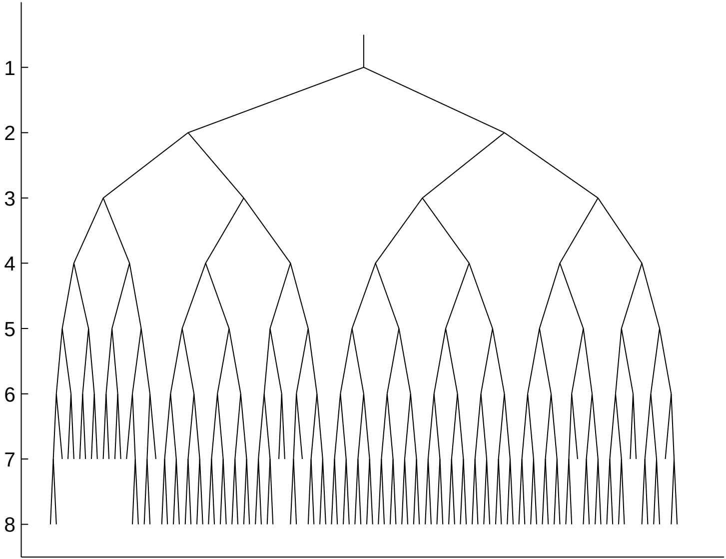
AMLS consists of two ingredients.

1. First, based on the substructuring the stiffness matrix K is transformed to block diagonal form by Gaussian elimination,
2. and secondly, the dimension is reduced substantially by modal condensation of the substructures.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Automated Multi-Level Substructring

If K_{ss} is a sub-matrix of K corresponding to a particular substructure, then after reordering rows and columns in (1) the pencil obtains the form

$$\left(\begin{bmatrix} K_{ss} & K_{sr} \\ K_{rs} & K_{rr} \end{bmatrix}, \begin{bmatrix} M_{ss} & M_{sr} \\ M_{rs} & M_{rr} \end{bmatrix} \right).$$

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With block Gaussian elimination, i.e. post- and premultiplying this pencil with

$$U_s = \begin{bmatrix} I & -K_{ss}^{-1} K_{sr} \\ O & I \end{bmatrix}$$

and U_s^T, respectively, K_{ss} is decoupled, and the pencil obtains the following form

$$(U_s^T K U_s, U_s^T M U_s) = \left(\begin{bmatrix} K_{ss} & O \\ O & \tilde{K}_{rr} \end{bmatrix}, \begin{bmatrix} M_{ss} & \tilde{M}_{sr} \\ \tilde{M}_{sr}^T & \tilde{M}_{rr} \end{bmatrix} \right).$$

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Repeating the block elimination for all substructures $1, \dots, m$ we get

$$\tilde{K} = U^T K U, \ \tilde{M} = U^T M U \text{ with } U = U_1 U_2 \dots U_m$$

where the transformed stiffness matrix $\tilde{K}$ has block diagonal form.

Automated Multi-Level Substructring

To reduce the dimension of the eigenproblem we determine for every substructure (after decoupling it from the remaining degrees of freedom in the stiffness matrix as above, and neglecting connections to other substructures in the mass matrix) all eigenvalues λ_{sj} not exceeding a cut off frequency λ_{cutoff} and corresponding eigenvectors z_{sj}, $j = 1, \dots, m_s$.

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Then with $Z_s = [z_{s1}, \dots, z_{sm_s}]$ and the global block diagonal projection matrix $Z = \text{diag}\{Z_1, \dots, Z_m\}$ we finally get the reduced eigenvalue problem

$$K_c x_c = \lambda M_c x_c \tag{2}$$

where $K_c = Z^T \tilde{K} Z = Z^T U^T K U Z$ is a diagonal matrix and $M_c = Z^T \tilde{M} Z = Z^T U^T M U Z$ has generalized block arrowhead form.

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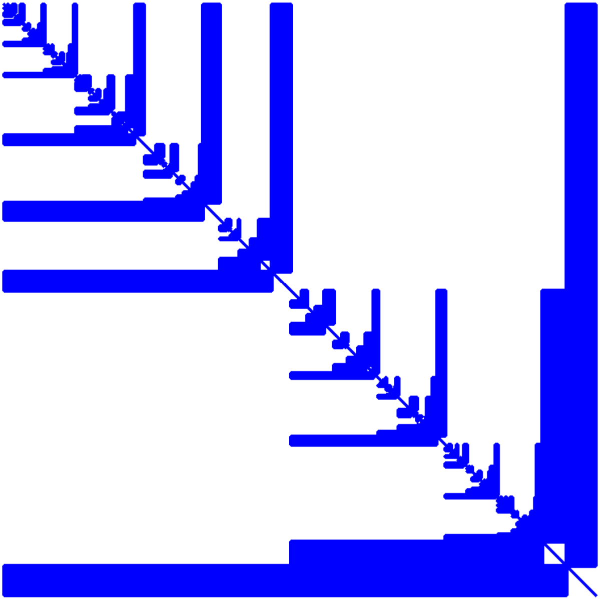
Important: In an implementation the block Gaussian eliminations and the condensations are performed in an interleaving way to avoid the storage of large dense sub-matrices of the transformed mass matrix which would occur in the course of the block elimination: as soon as a sub-matrix pencil $(\tilde{K}_{ss}, \tilde{M}_{ss})$ has been formed, the eigenproblem $\tilde{K}_{ss} Z_s = \tilde{M}_{ss} Z_s \Lambda_s$ is solved and the corresponding projection is executed.



Numerical example

Finite element model of a blade of a 1.5 MW wind turbine. 117990 DoF

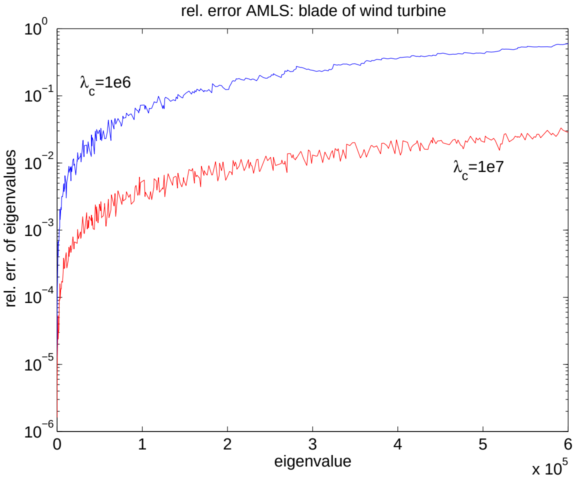


Figure: Blade of 1.5 MW wind turbine

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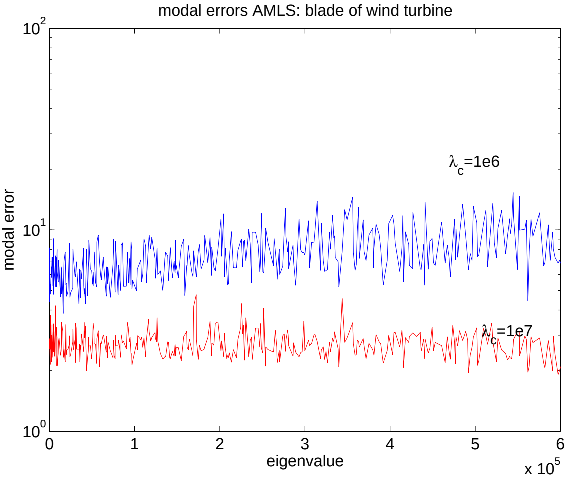

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However, in structural analysis its accuracy is not satisfactory because the precision of extracted eigenvector approximations are too low to meet the requirements in strain and stress computations.

In structural analysis, the modal errors are required to be as low as 10^{-3}. Otherwise, no sufficiently accurate strain or stress can be derived.

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Eigenvector approximation by AMLS

AMLS is a one shot projection method, i.e. after having chosen a cut-off frequency the method produces a fixed subspace $\mathcal{V} := \text{span}\{V\}$, $V := UZ$ and the corresponding projected eigenproblem.

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Differently from iterative projections methods such as Krylov subspace or Jacobi–Davidson methods there is no way to expand the subspace $\mathcal{V}$ further reusing the projected problem if the computed approximate eigenpairs turn out to be not accurate enough. One has to repeat the reduction with a higher cut-off frequency.

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Alternatively, one can improve the subspace $\mathcal{V}$ obtained with AMLS by subspace iteration.

Subspace iteration

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Then one step of subspace iteration requires to solve a linear system $(K - \sigma M)V_1 = MV_0$ for V_1 where σ is some shift close to the wanted eigenvalues. However, for huge matrices K and M a factorization of $K - \sigma M$ and a solution of this system is very costly.

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Alternatively, we may apply subspace iteration to the transformed problem

$$\tilde{K}z := U^T KUz = \lambda U^T MUz =: \lambda\tilde{M}z,$$

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Due to the interleaving implementation of AMLS the matrices $\tilde{K}$ and $\tilde{M}$ are usually not stored when computing the reduced model, but in principle this could be easily done. The matrix $\tilde{K}$ then obtains block diagonal form with moderate block sizes, but owing to fill in during the elimination process $\tilde{M}$ will contain many dense sub-matrices requiring a huge amount of storage. So, this approach is also not efficient.

Subspace iteration

The way out is to combine the benefits of both approaches, i.e. to apply subspace iteration to the transformed system, but to evaluate $\tilde{M}\tilde{V}$ taking advantage of the transformation matrix U and the sparse structure of the original mass matrix M.

Subspace iteration with AMLS

Require: Transformed eigenvectors $\tilde{V}$, the transformed stiffness matrix $\tilde{K}$, and the transformation matrix U from AMLS

initialize the iteration matrices $\tilde{Q}^{(0)} = \tilde{V}$

for $k = 1, 2, \dots, n_k$ **do**

  transform backward $Q^{(k-1)} = U\tilde{Q}^{(k-1)}$

  compute $R = MQ^{(k-1)}$

  transform forward $\tilde{R} = U^T R$

  solve for $\tilde{Q}^{(k)}$: $\tilde{K}\tilde{Q}^{(k)} = \tilde{R}$

end for

project transformed stiffness matrix $\tilde{K}_c = \tilde{R}^T \tilde{Q}^{(n_k)}$

transform backward $Q^{(n_k)} = U\tilde{Q}^{(n_k)}$

project transformed mass matrix $\tilde{M}_c = (Q^{(n_k)})^T MQ^{(n_k)}$

solve projected problem $\tilde{K}_c X_c = \tilde{M}_c X_c \Lambda$

compute eigenvector approximations $V^{(n_k)} = Q^{(n_k)} X_c$.

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Table: Computation time of NormalSIM and AMLS-SIM

AMLS reduction	100.2s
eigenvectors	7.6s
AMLS-SIM	65.6s
normal SIM	167.0s

Blade of 1.5 MW wind turbine

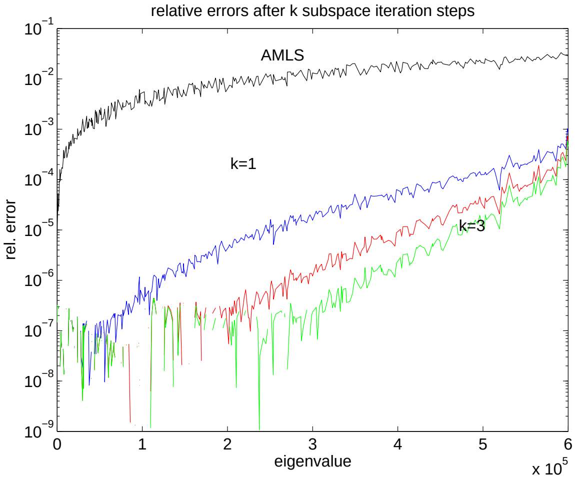

Figure: Relative errors of eigenvalue approximations

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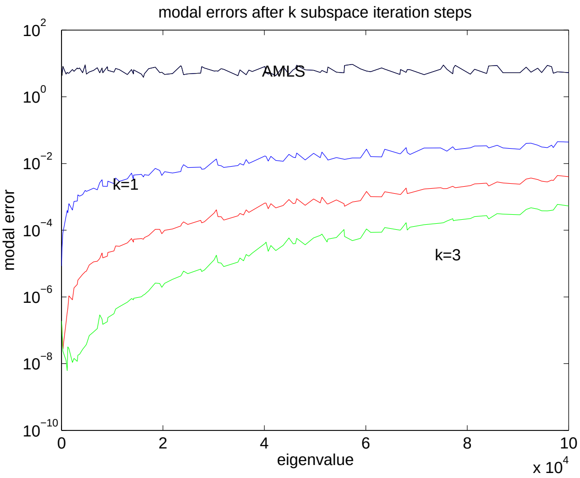

Figure: Modal errors

Gyroscopic eigenvalue problem

Consider the gyroscopic eigenproblem

$$Kx + i\omega Gx - \omega^2 Mx = 0$$

with $K = K^T > 0$, $M = M^T > 0$, and $G = -G^T$ describing the eigenproblem of a rotating structure.

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Hermitian linearization

$$\begin{bmatrix} iG & K \\ K & O \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \omega \begin{bmatrix} M & O \\ O & K \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}, \quad y = \omega x$$

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AMLS is applied to the linear eigenvalue problem

$$Kx = \lambda Mx$$

and then the gyroscopic problem is projected to the space spanned by its eigenvectors corresponding to eigenvalues of interest.

Example

Consider a rotating tire the FE model of which consists of 39204 brick elements with 124992 degrees of freedom and accounts for 20 different material groups. The speed is assumed to be 60 km/h. Our aim is to determine approximations to the smallest 200 eigenvalues and corresponding eigenvectors.

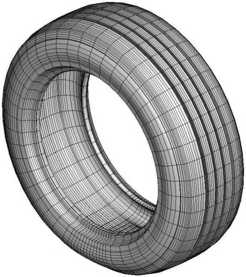

Figure: Continental AG 205/55R16-91H tire

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The AMLS method addressing the linear eigenvalue problem $Kx = \lambda Mx$ costs 881.2 seconds for the AMLS projection and 270.4 seconds for solving the projected linear eigenvalue problem of dimension 2263 by `eig`.

Numerical example

Table: Computation time of NormalSIM and AMLS-SIM with 208 iteration vectors

Computational Steps	NormalSIM(s)	AMLS-SIM(s)
Compute initial eigenvectors from AMLS	9.2	2.4
1 Iteration	2401.0	67.8
2 Iterations	Not computed	141.3
3 Iterations	Not computed	216.8
4 Iterations	Not computed	289.4
Compute $\hat{K}$ and $\hat{M}$ if $n_k = 1$	10.0	6.2
Compute $\hat{K}$ and $\hat{M}$ if $n_k > 1$	Not computed	8.7
Solve $\hat{K}Z = \hat{M}V\hat{\Lambda}$ by `eig`	2.6	2.6
Compute final eigenvectors	0.9	25.6

Numerical example

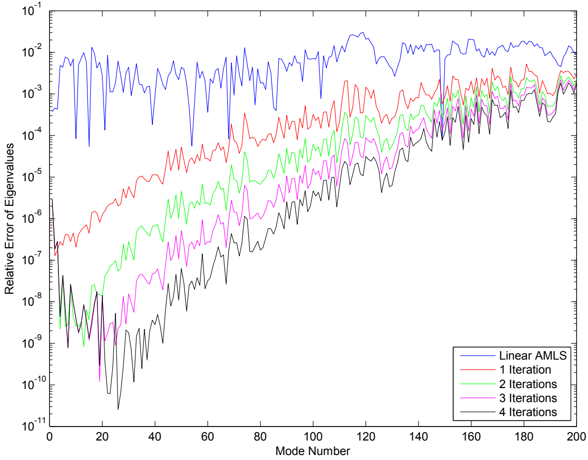

Figure: Relative errors of eigenvalues computed with subspace iteration with AMLS

Numerical example

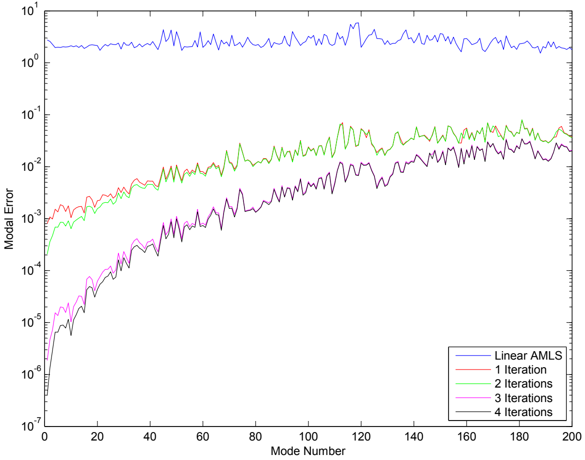

Figure: Modal errors computed with subspace iteration with AMLS

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Combining AMLS with subspace iteration taking advantage of the block structure of the transformed stiffness matrix, but avoiding the use of the highly populated transformed mass matrix, the accuracy can be improved efficiently.