Variational Principles for Nonlinear Eigenvalue Problems

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On the one hand: Parameter dependent nonlinear (with respect to the state variable $u$) operator equations

$$T(\lambda, u) = 0$$

are discussed concerning

— positivity of solutions
— multiplicity of solution
— dependence of solutions on the parameter; bifurcation
— (change of ) stability of solutions
For $\lambda \in D \subset \mathbb{C}$ let $T(\lambda)$ be a linear self-adjoint and bounded operator on a Hilbert space $\mathcal{H}$.
In this presentation

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Find $\lambda \in D$ and $x \neq 0$ such that

$$T(\lambda)x = 0. \quad (1)$$

Then $\lambda$ is called an eigenvalue of $T(\cdot)$, and $x$ a corresponding eigenelement.
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- vibration of fluid-solid structures
- vibration of sandwich plates
- accelerator design
- vibro-acoustics of piezoelectric/poroelastic structures
- nonlinear integrated optics
- regularization of total least squares problems
- stability of delay differential equations
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For \( \lambda \in D \subset \mathbb{C} \) let \( T(\lambda) \) be a linear self-adjoint and bounded operator on a Hilbert space \( \mathcal{H} \). Find \( \lambda \in D \) and \( x \neq 0 \) such that

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- stability of delay differential equations
Outline

1. Two examples

2. Variational characterization of eigenvalue problems
   - Overdamped Problems
   - Nonoverdamped problems

3. An unsymmetric linear eigenproblem

4. Concluding remarks
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The ultimate limit of low dimensional structures is the quantum dot, in which the carriers are confined in all three directions, thus reducing the degrees of freedom to zero. Therefore, a quantum dot can be thought of as an artificial atom.
Problem

Determine relevant energy states (i.e. eigenvalues) and corresponding wave functions (i.e. eigenfunctions) of a three-dimensional quantum dot embedded in a matrix.
Two examples

Problem ct.

Governing equation: *Schrödinger equation*

\[-\nabla \cdot \left( \frac{\hbar^2}{2m(x,E)} \nabla \Phi \right) + V(x)\Phi = E\Phi, \quad x \in \Omega_q \cup \Omega_m\]

where \(\hbar\) is the reduced Planck constant, \(m(x,E)\) is the electron effective mass, and \(V(x)\) is the confinement potential.
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**Boundary and interface conditions**

$$\Phi = 0 \quad \text{on outer boundary of matrix } \Omega_m$$

BenDaniel–Duke condition

$$\left. \frac{1}{m} \frac{\partial \Phi}{\partial n} \right|_{\partial \Omega_m} = \left. \frac{1}{m} \frac{\partial \Phi}{\partial n} \right|_{\partial \Omega_q} \quad \text{on interface}$$
Find $E \in \mathbb{R}$ and $\Phi \in H_0^1(\Omega)$, $\Phi \neq 0$, $\Omega := \overline{\Omega}_q \cup \Omega_m$, such that

$$a(\Phi, \psi; E) := \frac{\hbar^2}{2} \int_{\Omega_q} \frac{1}{m_q(x, E)} \nabla \Phi \cdot \nabla \psi \, dx + \frac{\hbar^2}{2} \int_{\Omega_m} \frac{1}{m_m(x, E)} \nabla \Phi \cdot \nabla \psi \, dx$$

$$+ \int_{\Omega_q} V_q(x) \Phi \psi \, dx + \int_{\Omega_m} V_m(x) \Phi \psi \, dx$$

$$= E \int_{\Omega} \Phi \psi \, dx =: Eb(\Phi, \psi) \quad \text{for every } \psi \in H_0^1(\Omega)$$
The dependence of $m(x, E)$ on $E$ can be derived from the eight-band $k \cdot p$ analysis and effective mass theory. Projecting the $8 \times 8$ Hamiltonian onto the conduction band results in the single Hamiltonian eigenvalue problem with

$$m(x, E) = \begin{cases} m_q(E), & x \in \Omega_q \\ m_m(E), & x \in \Omega_m \end{cases}$$

$$\frac{1}{m_j(E)} = \frac{P_j^2}{\hbar^2} \left( \frac{2}{E + g_j - V_j} + \frac{1}{E + g_j - V_j + \delta_j} \right), \quad j \in \{m, q\}$$

where $m_j$ is the electron effective mass, $V_j$ the confinement potential, $P_j$ the momentum, $g_j$ the main energy gap, and $\delta_j$ the spin-orbit splitting in the $j$th region.
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Other types of effective mass (taking into account the effect of strain, e.g.) appear in the literature. They are all rational functions of $E$ where $1/m(x, E)$ is monotonically decreasing with respect to $E$, and that’s all we need.
Two examples

Properties

- \( a(\cdot, \cdot, E) \) bilinear, symmetric, bounded, \( H^1_0(\Omega) \)-elliptic for \( E \geq 0 \)
- \( b(\cdot, \cdot) \) bilinear, positive definite, bounded, completely continuous

By the Lax–Milgram lemma the variational eigenproblem is equivalent to

\[
T(E)\Phi = 0
\]

where

\[
T(E) : H^1_0(\Omega) \to H^1_0(\Omega), \quad E \geq 0,
\]

is a family of self-adjoint and bounded operators.
Example 2: Free vibrations of fluid-solid structures can be modelled in terms of solid displacement and fluid pressure and one obtains the classical form of an eigenproblem

$$
\text{div } [\sigma(u)] + \omega^2 \rho_s u = 0 \text{ in } \Omega_s,
$$

$$
\Delta p + \frac{\omega^2}{c^2} p = 0 \text{ in } \Omega_f,
$$

$$
\sigma(u) \cdot n - pn = 0 \text{ on } \Gamma_I,
$$

$$
\nabla p \cdot n + \omega^2 \rho_f u \cdot n = 0 \text{ on } \Gamma_I,
$$

$$
u = 0 \text{ on } \Gamma_D,
$$

$$
\nabla p \cdot n = 0 \text{ on } \Gamma_N,
$$

- $u$: solid displacement
- $p$: fluid pressure
- $\lambda = \omega^2$: eigenparameter
- $\sigma(u)$: linearized stress tensor
- $\rho_s, \rho_f$: densities of solid and fluid

Interface conditions: equilibrium of accelerations and of force densities.
Find $\lambda := \omega^2 \in \mathbb{C}$ and $(u, p) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f)$ such that

$$a_s(v, u) + c(v, p) = \lambda b_s(v, u)$$
$$a_f(q, p) = \lambda (-c(u, q) + b_f(q, p)).$$

for every $(v, q) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f)$. 
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$$ a_s(v, u) + c(v, p) = \lambda b_s(v, u) \quad \text{and} \quad a_f(q, p) = \lambda(-c(u, q) + b_f(q, p)). $$

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which (using the Lax-Milgram Lemma) can be transformed into a linear (but not self-adjoint) eigenvalue problem

$$ K_S u + C p = \lambda M_S u \quad (2a) $$
$$ K_f p = \lambda(-C' u + M_f p) \quad (2b) $$

where $K_S : H^1_{\Gamma_D}(\Omega_s)^3 \to H^1_{\Gamma_D}(\Omega_s)^3$ is self-adjoint, elliptic, bounded, ...
Let $0 < \sigma_1 \leq \sigma_2 \leq \ldots$ denote the eigenvalues of the decoupled eigenproblem

$$K_S u = \sigma M_S u$$

and denote by $u_1, u_2, \ldots$ corresponding orthonormal eigenfunctions. Then the spectral theorem yields

$$(K_S - \lambda M_S)^{-1} u = \sum_{n=1}^{\infty} \frac{1}{\sigma_n - \lambda} \langle u_n, u \rangle u_n.$$
Rational form of fluid-solid eigenproblem

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If \( \lambda \) is not contained in the spectrum of the decoupled solid eigenproblem, then \( \lambda \) is an eigenvalue of the coupled fluid-solid problem if and only if it is an eigenvalue of the rational eigenvalue problem

\[
T(\lambda)p := -K_f p + \lambda M_f p + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} C_n p, \quad C_n p := \langle u_n, C p \rangle C' u_n.
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Rational form of fluid-solid eigenproblem

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If $\lambda$ is not contained in the spectrum of the decoupled solid eigenproblem, then $\lambda$ is an eigenvalue of the coupled fluid-solid problem if and only if it is an eigenvalue of the rational eigenvalue problem

$$T(\lambda) p := -K_f p + \lambda M_f p + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} C_n p, \quad C_n p := \langle u_n, C p \rangle C' u_n.$$ 

$T(\lambda) : H^1(\Omega_f) \to H^1(\Omega_f)$ is self-adjoint and bounded.
Another self-adjoint form of the fluid-solid eigenproblem is obtained if the second equation in (2) is multiplied by $\omega$ and $p$ is substituted by $p := \omega w$. 
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Then problem (2) is equivalent to the quadratic eigenvalue problem

$$
\begin{pmatrix}
K_s & 0 \\
0 & K_f
\end{pmatrix}
+ \omega
\begin{pmatrix}
O & C \\
C' & O
\end{pmatrix}
- \omega^2
\begin{pmatrix}
M_s & O \\
O & M_f
\end{pmatrix}
\begin{pmatrix}
u \\
w
\end{pmatrix} = 0.
$$
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   - Overdamped Problems
   - Nonoverdamped problems

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4. Concluding remarks
Let $A : \mathcal{H} \to \mathcal{H}$ a bounded linear and self-adjoint operator in a Hilbert space $\mathcal{H}$. Then those eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$ above the essential spectrum of $A$ (if there are any) can be characterized by three fundamental variational principles,

- Rayleigh's principle:
  \[ \lambda_n = \max \left\{ \frac{\langle Ax, x \rangle}{\langle x, x \rangle} : \langle x, x_i \rangle = 0, i = 1, \ldots, n-1 \right\}, \]

- the maxmin characterization due to Poincaré:
  \[ \lambda_n = \max \dim V = n \min_{x \in V, x \neq 0} R(x), \]

- the minmax characterization due to Courant, Fischer, and Weyl:
  \[ \lambda_n = \min \dim V = n-1 \max_{x \in V^\perp, x \neq 0} R(x). \]

Variational characterizations are very powerful tools when studying self-adjoint linear operators on a Hilbert space $\mathcal{H}$. Bounds for eigenvalues, comparison theorems, interlacing results and monotonicity of eigenvalues can be proved easily with these characterizations, to name just a few.
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Variational characterizations are very powerful tools when studying self-adjoint linear operators on a Hilbert space $\mathcal{H}$. Bounds for eigenvalues, comparison theorems, interlacing results and monotonicity of eigenvalues can be proved easily with these characterizations, to name just a few.
Rayleigh functional

Let

\[ f : \begin{cases} J \times \mathcal{H} & \to \mathbb{R} \\ (\lambda, x) & \mapsto \langle T(\lambda)x, x \rangle \end{cases} \]

be continuous, and assume that for every fixed \( x \in \mathcal{H}, x \neq 0 \), the real equation

\[ f(\lambda, x) = 0 \tag{3} \]

has at most one solution in \( J \).
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has at most one solution in \( \mathcal{J} \).

Then equation (3) implicitly defines a functional \( p \) on some subset \( \mathcal{D} \) of \( \mathcal{H} \setminus \{0\} \) which we call the Rayleigh functional, and which is exactly the Rayleigh quotient in case of a linear eigenproblem \( T(\lambda) = \lambda I - A \).
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We further assume that for every \( x \in \mathcal{D}, \ x \neq 0 \) and \( \lambda \in J, \ \lambda \neq p(x) \) it holds that

\[ f(\lambda, x)(\lambda - p(x)) > 0 \]

which generalizes the definiteness of the operator \( B \) for the generalized linear eigenproblem \( T(\lambda) := \lambda B - A \).
If the Rayleigh functional \( p \) is defined on the entire space \( \mathcal{H} \setminus \{0\} \) then the eigenproblem \( T(\lambda)x = 0 \) is called **overdamped**.
Overdamped problems

If the Rayleigh functional $p$ is defined on the entire space $\mathcal{H} \setminus \{0\}$ then the eigenproblem $T(\lambda)x = 0$ is called **overdamped**.

This notation is motivated by the finite dimensional quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda \alpha Cx + Kx = 0$$

where $M$, $C$ and $K$ are symmetric and positive definite matrices.
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- $\alpha = 0$: all eigenvalues on imaginary axis
- increase $\alpha$: eigenvalues go into left half plane as conjugate complex pairs
- all eigenvalues going to the left are smaller than all eigenvalues going to the right: system is overdamped
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- increase $\alpha$ all eigenvalues on the negative real axis
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where $M$, $C$ and $K$ are symmetric and positive definite matrices.

- $\alpha = 0$ all eigenvalues on imaginary axis
- increase $\alpha$ eigenvalues go into left half plane as conjugate complex pairs
- increase $\alpha$ complex pairs reach real axis, run in opposite directions
- increase $\alpha$ all eigenvalues on the negative real axis
- increase $\alpha$ all eigenvalues going to the left are smaller than all eigenvalues going to the right
- system is overdamped
For quadratic overdamped systems the two solutions

\[ p_{\pm}(x) = \frac{1}{2} \left( -\alpha \langle Cx, x \rangle \pm \sqrt{\alpha^2 \langle Cx, x \rangle^2 - 4 \langle Mx, x \rangle \langle Kx, x \rangle} \right) / \langle Mx, x \rangle. \]

of the quadratic equation

\[ \langle T(\lambda) x, x \rangle = \lambda^2 \langle Mx, x \rangle + \lambda \alpha \langle Cx, x \rangle + \langle Kx, x \rangle = 0 \quad (4) \]

are real, and they satisfy \( \sup_{x \neq 0} p_-(x) < \inf_{x \neq 0} p_+(x) \).
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are real, and they satisfy \( \sup_{x \neq 0} p_-(x) < \inf_{x \neq 0} p_+(x) \).

Hence, equation (4) defines two Rayleigh functionals \( p_- \) and \( p_+ \) corresponding to the intervals

\[ J_- := (-\infty, \inf_{x \neq 0} p_+(x)) \quad \text{and} \quad J_+ := (\sup_{x \neq 0} p_-(x), \infty). \]
Rayleigh’s principle

For general (not necessarily quadratic) overdamped problems Hadeler (1967 for the finite dimensional case, and 1968 for dim $\mathcal{H} = \infty$) generalized Rayleigh’s principle proving that the eigenvectors are orthogonal with respect to the generalized scalar product

$$[x, y] := \begin{cases} 
\langle \frac{T(p(x)) - T(p(y))}{p(x) - p(y)} x, y \rangle, & \text{if } p(x) \neq p(y) \\
\langle T'(p(x))x, y \rangle, & \text{if } p(x) = p(y)
\end{cases}$$

which is symmetric, definite and homogeneous, but in general is not bilinear.
Rayleigh’s principle (Hadeler 1967, 1968)

Let $T(\lambda) : \mathcal{H} \rightarrow \mathcal{H}$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators such that (1) is over-damped, and assume that for $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact.
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Let $T(\lambda) : \mathcal{H} \rightarrow \mathcal{H}$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators such that (1) is over-damped, and assume that for $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda) I$ is compact. Let $T(\cdot)$ be continuously differentiable and suppose that

$$\frac{\partial}{\partial \lambda} \left( \langle T(\lambda)x, x \rangle \right) \bigg|_{\lambda = \rho(x)} > 0 \quad \text{for ever } x \neq 0.$$
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$$\left. \frac{\partial}{\partial \lambda} \langle T(\lambda)x, x \rangle \right|_{\lambda=\rho(x)} > 0 \quad \text{for ever } x \neq 0.$$

Then problem $T(\lambda)x = 0$ has at most a countable set of eigenvalues in $J$ which we assume to be ordered by magnitude $\lambda_1 \leq \lambda_2 \leq \ldots$, where each eigenvalue is counted according to its multiplicity.
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Let $T(\lambda) : \mathcal{H} \to \mathcal{H}$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators such that (1) is over-damped, and assume that for $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact.

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The corresponding eigenvectors $x_1, x_2, \ldots$ can be chosen orthonormally with respect to the generalized scalar product (5), and the eigenvalues can be determined recurrently by

$$\lambda_n = \min \{ \rho(x) : [x, x_i] = 0, \ i = 1, \ldots, n-1, \ x \neq 0 \}.$$
Poincaré’s maxmin characterization was first generalized by Duffin (1955) to overdamped quadratic eigenproblems of finite dimension, and for more general overdamped problems of finite dimension it was proved by Rogers (1964).
Minmax principle

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Infinite dimensional eigenvalue problems were studied by Turner (1967), Langer (1968), and Weinberger (1969) who proved generalizations of both, the maxmin characterization of Poincaré and of the minmax characterization of Courant, Fischer and Weyl for quadratic (and by Turner (1968) for polynomial) overdamped problems.
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The corresponding generalizations for general overdamped problems of infinite dimension were derived by Hadeler (1968). Similar results (weakening the compactness or smoothness requirements) are contained in Rogers (1968), Werner (1971), Abramov (1973), Hadeler (1975), Markus (1985), Maksudov & Gasanov (1992), and Hasanov (2002).
Minmax and Maxmin principle (Haderler 1968)

Let $T(\lambda) : \mathcal{H} \to \mathcal{H}$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators such that (1) is over-damped, and assume that for $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact.
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Let $T(\cdot)$ be continuously differentiable and suppose that

$$\left. \frac{\partial}{\partial \lambda} \langle T(\lambda)x, x \rangle \right|_{\lambda = \rho(x)} > 0 \quad \text{for ever } x \neq 0.$$

Let the eigenvalues $\lambda_n$ of $T(\lambda)x = 0$ be numbered in non-decreasing order regarding their multiplicities. Then they can be characterized by the following two variational principles

$$\lambda_n = \min_{\dim V = n} \max_{x \in V, x \neq 0} p(x)$$

$$= \max_{\dim V = n-1} \min_{x \in V^\perp, x \neq 0} p(x).$$
Example 1: Quantum dot problem

For the quantum dot problem the family of operators $T(\lambda)$ satisfies the general conditions of the variational characterizations.
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For the quantum dot problem the family of operators \( T(\lambda) \) satisfies the general conditions of the variational characterizations.

For \( x \neq 0 \) it holds that \( a(x, x, 0) > 0, b(x, x) > 0 \) and \( \lambda \mapsto a(x, x, \lambda) \) is monotonically decreasing for \( \lambda \geq 0 \). Hence,

\[
 f(\lambda, x) = \langle T(\lambda)x, x \rangle = \lambda b(x, x) - a(x, x, \lambda) = 0
\]

has exactly one positive solution \( p(x) \), and

\[
 \frac{\partial}{\partial \lambda} f(\lambda, x) \bigg|_{\lambda=p(x)} > 0.
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Example 1: Quantum dot problem

For the quantum dot problem the family of operators $T(\lambda)$ satisfies the general conditions of the variational characterizations.

For $x \neq 0$ it holds that $a(x, x, 0) > 0$, $b(x, x) > 0$ and $\lambda \mapsto a(x, x, \lambda)$ is monotonically decreasing for $\lambda \geq 0$. Hence,

$$f(\lambda, x) = \langle T(\lambda)x, x \rangle = \lambda b(x, x) - a(x, x, \lambda) = 0$$

has exactly one positive solution $p(x)$, and

$$\frac{\partial}{\partial \lambda} f(\lambda, x) \bigg|_{\lambda=p(x)} > 0.$$

Thus, the quantum dot problem has a countable number of non-negative eigenvalues which allow for all three variational characterizations.
Example 2: Fluid-solid vibration

For the rational eigenproblem governing free vibrations the family of operators $T(\lambda)$ satisfies the general conditions of the variational characterizations in every interval $J_n := (\sigma_{n-1}, \sigma_n)$. 
Example 2: Fluid-solid vibration

For the rational eigenproblem governing free vibrations the family of operators $T(\lambda)$ satisfies the general conditions of the variational characterizations in every interval $J_n := (\sigma_{n-1}, \sigma_n)$.

$$f(\lambda, p) := \langle T(\lambda)p, p \rangle = -\langle K_f p, p \rangle + \lambda \langle M_f p, p \rangle + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} |\langle u_n, C p \rangle|^2$$

is monotonically increasing, such that $f(\lambda, p) = 0$ has at most one solution in $J_n$, but the Rayleigh functional is not defined on the entire space.
Outline

1. Two examples

2. Variational characterization of eigenvalue problems
   - Overdamped Problems
   - Nonoverdamped problems

3. An unsymmetric linear eigenproblem

4. Concluding remarks
For nonoverdamped eigenproblems (i.e. $\mathcal{D}(p) \neq \mathcal{H} \setminus \{0\}$) the natural ordering to call the smallest eigenvalue the first one, the second smallest the second one, etc., is not appropriate.
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This is obvious if we make a linear eigenvalue

$$T(\lambda)x := (\lambda I - A)x = 0$$

nonlinear by restricting it to an interval $J$ which does not contain the smallest eigenvalue of $A$. 
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nonlinear by restricting it to an interval $J$ which does not contain the smallest eigenvalue of $A$.

Then all conditions are satisfied, $p$ is the restriction of the Rayleigh quotient $R_A$ to $\mathcal{D}(p) := \{ x \neq 0 : R_A(x) \in J \}$, and $\inf_{x \in \mathcal{D}(p)} p(x)$ will not be an eigenvalue.
Enumeration of eigenvalues

\( \lambda \in J \) is an eigenvalue of \( T(\cdot) \) if and only if \( \mu = 0 \) is an eigenvalue of the linear problem \( T(\lambda)y = \mu y \). The key idea is to orientate the number of \( \lambda \) on the location on the eigenvalue \( \mu = 0 \) in the spectrum of the linear operator \( T(\lambda) \).
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We assume that for every \( \lambda \in J \) it holds that the supremum of the essential spectrum of \( T(\lambda) \) is negative (for instance: there exists \( \nu(\lambda) > 0 \) such that \( T(\lambda) + \nu(\lambda)I \) is compact).
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If \( \lambda \in J \) is an eigenvalue of \( T(\cdot) \) then there exists \( n \in \mathbb{N} \) such that

\[
0 = \max_{\dim V = n} \min_{x \in V, x \neq 0} \frac{\langle T(\lambda)x, x \rangle}{\langle x, x \rangle}.
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\]

In this case we assign \( n \) to the eigenvalue \( \lambda \) of problem \( T(\lambda)x = 0 \) as its number and call \( \lambda \) an \( n \)-th eigenvalue of \( T(\cdot) \).
Minmax characterization (V.&B.Werner 1982, V. 2010)

Let $T(\lambda), \lambda \in J$ be a family of linear self-adjoint and bounded operators on a Hilbert space $\mathcal{H}$ depending continuously on a parameter $\lambda \in J$ where $J$ is an open real (not necessarily bounded) interval.
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Assume that

- for $x \neq 0$ $f(\lambda, x) := \langle T(\lambda)x, x \rangle = 0$ has at most one solution $p(x) \in J$, and let $\mathcal{D}$ be the domain of the Rayleigh functional $p$,
- $(\lambda - p(x))f(\lambda, x) > 0$ for every $x \in \mathcal{D}$ and $\lambda \neq p(x)$,
- and that the supremum of the essential spectrum of $T(\lambda)$ is negative for every $\lambda \in J$. 

Then the nonlinear eigenvalue problem $T(\lambda)x = 0$ has at most a countable set of eigenvalues in $J$, and it holds that:

If $\lambda_n \in J$ is an $n$-th eigenvalue then 
$$\lambda_n = \min \{ \dim V = n, V \cap \mathcal{D} = \emptyset \} \sup_{x \in \mathcal{D} \cap V} p(x).$$ (6)

If conversely $\lambda_n = \inf \{ \dim V = n, V \cap \mathcal{D} = \emptyset \} \sup_{x \in \mathcal{D} \cap V} p(x)$ then $\lambda_n$ is an $n$-th eigenvalue of $T(\lambda)$ and (6) holds.
Let $T(\lambda)$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators on a Hilbert space $\mathcal{H}$ depending continuously on a parameter $\lambda \in J$ where $J$ is an open real (not necessarily bounded) interval.

Assume that
- for $x \neq 0 \ f(\lambda, x) := \langle T(\lambda)x, x \rangle = 0$ has at most one solution $p(x) \in J$, and let $D$ be the domain of the Rayleigh functional $p$,
- $(\lambda - p(x))f(\lambda, x) > 0$ for every $x \in D$ and $\lambda \neq p(x)$,
- and that the supremum of the essential spectrum of $T(\lambda)$ is negative for every $\lambda \in J$.

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If conversely

$$\lambda_n = \inf_{\dim V = n, \ V \cap \mathcal{D} \neq \emptyset} \sup_{x \in \mathcal{D} \cap V} p(x) \in J$$

then $\lambda_n$ is an $n$-th eigenvalue of $T(\lambda)x = 0$ and (6) holds.
Sketch of proof

Step 1 (technical): Let $\lambda \in J$, and assume that $V$ is a finite dimensional subspace of $\mathcal{H}$ such that $V \cap D \neq \emptyset$. Then it holds that

$$
\lambda \left\{ \begin{array}{l}
< \\
= \\
> 
\end{array} \right\} \sup_{x \in V \cap D(p)} p(x) \iff \min_{x \in V} \langle T(\lambda)x, x \rangle \left\{ \begin{array}{l}
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\[
\lambda \begin{cases} < \\ = \\ > \end{cases} \quad \sup_{x \in V \cap D(\rho)} p(x) \quad \Leftrightarrow \quad \min_{x \in V} \langle T(\lambda)x, x \rangle \begin{cases} < \\ = \\ > \end{cases} 0 \quad (7)
\]

Step 2: If \( \lambda_n \) is an \( n \)-th eigenvalue, then \( \mu_n(\lambda_n) = 0 \), and

\[
\mu_n(\lambda_n) = \max_{\text{dim } V = n} \min_{x \in V, \|x\| = 1} \langle T(\lambda_n)x, x \rangle = \min_{x \in \overline{V}, \|x\| = 1} \langle T(\lambda_n)x, x \rangle.
\]
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Hence, $\min_{x \in V, \|x\|=1} \langle T(\lambda_n)x, x \rangle \leq 0$ for every $V$ with $\dim V = n$, and (7) implies

$$\sup_{x \in V \cap D} p(x) \geq \lambda_n = \sup_{x \in \bar{V} \cap D} p(x).$$

Hence, $\lambda_n$ is a minmax value of $p$. 

Theorem (minmax for extreme eigenvalues)

Assume that the conditions of the minmax characterization hold and that

\[ \inf_{x \in D} p(x) \in J. \]
Theorem (minmax for extreme eigenvalues)

Assume that the conditions of the minmax characterization hold and that

$$\inf_{x \in \mathcal{D}} p(x) \in J.$$ 

If \( \lambda_n \in J \) for some \( n \in \mathbb{N} \) then every \( V \in H_j \) with \( V \cap \mathcal{D}(p) \neq \emptyset \) and \( \lambda_j = \sup_{x \in V \cap \mathcal{D}(p)} p(x) \) is contained in \( \mathcal{D} \cup \{0\} \), and the characterization (6) can be replaced by

$$\lambda_j = \min_{\dim V = j} \max_{\substack{v \in V, \ x \neq 0 \ \forall V \subset \mathcal{D} \cup \{0\}}} p(v), \quad j = 1, \ldots, n. \quad (8)$$
Maxmin characterization (V. 2003)

Assume that the conditions of the minmax characterization are satisfied.
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If there is an $n$-th eigenvalue $\lambda_n \in J$ of $T(\lambda)x = 0$, then

$$\lambda_n = \max_{\substack{V \in H_{n-1} \setminus \mathcal{D} \\ V\perp \cap \mathcal{D} \neq \emptyset}} \inf_{v \in V \cap \mathcal{D}} p(v),$$

and the maximum is attained by $W := \text{span}\{u_1, \ldots, u_{n-1}\}$ where $u_j$ denotes an eigenvector corresponding to the $j$-largest eigenvalue $\mu_j(\lambda_n)$ of $T(\lambda_n)$. 
Maxmin characterization (V. 2003)

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If there is an $n$-th eigenvalue $\lambda_n \in J$ of $T(\lambda)x = 0$, then

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\lambda_n = \max_{\substack{V \in H_{n-1} \\
v \in V^\perp \cap \mathcal{D} \neq \emptyset}} \inf_{\substack{V \in H_{n-1} \\
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$$

and the maximum is attained by $W := \text{span}\{u_1, \ldots, u_{n-1}\}$ where $u_j$ denotes an eigenvector corresponding to the $j$-largest eigenvalue $\mu_j(\lambda_n)$ of $T(\lambda_n)$.

Proof takes advantage of the following Lemma:
Let $\lambda \in J$, and let $V$ be a finite dimensional subspace of $\mathcal{H}$ such that $V^\perp \cap \mathcal{D} \neq \emptyset$. Then it holds that

$$
\lambda \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \inf_{x \in V^\perp \cap \mathcal{D}(p)} p(x) \iff \max_{x \in V^\perp, \|x\|=1} \langle T(\lambda)x, x \rangle \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 0
$$
Under the conditions of the minmax characterization assume that $J$ contains $n \geq 1$ eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ (where $\lambda_i$ is an $i$th eigenvalue) with corresponding $[\cdot, \cdot]$ orthogonal eigenvectors $x_1, \ldots, x_n$.

If there exists $x \in D$ with $[x_i, x] = 0$ for $i = 1, \ldots, n$ then $J$ contains an $(n+1)$th eigenvalue, and

$$
\lambda_{n+1} = \inf \{ p(x) : [x_j, x] = 0, \ i = 1, \ldots, n \}.
$$

(9)
Rayleigh’s principle for nonoverdamped problems

Under the conditions of the minmax characterization assume that $J$ contains $n \geq 1$ eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ (where $\lambda_i$ is an $i$th eigenvalue) with corresponding $[\cdot, \cdot]$ orthogonal eigenvectors $x_1, \ldots, x_n$. If there exists $x \in D$ with $[x_i, x] = 0$ for $i = 1, \ldots, n$ then $J$ contains an $(n + 1)$th eigenvalue, and

$$
\lambda_{n+1} = \inf \{ p(x) : [x_j, x] = 0, \ i = 1, \ldots, n \}.
$$

The generalized scalar product (5) has to be modified according to

$$
[x, y] := \begin{cases} 
\left\langle \frac{T(p(x)) - T(p(y))}{p(x) - p(y)} x, y \right\rangle, & \text{if } p(x) \neq p(y) \\
\langle x, y \rangle, & \text{if } p(x) = p(y)
\end{cases}
$$

if $T$ is not differentiable at $p(x) = p(y)$. 
Sketch of proof

Let \( \{y_k\} \) be a minimizing sequence of (9) such that

\[
\|y_k\| = 1, \quad [x_j, y_k] = 0, \quad j = 1, \ldots, n, \quad p(y_k) \to \lambda_{n+1}.
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Sketch of proof

Let \( \{y_k\} \) be a minimizing sequence of (9) such that

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\]

There exist unique

\[
\tilde{y}_k = y_k + \sum_{j=1}^{n} c_{kj} x_j
\]

such that \( \langle T(\lambda_{n+1}) x_j, \tilde{y}_k \rangle = 0, \quad j = 1, \ldots, n \), and it holds that

\[
\|y_k - \tilde{y}_k\| \to 0 \quad \text{and} \quad p(\tilde{y}_k) \to \lambda_{n+1}.
\]
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such that \( \langle T(\lambda_{n+1}) x_j, \tilde{y}_k \rangle = 0, \quad j = 1, \ldots, n, \) and it holds that

\[
\|y_k - \tilde{y}_k\| \to 0 \quad \text{and} \quad p(\tilde{y}_k) \to \lambda_{n+1}.
\]

For \( V_k := \text{span}\{x_1, \ldots, x_n, \tilde{y}_k\} \) it can be shown that

\[
\lim_{k \to \infty} \sup \{p(z) : z \in V_k\} = \lambda_{n+1},
\]

and it follows from the minmax characterization that \( \lambda_{n+1} \) is an eigenvalue of \( T(\cdot) \).
Minmax characterizations for non-overdamped nonlinear eigenvalue problems were proved (independently from our work) by

- **Barston (1974)** for some extreme eigenvalues of finite dimensional quadratic eigenproblems
Further literature

Minmax characterizations for non-overdamped nonlinear eigenvalue problems were proved (independently from our work) by

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Recall

\[ T(\lambda)u := -K_f u + \lambda M_f u + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} C_n u, \quad C_n u := \langle u_n, Cu \rangle C' u_n. \] (10)
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(10)

\( f(\cdot, u) = \langle T(\cdot) u, u \rangle \) is strictly monotonically increasing and \( f(0, u) < 0 \) for every \( u \neq 0 \) and \( \lambda \neq \sigma_\ell, \ell \in \mathbb{N} \).

Hence, in each interval \( J_\ell := (\sigma_{\ell-1}, \sigma_\ell) \) (with \( \sigma_0 := 0 \)) the eigenvalues allow for a minmax and maxmin characterization.
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Hence, in each interval \( J_\ell := (\sigma_{\ell-1}, \sigma_\ell) \) (with \( \sigma_0 := 0 \)) the eigenvalues allow for a minmax and maxmin characterization.

For \( \kappa \in J_\ell \) consider the linear comparison problem

\[(K_f + \sum_{j=1}^{\ell-1} \frac{\kappa}{\kappa - \sigma_j} C_j)u = \mu (M_f + \sum_{j=\ell}^{\infty} \frac{1}{\sigma_j - \kappa} C_j)u \]  

(11)

and denote by \( R_\kappa(u) \) the corresponding Rayleigh quotient.
Number of eigenvalues

Let $\mu_n(\kappa)$ be the $n$ smallest eigenvalue of the comparison problem (11). Then $\kappa \mapsto \mu_n(\kappa)$ is monotonically increasing and $\lambda$ is an $n$-th eigenvalue of the rational eigenproblem (10) if and only if $\lambda$ is a fixed point of $\mu_n(\cdot)$. 
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**Theorem**

Let $\sigma_{\ell-1} < \kappa_1 < \kappa_2 < \sigma_{\ell}$ and $N(\kappa) := \max\{n \in \mathbb{N} : \mu_n(\kappa) \leq \kappa\}$.

Then the nonlinear eigenvalue problem (10) has exactly $N(\kappa_2) - N(\kappa_1)$ eigenvalues in $(\kappa_1, \kappa_2]$. 

Bounds to Rayleigh functional

Lemma
Assume that $\kappa \in J_\ell$ and $R_\kappa(u) \in J_\ell$ for some $u \in \mathcal{H}$, $u \neq 0$. Then $u \in \mathcal{D}_\ell$, and

$$\min\{\kappa, R_\kappa(u)\} \leq p_\ell(u) \leq \max\{\kappa, R_\kappa(u)\}.$$
Lemma
Assume that \( \kappa \in J_\ell \) and \( R_\kappa(u) \in J_\ell \) for some \( u \in \mathcal{H} \), \( u \neq 0 \). Then \( u \in D_\ell \), and
\[
\min\{\kappa, R_\kappa(u)\} \leq p_\ell(u) \leq \max\{\kappa, R_\kappa(u)\}.
\]

Proof:
\[
f(R_\kappa(u), u) = \cdots = \sum_{j=1}^{\ell-1} \frac{\sigma_\ell(R_\kappa(u) - \kappa)}{(\sigma_\ell - R_\kappa(u))(\sigma_\ell - \kappa)} \langle u, C_j u \rangle + \sum_{j=\ell}^{\infty} \frac{R_\kappa(u)(R_\kappa(u) - \kappa)}{(\sigma_\ell - R_\kappa(u))(\sigma_\ell - \kappa)} \langle u, C_j u \rangle
\]
i.e. \( f(R_\kappa(u), u) \geq 0 \) for \( R_\kappa(u) \geq \kappa \), and \( f(R_\kappa(u), u) \leq 0 \) for \( R_\kappa(u) \leq \kappa \).
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Proof:

\[ f(R_{\kappa}(u), u) = \cdots = \]

\[ = \sum_{j=1}^{\ell-1} \frac{\sigma_\ell(R_{\kappa}(u) - \kappa)}{(\sigma_\ell - R_{\kappa}(u))(\sigma_\ell - \kappa)} \langle u, C_j u \rangle + \sum_{j=\ell}^{\infty} \frac{R_{\kappa}(u)(R_{\kappa}(u) - \kappa)}{(\sigma_\ell - R_{\kappa}(u))(\sigma_\ell - \kappa)} \langle u, C_j u \rangle \]

i.e. $f(R_{\kappa}(u), u) \geq 0$ for $R_{\kappa}(u) \geq \kappa$, and $f(R_{\kappa}(u), u) \leq 0$ for $R_{\kappa}(u) \leq \kappa$. And

\[ f(\kappa, u) = \cdots = (\kappa - R_{\kappa}(u)) \left( \sum_{j=\ell}^{\infty} \frac{1}{\sigma_j - \kappa} \langle u, C_j u \rangle + \langle u, M_f u \rangle \right), \]

i.e. $f(R_{\kappa}(u), u) \leq 0$ for $R_{\kappa}(u) \geq \kappa$, and $f(R_{\kappa}(u), u) \geq 0$ for $R_{\kappa}(u) \leq \kappa$. 
Theorem
Assume that for $\kappa \in J_1 = (0, \sigma_1)$ the comparison problem (11) has $r$ eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_r$ in $J_1$.
Then the rational eigenvalue problem (10) has $r$ eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$ (where $\lambda_j$ is a $j$-th eigenvalue), and it holds that

$$\min\{\mu_j, \kappa\} \leq \lambda_j \leq \max\{\mu_j, \kappa\} \quad \text{for } j = 1, \ldots, r. \quad (12)$$
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Proof: $\inf_{u \in D_1} p_1(u) > 0$ follows by contradiction from a comparison with (11) for $\kappa = \mu_1$.
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The existence of $\lambda_j$, $j = 1, \ldots, r$ follows from minmax Theorem for extreme eigenvalues since the invariant subspace $W$ of (11) corresponding to $\mu_r$ is contained in $D_1$ and $\max_{v \in W, \not{x=0}} R_\kappa(v) \in J_1$. 


Theorem
Assume that for \( \kappa \in J_1 = (0, \sigma_1) \) the comparison problem (11) has \( r \) eigenvalues \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_r \) in \( J_1 \).
Then the rational eigenvalue problem (10) has \( r \) eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r \) (where \( \lambda_j \) is a \( j \)-th eigenvalue), and it holds that

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\min\{\mu_j, \kappa\} \leq \lambda_j \leq \max\{\mu_j, \kappa\} \quad \text{for } j = 1, \ldots, r.
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The existence of \( \lambda_j, j = 1, \ldots, r \) follows from minmax Theorem for extreme eigenvalues since the invariant subspace \( W \) of (11) corresponding to \( \mu_r \) is contained in \( D_1 \) and \( \max_{v \in W, x \neq 0} R_{\kappa}(v) \in J_1 \).

(12) follows from the last Lemma and the minmax Theorem for extreme eigenvalues by comparison with an invariant subspace corresponding to \( \mu_1, \ldots, \mu_j \).
Theorem
Assume that for $\kappa \in J_\ell$ the $n$-th eigenvalue of the comparison problem (11) satisfies $\mu_n(\kappa) \in (\sigma_{\ell-1}, \sigma_\ell)$. Then the rational eigenvalue problem (10) has an $n$-th eigenvalue $\lambda_n \in (\sigma_{\ell-1}, \sigma_\ell)$, and it holds that

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$$\min\{\kappa, \mu_n(\kappa)\} \leq \lambda_n \leq \max\{\kappa, \mu_n(\kappa)\}.$$  \hspace{1cm} (13)

Proof: Show that

- There exist $V$ with $\dim V = n$ such that $V \cap D_\ell \neq \emptyset$ and $\sup_{u \in V \cap D_\ell} p_\ell(u) \leq \max\{\kappa, \mu_n(\kappa)\}$. 

...
Interior intervals $J_\ell$

**Theorem**
Assume that for $\kappa \in J_\ell$ the $n$-th eigenvalue of the comparison problem (11) satisfies $\mu_n(\kappa) \in (\sigma_{\ell-1}, \sigma_\ell)$. Then the rational eigenvalue problem (10) has an $n$-th eigenvalue $\lambda_n \in (\sigma_{\ell-1}, \sigma_\ell)$, and it holds that

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- There exist $V$ with $\dim V = n$ such that $V \cap \mathcal{D}_\ell \neq \emptyset$ and $\sup_{u \in V \cap \mathcal{D}_\ell} p_\ell(u) \leq \max\{\kappa, \mu_n(\kappa)\}$
- $\sup_{u \in V \cap \mathcal{D}_\ell} p_\ell(u) \geq \min\{\kappa, \mu_n(\kappa)\}$ for every $V$ with $\dim V = n$ such that $V \cap \mathcal{D}_\ell \neq \emptyset$. 


Theorem
Assume that for $\kappa \in J_\ell$ the $n$-th eigenvalue of the comparison problem (11) satisfies $\mu_n(\kappa) \in (\sigma_{\ell-1}, \sigma_\ell)$. Then the rational eigenvalue problem (10) has an $n$-th eigenvalue $\lambda_n \in (\sigma_{\ell-1}, \sigma_\ell)$, and it holds that

$$\min\{\kappa, \mu_n(\kappa)\} \leq \lambda_n \leq \max\{\kappa, \mu_n(\kappa)\}. \quad (13)$$

Proof: Show that

- There exist $V$ with $\dim V = n$ such that $V \cap D_\ell \neq \emptyset$ and
  $$\sup_{u \in V \cap D_\ell} p_\ell(u) \leq \max\{\kappa, \mu_n(\kappa)\}$$
- $\sup_{u \in V \cap D_\ell} p_\ell(u) \geq \min\{\kappa, \mu_n(\kappa)\}$ for every $V$ with $\dim V = n$ such that $V \cap D_\ell \neq \emptyset$.

Then

$$\lambda_n = \inf_{\dim V=n, \ V \cap D_\ell \neq \emptyset} \sup_{u \in V \cap D_\ell} p_\ell(u) \in J_\ell,$$

i.e. $\lambda_n$ is an $n$-th eigenvalue of (10) and inequalities (13) holds.
Outline

1. Two examples

2. Variational characterization of eigenvalue problems
   - Overdamped Problems
   - Nonoverdamped problems

3. An unsymmetric linear eigenproblem

4. Concluding remarks
Recall

Find $\lambda := \omega^2 \in \mathbb{C}$ and $(u, p) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f)$, $(u, p) \neq (0, 0)$ such that

$$a_s(v, u) + c(v, p) = \lambda b_s(v, u) \quad \text{and} \quad a_f(q, p) = \lambda (-c(u, q) + b_f(q, p)).$$  \hfill (14a) \hfill (14b)

for every $(v, q) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f)$. 

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for every $(v, q) \in H^1_D(\Omega_s)^3 \times H^1(\Omega_f)$.

Then its adjoint problem is
Find $\lambda \in \mathbb{C}$ and nonzero $(u, p) \in H^1_D(\Omega_s)^3 \times H^1(\Omega_f)$ such that

$$a_s(u, v) = \lambda (b_s(u, v) - c(v, p)) \quad (15a)$$
$$c(u, q) + a_f(p, q) = \lambda b_f(p, q). \quad (15b)$$

for all $(v, q) \in H^1_D(\Omega_s)^d \times H^1(\Omega_f)$. 
Lemma

\((u, p)\) is an eigensolution of (14) corresponding to an eigenvalue \(\lambda\) if and only if \((\lambda u, p)\) is an eigenvalue of the adjoint problem (15) corresponding to the same eigenvalue \(\lambda\).
Lemma

- $(u, p)$ is an eigensolution of (14) corresponding to an eigenvalue $\lambda$ if and only if $(\lambda u, p)$ is an eigenvalue of the adjoint problem (15) corresponding to the same eigenvalue $\lambda$.

- Eigensolutions $(u_1, p_1)$ and $(u_2, p_2)$ corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal with respect to the inner product

$$\langle(u, p), (v, q)\rangle := a_s(u, v) + b_f(p, q).$$
Lemma

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- Eigensolutions $(u_1, p_1)$ and $(u_2, p_2)$ corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal with respect to the inner product

$$\langle (u, p), (v, q) \rangle := a_s(u, v) + b_f(p, q).$$

- The eigenvalue problem (14) has only real non-negative eigenvalues.
Using the adjoint eigenfunction as a test function in equation (14) one obtains
\[ \lambda a_s(u, u) + \lambda c(u, p) + a_f(p, p) = \lambda^2 b_s(u, u) - \lambda c(u, p) + \lambda b_f(p, p) \]
for any eigensolution \((\lambda, (u, p)))\), i.e. \(\lambda\) is the positive root of
\[ g(\lambda, (u, p)) := \lambda^2 b_s(u, u) + \lambda (b_f(p, p) - a_s(u, u) - 2c(u, p)) - a_f(p, p). \]
An unsymmetric linear eigenproblem

Rayleigh functional of unsymmetric fluid-solid problem

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$$g(\lambda, (u, p)) := \lambda^2 b_s(u, u) + \lambda (b_f(p, p) - a_s(u, u) - 2c(u, p)) - a_f(p, p).$$

**Definition:** The functional \(r : H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \setminus \{0\} \to \mathbb{R}\), where any nonzero \((u, p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)\) is mapped to the maximal root of \(g(\cdot, (u, p))\) is called the **nonlinear Rayleigh functional** of (14), i.e.

$$r(u, p) = \begin{cases} \Delta + \sqrt{\Delta^2 + \frac{a_f(p, p)}{b_s(u, u)}} & \text{if } b_s(u, u) \neq 0, \\ \frac{a_f(p, p)}{b_f(p, p)} & \text{if } b_s(u, u) = 0, \end{cases}$$

where

$$\Delta = \frac{1}{2} \frac{-b_f(p, p) + a_s(u, u) + 2c(u, p)}{b_s(u, u)}.$$
Lemma
Let $I = \mathbb{N}$ or $I = \{1, \ldots, m\}$ an index set, $(u_i, p_i)_{i \in I}$ linearly independent eigenfunctions of (14) corresponding to distinct eigenvalues $(\lambda_i)_{i \in I}$ enumerated in ascending order, $\lambda_i \leq \lambda_j$ if $i < j$ and

$$(u, p) = \sum_{i \in I} (u_i, p_i).$$
Lemma

Let \( I = \mathbb{N} \) or \( I = \{1, \ldots, m\} \) an index set, \((u_i, p_i)_{i \in I}\) linearly independent eigenfunctions of (14) corresponding to distinct eigenvalues \((\lambda_i)_{i \in I}\) enumerated in ascending order, \( \lambda_i \leq \lambda_j \) if \( i < j \) and

\[
(u, \bar{p}) = \sum_{i \in I} (u_i, p_i).
\]

(i) It holds for any \( j \in I \) that

\[
g(\lambda_j, (u, \bar{p})) = g(\lambda_j, (u, \bar{p}) - (u_j, p_j)).
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Lemma
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(i) It holds for any $j \in I$ that

$$g(\lambda_j, (u, p)) = g(\lambda_j, (u, p) - (u_j, p_j)).$$

(ii) It holds that

$$\lambda_1 \leq r((u, p)) \leq \sup_{i \in I} \lambda_i.$$
Variational characterization

Theorem (Stammberger, V. 2013)
Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of the unsymmetric eigenproblem (14) in ascending order and $(u_1, p_1), (u_2, p_2), \ldots$ corresponding eigenfunctions. Then it holds that

(i) (Rayleigh's principle) $\lambda_k = \min \{ r(u, p) : a_s(u, u_j) + b_f(p, p_j) = 0, j = 1, \ldots, k - 1 \}$,

(ii) (minmax and maxmin characterizations) $\lambda_k = \min_{S_k \subset H_\Gamma D(\Omega_s)} \{ r(u, p) : \dim S_k = k \max_{0 \neq (u, p) \in S_k} r(u, p) \}$ $\lambda_{k-1} = \max_{S_{k-1} \subset H_\Gamma D(\Omega_s)} \{ r(u, p) : \dim S_{k-1} = k - 1 \min_{0 \neq (u, p) \in S_{k-1}} r(u, p) \}$.
Variational characterization

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\]

(ii) (minmax and maxmin characterizations)

\[
\lambda_k = \min_{S_k \subset H^1_D(\Omega_s)^d \times H^1(\Omega_f)} \max_{0 \neq (u, p) \in S_k} r(u, p) \\
= \max_{S_{k-1} \subset H^1_D(\Omega_s)^d \times H^1(\Omega_f)} \min_{0 \neq (u, p) \in S_{k-1}^\bot} r(u, p).
\]
Outline

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4. Concluding remarks
Applications of variational principles

Infinite dimensional problems

- locating eigenvalues of fluid–solid type structures (V. 2003, 2005)
- quadratic problems of restricted rank (V. 2004)
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Infinite dimensional problems
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Finite dimensional problems
- safeguarded iteration (V. & Werner 1982)
- a priori bounds for condensation methods (V. 1984)
- dynamic element methods (V. 1987)
- nonlinear Arnoldi method (V. 2003)
- Jacobi–Davidson type method for nonlinear evps (T. Betcke & V. 2004)
- a priori bounds for AMLS (Elssel & V. 2006)
- regularized total least squares problems (Lampe & V. 2007, 2008)
- detecting hyperbolic and definite matrix polynomials (Niendorf & V. 2010)
- Low rank modifications of symmetric evps (V., Yildiztekin & Huang 2011)
- generalization of Sylvester’s law (Kostic & V. 2013)
Discretization of rational eigenproblems

Orthogonal projection of the rational eigenvalue problem describing the electronic stats of a quantum dot yields a rational matrix eigenvalue problem which inherits the variational characterizations of its eigenvalues.
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This enables its efficient numerical solution (V. 2006) by iterative projection methods of Nonlinear Arnoldi (V. 2003,2004) and Jacobi-Davidson type (T. Betcke & V. 2004).

Refined quantum dot models were considered including wetting layers (Markiewicz & V. 2006), spin orbit splitting (M. Betcke & V. 2007), and a magnetic fields (M. Betcke & V. 2012). The last two problems are no longer overdamped.

The same holds true for the rational eigenvalue problem governing free vibrations of fluid-solid structures. Here the additional problem arises how to initialize the method when looking for eigenvalues right to a pole (V. 2003), or restarting the iterative projection method locally if a large number of eigenpairs is sought in the interior of the spectrum (Markiewicz & V. 2005, M. Betcke & V. 2013).
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Rayleigh’s principle for the unsymmetric eigenproblem suggests a Rayleigh functional iteration which is cubically convergent.
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This suggests efficient structure preserving iterative projection methods (Stammberger & V. 2010, 2013) and an AMLS methods for vibrations of fluid–solid structures (Stammberger & V. 2011).