Variationsprinzipien für nichtlineare Eigenwertaufgaben

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On the one hand: Parameter dependent nonlinear (with respect to the state variable) operator equations

$$T(\lambda, u) = 0$$

are discussed concerning

- positivity of solutions
- multiplicity of solution
- dependence of solutions on the parameter; bifurcation
- (change of) stability of solutions

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- nonlinear integrated optics

Example 1: Electronic behavior of quantum dots

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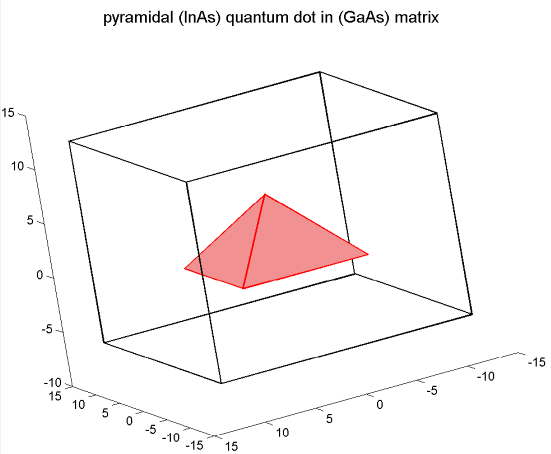
The ultimate limit of low dimensional structures is the quantum dot, in which the carriers are confined in all three directions, thus reducing the degrees of freedom to zero.
Therefore, a quantum dot can be thought of as an artificial atom.

Problem

Determine relevant energy states (i.e. eigenvalues) and corresponding wave functions (i.e. eigenfunctions) of a three-dimensional quantum dot embedded in a matrix.



Problem ct.

Governing equation: Schrödinger equation

$$-\nabla \cdot \left(\frac{\hbar^2}{2m(x,E)} \nabla \Phi \right) + V(x)\Phi = E\Phi, \quad x \in \Omega_q \cup \Omega_m$$

where $\hbar$ is the reduced Planck constant, $m(x,E)$ is the electron effective mass, and $V(x)$ is the confinement potential.

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Boundary and interface conditions

$$\Phi = 0 \quad \text{on outer boundary of matrix } \Omega_m$$

BenDaniel–Duke condition $\quad \frac{1}{m_m} \frac{\partial \Phi}{\partial n}\Big|_{\partial \Omega_m} = \frac{1}{m_q} \frac{\partial \Phi}{\partial n}\Big|_{\partial \Omega_q}$ on interface

Variational form

Find $E \in \mathbb{R}$ *and* $\Phi \in H_0^1(\Omega)$, $\Phi \neq 0$, $\Omega := \overline{\Omega}_q \cup \Omega_m$, *such that*

$$
\begin{aligned}
a(\Phi, \Psi; E) &:= \frac{\hbar^2}{2} \int\limits_{\Omega_q} \frac{1}{m_q(x,E)} \nabla\Phi \cdot \nabla\Psi \, dx + \frac{\hbar^2}{2} \int\limits_{\Omega_m} \frac{1}{m_m(x,E)} \nabla\Phi \cdot \nabla\Psi \, dx \\
&\quad + \int\limits_{\Omega_q} V_q(x)\Phi\Psi \, dx + \int\limits_{\Omega_m} V_m(x)\Phi\Psi \, dx \\
&= E \int\limits_{\Omega} \Phi\Psi \, dx =: Eb(\Phi, \Psi) \qquad \textit{for every } \Psi \in H_0^1(\Omega)
\end{aligned}
$$

Electron effective mass

The dependence of $m(x, E)$ on E can be derived from the eight-band $k \cdot p$ analysis and effective mass theory. Projecting the 8×8 Hamiltonian onto the conduction band results in the single Hamiltonian eigenvalue problem with

$$m(x, E) = \begin{cases} m_q(E), & x \in \Omega_q \\ m_m(E), & x \in \Omega_m \end{cases}$$

$$\frac{1}{m_j(E)} = \frac{P_j^2}{\hbar^2} \left(\frac{2}{E + g_j - V_j} + \frac{1}{E + g_j - V_j + \delta_j} \right), \; j \in \{m, q\}$$

where m_j is the electron effective mass, V_j the confinement potential, P_j the momentum, g_j the main energy gap, and δ_j the spin-orbit splitting in the jth region.

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Other types of effective mass (taking into account the effect of strain, e.g.) appear in the literature. They are all rational functions of E where $1/m(x, E)$ is monotonically decreasing with respect to E, and that's all we need.

Properties

- $a(\cdot,\cdot,E)$ bilinear, symmetric, bounded, $H_0^1(\Omega)$–elliptic for $E \geq 0$
- $b(\cdot,\cdot)$ bilinear, positive definite, bounded, completely continuous

By the Lax–Milgram lemma the variational eigenproblem is equivalent to

$$T(E)\Phi = 0$$

where

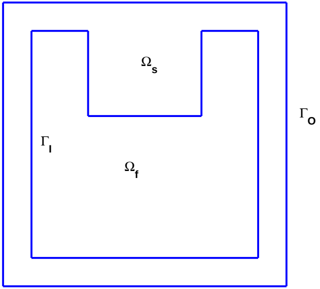
$$T(E) \, : \, H_0^1(\Omega) \to H_0^1(\Omega), \quad E \geq 0,$$

is a family of self-adjoint and bounded operators.

Example 2: Free vibrations of fluid-solid structures

can be modelled in terms of solid displacement and fluid pressure and one obtains the classical form of an eigenproblem

$$\begin{aligned} \operatorname{div}\,[\sigma(u)] + \omega^2 \rho_s u &= 0 \text{ in } \Omega_s, \\ \Delta p + \frac{\omega^2}{c^2} p &= 0 \text{ in } \Omega_f, \\ \sigma(u)\cdot n - pn &= 0 \text{ on } \Gamma_I, \\ \nabla p \cdot n + \omega^2 \rho_f u \cdot n &= 0 \text{ on } \Gamma_I, \\ u &= 0 \text{ on } \Gamma_D, \\ \nabla p \cdot n &= 0 \text{ on } \Gamma_N, \end{aligned}$$



where

- u: solid displacement
- p: fluid pressure
- $\lambda = \omega^2$: eigenparameter
- $\sigma(u)$: linearized stress tensor
- ρ_s, ρ_f: densities of solid and fluid

Variational and operator form

Find $\lambda := \omega^2 \in \mathbb{C}$ and $(u, p) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f)$ such that

$$\begin{aligned} a_s(v, u) + c(v, p) &= \lambda b_s(v, u) \text{ and} \\ a_f(q, p) &= \lambda(-c(u, q) + b_f(q, p)). \end{aligned}$$

for every $(v, q) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f)$.

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which (using the Lax-Milgram Lemma) can be transformed into a linear (but not self-adjoint) eigenvalue problem

$$K_s u + Cp = \lambda M_s u \tag{2a}$$

$$K_f p = \lambda(-C' u + M_f p) \tag{2b}$$

where $K_s : H^1_{\Gamma_D}(\Omega_s)^3 \to H^1_{\Gamma_D}(\Omega_s)^3$ is self-adjoint, elliptic, bounded, ...

Rational form of fluid-solid eigenproblem

Let $0 < \sigma_1 \leq \sigma_2 \leq \dots$ denote the eigenvalues of the decoupled eigenproblem

$$K_s u = \sigma M_s u$$

and denote by $u_1, u_2, \dots$ corresponding orthonormal eigenfunctions. Then the spectral theorem yields

$$(K_s - \lambda M_s)^{-1} u = \sum_{n=1}^{\infty} \frac{1}{\sigma_n - \lambda} \langle u_n, u \rangle u_n.$$

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If λ is not contained in the spectrum of the decoupled solid eigenproblem, then λ is an eigenvalue of the coupled fluid-solid problem if and only if it is an eigenvalue of the rational eigenvalue problem

$$T(\lambda)p := -K_f p + \lambda M_f p + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} C_n p, \quad C_n p := \langle u_n, Cp \rangle C' u_n.$$

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$T(\lambda) : H^1(\Omega_f) \to H^1(\omega_f)$ is self-adjoint and bounded.

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Then problem (2) is equivalent to the quadratic eigenvalue problem

$$\left(\begin{pmatrix} K_s & O \\ O & K_f \end{pmatrix} + \omega \begin{pmatrix} O & C \\ C' & O \end{pmatrix} - \omega^2 \begin{pmatrix} M_s & O \\ O & M_f \end{pmatrix}\right) \begin{pmatrix} u \\ w \end{pmatrix} = 0.$$

Variational characterization for linear eigenproblems

Let $A : \mathcal{H} \to \mathcal{H}$ a bounded linear and self-adjoint operator in a Hilbert space $\mathcal{H}$. Then those eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$ above the essential spectrum of A can be characterized by three fundamental variational principles,

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- Rayleigh's principle

$$\lambda_n = \max\{R(x) : \langle x, x_i \rangle = 0,\ i = 1, \dots, n-1\},\ R(x) := \frac{\langle Ax, x \rangle}{\langle x, x \rangle}.$$

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These variational characterizations of eigenvalues are very powerful tools when studying self-adjoint linear operators on a Hilbert space $\mathcal{H}$.
Bounds for eigenvalues, comparison theorems, interlacing results and monotonicity of eigenvalues can be proved easily with these characterizations, to name just a few.

Rayleigh functional

Let

$$f : \begin{cases} J \times H & \to & \mathbb{R} \\ (\lambda, x) & \mapsto & \langle T(\lambda)x, x\rangle \end{cases}$$

be continuous, and assume that for every fixed $x \in \mathcal{H}$, $x \neq 0$, the real equation

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We further assume that for every $x \in D$, $x \neq 0$ and $\lambda \in J$, $\lambda \neq p(x)$ it holds that

$$f(\lambda, x)(\lambda - p(x)) > 0$$

which generalizes the definiteness of the operator B for the generalized linear eigenproblem $T(\lambda) := \lambda B - A$.

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increase α	eigenvalues go into left half plane as conjugate complex pairs
increase α	complex pairs reach real axis, run in opposite directions
increase α	all eigenvalues on the negative real axis
increase α	all eigenvalues going to the left are smaller than all eigenvalues going to the right system is overdamped

Quadratic overdamped problems

For quadratic overdamped systems the two solutions

$$p_{\pm}(x) = \frac{1}{2}\left(-\alpha\langle Cx, x\rangle \pm \sqrt{\alpha^2\langle Cx, x\rangle^2 - 4\langle Mx, x\rangle\langle Kx, x\rangle}\right) / \langle Mx, x\rangle.$$

of the quadratic equation

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Hence, equation (4) defines two Rayleigh functionals p_- and p_+ corresponding to the intervals

$$J_- := (-\infty, \inf_{x\neq 0} p_+(x)) \quad \text{and} \quad J_+ := (\sup_{x\neq 0} p_-(x), \infty).$$

Rayleigh's principle

For general (not necessarily quadratic) overdamped problems Hadeler (1967 for the finite dimensional case, and 1968 for $\dim \mathcal{H} = \infty$) generalized Rayleigh's principle proving that the eigenvectors are orthogonal with respect to the generalized scalar product

$$[x, y] := \begin{cases} \langle \frac{T(p(x)) - T(p(y))}{p(x) - p(y)} x, y \rangle, & \text{if} \quad p(x) \neq p(y) \\ \langle T'(p(x)) x, y \rangle, & \text{if} \quad p(x) = p(y) \end{cases} \tag{5}$$

which is symmetric, definite and homogeneous, but in general is not bilinear.

Rayleigh's principle (Hadeler 1967,1968)

Let $T(\lambda) : \mathcal{H} \to \mathcal{H}$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators such that (1) is over-damped, and assume that for $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact.

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The corresponding eigenvectors $x_1, x_2, \dots$ can be chosen orthonormally with respect to the generalized scalar product (5), and the eigenvalues can be determined recurrently by

$$\lambda_n = \min\{p(x) : [x, x_i] = 0,\ i = 1, \dots, n-1,\ x \neq 0\}.$$

Minmax principle

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The corresponding generalizations for general overdamped problems of infinite dimension were derived by Hadeler (1968). Similar results (weakening the compactness or smoothness requirements) are contained in Rogers (1968), Werner (1971), Abramov (1973), Hadeler (1975), Markus (1985), Maksudov & Gasanov (1992), and Hasanov (2002).

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Let the eigenvalues λ_n of $T(\lambda)x = 0$ be numbered in non-decreasing order regarding their multiplicities. Then they can be characterized by the following two variational principles

$$\begin{aligned}\lambda_n &= \min_{\dim V=n}\ \max_{x\in V,\ x\neq 0}\ p(x)\\ &= \max_{\dim V=n-1}\ \min_{x\in V^{\perp},\ x\neq 0}\ p(x).\end{aligned}$$

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For $x \neq 0$ it holds that $a(x, x, 0) > 0$, $b(x, x) > 0$ and $\lambda \mapsto a(x, x, \lambda)$ is monotonically decreasing for $\lambda \geq 0$. Hence,

$$f(\lambda, x) = \langle T(\lambda)x, x \rangle = \lambda b(x, x) - a(x, x, \lambda) = 0$$

has exactly one positive solution $p(x)$, and

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Thus, the quantum dot problem has a countable number of non-negative eigenvalues which allow for all three variational characterizations.

Example 2: Fluid-solid vibration

For the rational eigenproblem governing free vibrations the family of operators $T(\lambda)$ satisfies the general conditions of the variational characterizations in every interval $J_n := (\sigma_{n-1}, \sigma_n)$.

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$$f(\lambda, p) := \langle T(\lambda)p, p\rangle = -\langle K_f p, p\rangle + \lambda\langle M_f p, p\rangle + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} |\langle u_n, Cp\rangle|^2$$

is monotonically increasing, such that $f(\lambda, p) = 0$ has at most one solution in J_n, but the Rayleigh functional is not defined on the entire space.

Non-overdamped problems

For nonoverdamped eigenproblems (i.e. $\mathcal{D}(p) \neq \mathcal{H} \setminus \{0\}$) the natural ordering to call the smallest eigenvalue the first one, the second smallest the second one, etc., is not appropriate.

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Then all conditions are satisfied, p is the restriction of the Rayleigh quotient R_A to $\mathcal{D}(p) := \{x \neq 0 \,:\, R_A(x) \in J\}$, and $\inf_{x \in \mathcal{D}(p)} p(x)$ will not be an eigenvalue.

Enumeration of eigenvalues

$\lambda \in J$ is an eigenvalue of $T(\cdot)$ if and only if $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$. The key idea is to orientate the number of λ on the location on the eigenvalue $\mu = 0$ in the spectrum of the linear operator $T(\lambda)$.

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We assume that for every $\lambda \in J$ it holds that the supremum of the essential spectrum of $T(\lambda)$ is negative (for instance: there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact).

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In this case we assign n to the eigenvalue λ of problem $T(\lambda)x = 0$ as its number and call λ an n-th eigenvalue of $T(\cdot)$.

Minmax characterization (V.&B.Werner 1982, V. 2010)

Let $T(\lambda)$, $\lambda \in J$ be a family of linear self-adjoint and bounded operator on a Hilbert space $\mathcal{H}$ depending continuously on a parameter $\lambda \in J$ where J is an open real (not necessarily bounded) interval.

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- $(\lambda - p(x))f(\lambda, x) > 0$ for every $x \in \mathcal{D}$ and $\lambda \neq p(x)$,
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Then the nonlinear eigenvalue problem $T(\lambda)x = 0$ has at most a countable set of eigenvalues, and it holds that:
If $\lambda_n \in J$ is an n-th eigenvalue then

$$\lambda_n = \min_{\dim V = n,\ V \cap \mathcal{D} \neq \emptyset} \ \sup_{x \in \mathcal{D} \cap V} p(x). \tag{6}$$

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If conversely

$$\lambda_n = \inf_{\dim V = n,\ V \cap \mathcal{D} \neq \emptyset} \ \sup_{x \in \mathcal{D} \cap V} p(x) \in J$$

then λ_n is an n-th eigenvalue of $T(\lambda)x = 0$ and (6) holds.

Sketch of proof

Step 1 (technical): Let $\lambda \in J$, and assume that V is a finite dimensional subspace of $\mathcal{H}$ such that $V \cap \mathcal{D} \neq \emptyset$. Then it holds that

$$\lambda \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \sup_{x \in V \cap \mathcal{D}(p)} p(x) \quad \Leftrightarrow \quad \min_{x \in V} \langle T(\lambda)x, x \rangle \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 0 \qquad (7)$$

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$$\mu_n(\lambda_n) = \max_{\dim V = n} \min_{x \in V,\ \|x\|=1} \langle T(\lambda_n)x, x \rangle = \min_{x \in \bar{V},\ \|x\|=1} \langle T(\lambda_n)x, x \rangle.$$

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Hence, $\min_{x \in V, \|x\|=1} \langle T(\lambda_n)x, x \rangle \leq 0$ for every V with $\dim V = n$, and (7) implies

$$\sup_{x \in V \cap \mathcal{D}} p(x) \geq \lambda_n = \sup_{x \in \bar{V} \cap \mathcal{D}} p(x).$$

Hence, λ_n is a minmax value of p.

Theorem (minmax for extreme eigenvalues)

Assume that the conditions of the minmax characterization holds and that

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If $\lambda_n \in J$ for some $n \in \mathbb{N}$ then every $V \in H_j$ with $V \cap \mathcal{D}(p) \neq \emptyset$ and $\lambda_j = \sup_{u\in V\cap\mathcal{D}(p)} p(u)$ is contained in $\mathcal{D} \cup \{0\}$, and the characterization (6) can be replaced by

$$\lambda_j = \min_{\substack{\dim V=j \\ V\subset\mathcal{D}\cup\{0\}}} \max_{v\in V,\ x\neq 0} p(v), \quad j = 1, \dots, n. \tag{8}$$

Maxmin characterization (V. 2003)

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If there is an n-th eigenvalue $\lambda_n \in J$ of $T(\lambda)x = 0$, then

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and the maximum is attained by $W := \text{span}\{u_1, \dots, u_{n-1}\}$ where u_j denotes an eigenvector corresponding to the j-largest eigenvalue $\mu_j(\lambda_n)$ of $T(\lambda_n)$.

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Proof takes advantage of the following Lemma:
Let $\lambda \in J$, and let V be a finite dimensional subspace of $\mathcal{H}$ such that $V^\perp \cap \mathcal{D} \neq \emptyset$. Then it holds that

$$\lambda \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \inf_{x \in V^\perp \cap \mathcal{D}(p)} p(x) \quad \Leftrightarrow \quad \max_{x \in V^\perp,\, \|x\|=1} \langle T(\lambda)x, x \rangle \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 0$$

Further literature

Variational characterizations for non-overdamped nonlinear eigenvalue problems were proved (independently from our work) by

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Free vibrations of fluid-solid structures

Recall

$$T(\lambda)u := -K_f u + \lambda M_f u + \sum_{n=1}^{\infty} \frac{\lambda}{\sigma_n - \lambda} C_n u, \quad C_n u := \langle u_n, Cu \rangle C' u_n. \qquad (9)$$

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$f(\cdot, u) = \langle T(\cdot)u, u\rangle$ is strictly monotonically increasing and $f(0, u) < 0$ for every $u \neq 0$ and $\lambda \neq \sigma_\ell$, $\ell \in \mathbb{N}$.
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Hence, in each interval $J_\ell := (\sigma_{\ell-1}, \sigma_\ell)$ (with $\sigma_0 := 0$) the eigenvalues allow for a minmax and maxmin characterization.

For $\kappa \in J_\ell$ consider the linear comparison problem

$$(K_f + \sum_{j=1}^{\ell-1} \frac{\kappa}{\kappa - \sigma_j} D_j)u = \mu(M_f + \sum_{j=\ell}^{\infty} \frac{1}{\sigma_j - \kappa} D_j)u \tag{10}$$

and denote by $R_\kappa(u)$ the corresponding Rayleigh quotient.

Number of eigenvalues

Let $\mu_n(\kappa)$ be the *n* smallest eigenvalue of the comparison problem (10).

Then $\kappa \mapsto \mu_n(\kappa)$ is monotonically increasing and λ is an *n*-th eigenvalue of the rational eigenproblem (9) if and only if λ is a fixed point of $\mu_n(\cdot)$.

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Theorem
Let $\sigma_{\ell-1} < \kappa_1 < \kappa_2 < \sigma_\ell$ and $N(\kappa) := \max\{n \in \mathbb{N} : \mu_n(\kappa) \leq \kappa\}$.
Then the nonlinear eigenvalue problem (9) has exactly $N(\kappa_2) - N(\kappa_1)$ eigenvalues in $(\kappa_1, \kappa_2]$.

Bounds to Rayleigh functional

Lemma

Assume that $\kappa \in J_\ell$ and $R_\kappa(u) \in J_\ell$ for some $u \in \mathcal{H}$, $u \neq 0$.
Then $u \in \mathcal{D}_\ell$, and

$$\min\{\kappa, R_\kappa(u)\} \leq p_\ell(u) \leq \max\{\kappa, R_\kappa(u)\}.$$

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Proof:

$$f(R_\kappa(u), u) = \cdots =$$

$$= \sum_{j=1}^{\ell-1} \frac{\sigma_\ell(R_\kappa(u) - \kappa)}{(\sigma_\ell - R_\kappa(u))(\sigma_\ell - \kappa)} \langle u, D_j u\rangle + \sum_{j=\ell}^{\infty} \frac{R_\kappa(u)(R_\kappa(u) - \kappa)}{(\sigma_\ell - R_\kappa(u))(\sigma_\ell - \kappa)} \langle u, D_j u\rangle$$

i.e. $f(R_\kappa(u), u) \geq 0$ for $R_\kappa(u) \geq \kappa$, and $f(R_\kappa(u), u) \leq 0$ for $R_\kappa(u) \leq \kappa$.

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i.e. $f(R_\kappa(u), u) \geq 0$ for $R_\kappa(u) \geq \kappa$, and $f(R_\kappa(u), u) \leq 0$ for $R_\kappa(u) \leq \kappa$. And

$$f(\kappa, u) = \cdots = (\kappa - R_\kappa(u)) \left(\sum_{j=\ell}^{\infty} \frac{1}{\sigma_j - \kappa} \langle u, D_j u\rangle + \langle u, M_f u\rangle \right),$$

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Lower end of spectrum

Theorem

Assume that for $\kappa \in J_1 = (0, \sigma_1)$ the comparison problem (10) has r eigenvalues $\mu_1 \le \mu_2 \le \dots \le \mu_n$ in J_1.

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(11) follows from the last Lemma and the minmax Theorem for extreme eigenvalues by comparison with an invariant subspace corresponding to $\mu_1, \dots, \mu_j$.

Interior intervals J_ℓ

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Assume that for $\kappa \in J_\ell$ the n-th eigenvalue of the comparison problem (10) satisfies $\mu_n(\kappa) \in (\sigma_{\ell-1}, \sigma_\ell)$. Then the rational eigenvalue problem (9) has an n-th eigenvalue $\lambda_n \in (\sigma_{\ell-1}, \sigma_\ell)$, and it holds that

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Then

$$\lambda_n = \inf_{\dim V = n,\ V \cap \mathcal{D}_\ell \neq \emptyset} \ \sup_{u \in V \cap \mathcal{D}_\ell} p_\ell(u) \in J_\ell,$$

i.e. λ_n is an n-th eigenvalue of (9) and inequalities (12) holds.

Unsymmetric fluid-solid eigenproblem

Recall
Find $\lambda := \omega^2 \in \mathbb{C}$ and $(u, p) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f)$ such that

$$a_s(v, u) + c(v, p) = \lambda b_s(v, u) \text{ and} \tag{13a}$$
$$a_f(q, p) = \lambda(-c(u, q) + b_f(q, p)). \tag{13b}$$

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Then its adjoint problem is
Find $\lambda \in \mathbb{C}$ and nonzero $(u, p) \in H^1_{\Gamma_D}(\Omega_s)^3 \times H^1(\Omega_f)$ such that

$$a_s(u, v) = \lambda(b_s(u, v) - c(v, p)) \tag{14a}$$
$$c(u, q) + a_f(p, q) = \lambda b_f(p, q). \tag{14b}$$

for all $(v, q) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$.

Properties of unsymmetric fluid-solid eigenproblem

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- (u, p) is an eigensolution of (13) corresponding to an eigenvalue λ if and only if $(\lambda u, p)$ is an eigenvalue of the adjoint problem (14) corresponding to the same eigenvalue λ.

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- The eigenvalue problem (13) has only real non-negative eigenvalues.

Rayleigh functional of unsymmetric fluid-solid problem

Using the adjoint eigenfunction as a test function in equation (13) one obtains

$$\lambda a_s(u,u) + \lambda c(u,p) + a_f(p,p) = \lambda^2 b_s(u,u) - \lambda c(u,p) + \lambda b_f(p,p)$$

for any eigensolution $(\lambda,(u,p))$, i.e. λ is the positive root of

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Definition: The functional $r : H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)\setminus\{0\} \to \mathbb{R}$, where any nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ is mapped to the maximal root of $g(\cdot, (u,p))$ is called the nonlinear Rayleigh functional of (13), i.e.

$$r(u,p) = \begin{cases} \Delta + \sqrt{\Delta^2 + \frac{a_f(p,p)}{b_s(u,u)}} & \text{if } b_s(u,u) \neq 0, \\ \frac{a_f(p,p)}{b_f(p,p)} & \text{if } b_s(u,u) = 0, \end{cases}$$

where

$$\Delta = \frac{1}{2}\frac{-b_f(p,p) + a_s(u,u) + 2c(u,p)}{b_s(u,u)}.$$

Unsymmetric fluid-solid eigenproblem

Lemma

Let $I = \mathbb{N}$ or $I = \{1, \dots, m\}$ an index set, $(u_i, p_i)_{i \in I}$ linearly independent eigenfunctions of (13) corresponding to distinct eigenvalues $(\lambda_i)_{i \in I}$ enumerated in ascendending order, $\lambda_i \leq \lambda_j$ if $i < j$ and

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(ii) It holds that

$$\lambda_1 \leq r\left((u, p)\right) \leq \sup_{i \in I} \lambda_i.$$

Variational characterization

Theorem
Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of the unsymmetric eigenproblem (13) in ascending order and $(u_1, p_1), (u_2, p_2), \ldots$ corresponding eigenfunctions. Then it holds that

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(i) (Rayleigh's principle)

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(ii) (Min-max Characterization)

$$\lambda_k = \min_{\substack{S_k \subset H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \\ \dim S_k = k}} \max_{0 \neq (u,p) \in S_k} r(u,p).$$

Discretization of rational eigenproblems

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The same holds true for the rational eigenvalue problem governing free vibrations of fluid-solid structures. Here the additional problem arises how to initialize the method when looking for eigenvalues right to a pole (V. 2003), or restarting the iterative projection method locally if a large number of eigenpairs is sought in the interior of the spectrum (Markiewicz & V. 2005, M. Betcke & V. 2008).

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