Solving huge gyroscopic eigenproblems with AMLS and subspace iteration

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Outline

Introduction

Simulation of acoustic properties has gained increasing importance in the engineering design process, in particular in automobile industries.

For passenger cars at speeds above 40 km/h, and for trucks above 60 km/h the major source of traffic noise is due to the sound radiation of rolling tires.

The simulation of the tire noise is performed in three steps.

First, the nonlinear tire deflections under steady state conditions are computed using an Arbitrary Lagrangian Eulerian (ALE) approach.
Next, the transient vibrations governed by the eigenpairs of a gyroscopic eigenvalue problem

$$Q(\omega)x := Kx + i\omega Gx - \omega^2 Mx = 0. \tag{1}$$

are assumed to be superimposed onto the nonlinear deflections.
Finally, the acoustic analysis is carried out solving Helmholtz's equation where the normal velocities at the wheel surface, extracted from the vibration analysis, are taken as boundary conditions.

Introduction

In this presentation we consider only the second step, i.e. the numerical solution of the gyroscopic eigenproblem (1) where K is the stiffness matrix modified by the presence of centripetal forces, M is the mass matrix, and G is the gyroscopic matrix stemming from the Coriolis force.

Clearly, K and M are symmetric and positive definite, and G is skew–symmetric.

Due to the complicated interior structure of a belted tire the matrices K, M and G of a sufficiently accurate FE model are very large and sparse.

Moreover, for the acoustic analysis many eigenpairs not necessarily at the end of the spectrum are needed.
Therefore, well-established sparse eigensolvers of Arnoldi type with shift and invert techniques for a linearization of problem (1), methods which are based on structure preserving linearizations like SHIRA, and iterative projection methods for nonlinear eigenproblems are very costly since LU factorizations of complex valued matrices $Q(\omega_j)$ for several parameters ω_j are required.

Automated Multi-Level Substructring

AMLS was introduced by Bennighof (1998) and was applied to huge problems of frequency response analysis.

The large finite element model is recursively divided into very many substructures on several levels based on the sparsity structure of the system matrices.

Assuming that the interior degrees of freedom of substructures depend quasistatically on the interface degrees of freedom, and modeling the deviation from quasistatic dependence in terms of a small number of selected substructure eigenmodes the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest.

Recent studies in vibro-acoustic analysis of passenger car bodies where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed have shown that AMLS is considerably faster than Lanczos type approaches.

Automated Multi-Level Substructring

Consider

$$Kx = \lambda Mx \tag{2}$$

where K and M are symmetric and positive definite (typically the stiffness and mass matrices of a FE model of a structure).

Similarly as in the component mode synthesis method (CMS) the structure is partitioned into a small number of substructures based on the sparsity pattern of the system matrices, but more generally than in CMS these substructures in turn are sub-structured on a number of levels yielding a tree topology for the substructures.

AMLS consists of two ingredients.

1. First, based on the substructuring the stiffness matrix K is transformed to block diagonal form by Gaussian elimination,
2. and secondly, the dimension is reduced substantially by modal condensation of the substructures.

Automated Multi-Level Substructring

If K_{ss} is a sub-matrix of K corresponding to a particular substructure, then after reordering rows and columns in (2) the pencil obtains the form

$$\left(\begin{bmatrix} K_{ss} & K_{sr} \\ K_{rs} & K_{rr} \end{bmatrix}, \begin{bmatrix} M_{ss} & M_{sr} \\ M_{rs} & M_{rr} \end{bmatrix} \right).$$

With block Gaussian elimination, i.e. post- and premultiplying this pencil with

$$U_s = \begin{bmatrix} I & -K_{ss}^{-1} K_{sr} \\ O & I \end{bmatrix}$$

and U_s^T, respectively, K_{ss} is decoupled, and the pencil obtains the following form

$$(U_s^T K U_s, U_s^T M U_s) = \left(\begin{bmatrix} K_{ss} & O \\ O & \tilde{K}_{rr} \end{bmatrix}, \begin{bmatrix} M_{ss} & \tilde{M}_{sr} \\ \tilde{M}_{sr}^T & \tilde{M}_{rr} \end{bmatrix} \right).$$

Repeating the block elimination for all substructures $1, \ldots, m$ we get

$$\tilde{K} = U^T K U, \ \tilde{M} = U^T M U \text{ with } U = U_1 U_2 \ldots U_m$$

where the transformed stiffness matrix $\tilde{K}$ has block diagonal form.

Automated Multi-Level Substructring

To reduce the dimension of the eigenproblem we determine for every substructure (after decoupling it from the remaining degrees of freedom in the stiffness matrix as above, and neglecting connections to other substructures in the mass matrix) all eigenvalues λ_{sj} not exceeding a cut off frequency λ_{cutoff} and corresponding eigenvectors z_{sj}, $j = 1, \ldots, m_s$.

Then with $Z_s = [z_{s1}, \ldots, z_{sm_s}]$ and the global block diagonal projection matrix $Z = \text{diag}\{Z_1, \ldots, Z_m\}$ we finally get the reduced eigenvalue problem

$$K_c x_c = \lambda M_c x_c \tag{3}$$

where $K_c = Z^T \tilde{K} Z = Z^T U^T K U Z$ is a diagonal matrix and $M_c = Z^T \tilde{M} Z = Z^T U^T M U Z$ has generalized block arrowhead form.

Important: In an implementation the block Gaussian eliminations and the condensations are performed in an interleaving way to avoid the storage of large dense sub-matrices of the transformed mass matrix which would occur in the course of the block elimination: as soon as a sub-matrix pencil $(\tilde{K}_{ss}, \tilde{M}_{ss})$ has been formed, the eigenproblem $\tilde{K}_{ss} Z_s = \tilde{M}_{ss} Z_s \Lambda_s$ is solved and the corresponding projection is executed.

Eigenvector approximation by AMLS

AMLS is a one shot projection method, i.e. after having chosen a cut-off frequency the method produces a fixed subspace $\mathcal{V} := \text{span}\{V\}$, $V := UZ$ and the corresponding projected eigenproblem.

Differently from iterative projections methods such as Krylov subspace or Jacobi–Davidson methods there is no way to expand the subspace $\mathcal{V}$ further reusing the projected problem if the computed approximate eigenpairs turn out to be not accurate enough. One has to repeat the reduction with a higher cut-off frequency.

Alternatively, one can improve the subspace $\mathcal{V}$ obtained with AMLS by subspace iteration.

Subspace iteration

Let the columns of $V_0 \in R^{n\times p}$ form an approximate basis of the invariant subspace of the pencil (K, M) corresponding to the wanted eigenvalues.

Then one step of subspace iteration requires to solve a linear system $(K - \sigma M)V_1 = MV_0$ for V_1 where σ is some shift close to the wanted eigenvalues. However, for huge matrices K and M a factorization of $K - \sigma M$ and a solution of this system is very costly.

Alternatively, we may apply subspace iteration to the transformed problem

$$\tilde{K}z := U^T KUz = \lambda U^T MUz =: \tilde{M}z,$$

where $U = U_1, \ldots, U_m$ is the matrix constructed in the AMLS process that transforms K to block diagonal form.

Due to the interleaving implementation of AMLS the matrices $\tilde{K}$ and $\tilde{M}$ are usually not stored when computing the reduced model, but in principle this could be easily done. The matrix $\tilde{K}$ then obtains block diagonal form with moderate block sizes, but owing to fill in during the elimination process $\tilde{M}$ will contain many dense sub-matrices requiring a huge amount of storage. So, this approach is also not efficient.

Subspace iteration

The way out is to combine the benefits of both approaches, i.e. to apply subspace iteration to the transformed system, but to evaluate $\tilde{M}\tilde{V}$ taking advantage of the transformation matrix U and the sparse structure of the original mass matrix M.

This procedure was proposed for definite linear eigenvalue problems $Kx = \lambda Mx$ and submitted to Computers & Structures.

Taking advantage of the special structure of the linearization

$$\begin{bmatrix} iG & K \\ K & O \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \omega \begin{bmatrix} M & O \\ O & K \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}, \quad y = \omega x$$

of the gyroscopic eigenproblem one obtains an efficient method for improving the eigenvector approximations from AMLS.

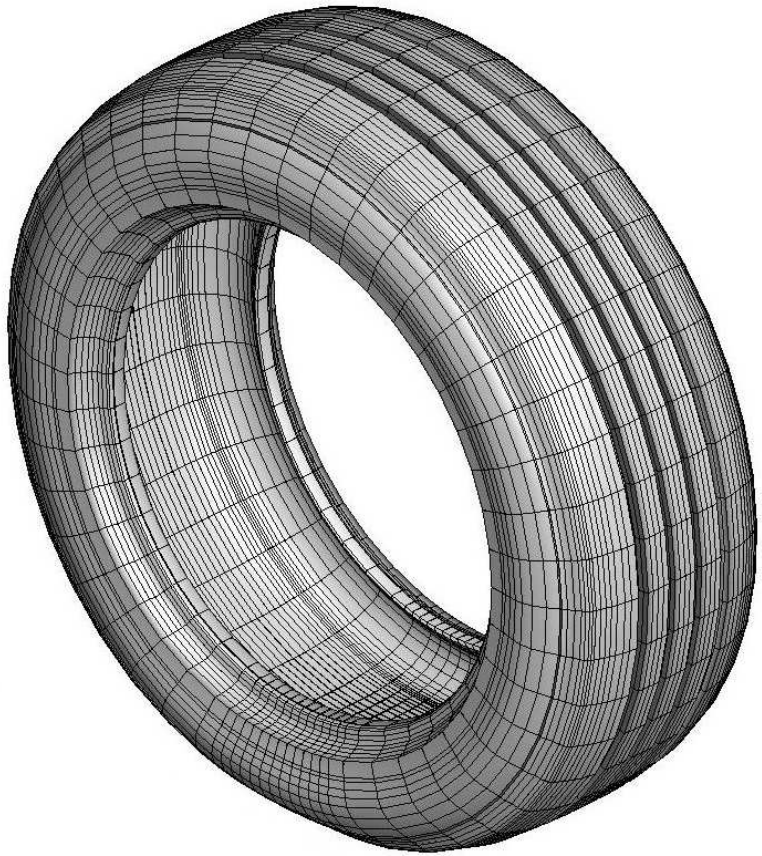
Subspace iteration

Require: Diagonal matrix $\tilde{\Lambda}$ with eigenv. approx. from AMLS, transf. eigenvec. $\tilde{V}$, transf. stiffness matrix $\tilde{K}$, transformation matrix U

initialize the iteration matrices $\tilde{Q}_0 = \tilde{V}$ and $\tilde{P}_0 = \tilde{V}\tilde{\Lambda}^{1/2}$
transform backward $P_0 = U\tilde{P}_0$
for $k = 1, 2, \ldots, n_k$ **do**
    transform backward $Q_{k-1} = U\tilde{Q}_{k-1}$
    compute $R = MP_{k-1} - iGQ_{k-1}$
    transform forward $\tilde{R} = U^T R$
    $P_k = Q_{k-1}$
    solve for $\tilde{Q}_k$: $\tilde{K}\tilde{Q}_k = \tilde{R}$
end for
$T = [\tilde{R}, -\overline{\tilde{R}}]^H [\tilde{Q}_{n_k}, -\overline{\tilde{Q}_{n_k}}]$
projected mass matrix $\hat{M} = [P_{n_k}, \overline{P_{n_k}}]^H M [P_{n_k}, \overline{P_{n_k}}] + T$
reload R and compute $S = [R, -\overline{R}]^H [P_{n_k}, \overline{P_{n_k}}]$
projected stiffness matrix $\hat{K} = i[P_{n_k}, \overline{P_{n_k}}]^H G[Pn_k, \overline{P_{n_k}}] + S + S^H$.
solve projected problem $\hat{K}Z = \hat{M}Z\hat{\Lambda}$
sort out positive eigenvalue $\hat{\Lambda}_+$ and corresponding eigenvectors Z_+
compute improved eigenvectors $V^{(n_k)} = [Q_{n_k}, -\overline{Q_{n_k}}]Z_+$.

Numerical example

Consider a rotating tire the FE model of which consists of 39204 brick elements with 124992 degrees of freedom and accounts for 20 different material groups. The speed is assumed to be 60 km/h. Our aim is to determine approximations to the smallest 200 eigenvalues and corresponding eigenvectors.



Numerical example

The numerical tests were performed on a 64-bit HP workstation with an Intel Xeon CPU (3.20 GHz, 2 cores) and 24GB memory. AMLS and the two subspace iteration algorithms were implemented with Matlab R2009a.

The AMLS method addressing the linear eigenvalue problem $Kx = \lambda Mx$ costs 881.2 seconds for the AMLS projection and 270.4 seconds for solving the projected linear eigenvalue problem of dimension 2263 by `eig`.

Numerical example

Table: Computation time of NormalSIM and AMLS-SIM with 208 iteration vectors

Computational Steps	NormalSIM(s)	AMLS-SIM(s)
Compute initial eigenvectors from AMLS	9.2	2.4
1 Iteration	2401.0	148.9
2 Iterations	Not computed	304.9
3 Iterations	Not computed	462.1
4 Iterations	Not computed	621.8
Compute $\hat{K}$ and $\hat{M}$ if $n_k = 1$	10.0	9.7
Compute $\hat{K}$ and $\hat{M}$ if $n_k > 1$	Not computed	19.6
Solve $\hat{K}Z = \hat{M}V\hat{\Lambda}$ by `eig`	2.6	2.6
Compute final eigenvectors	0.9	25.6

Numerical example

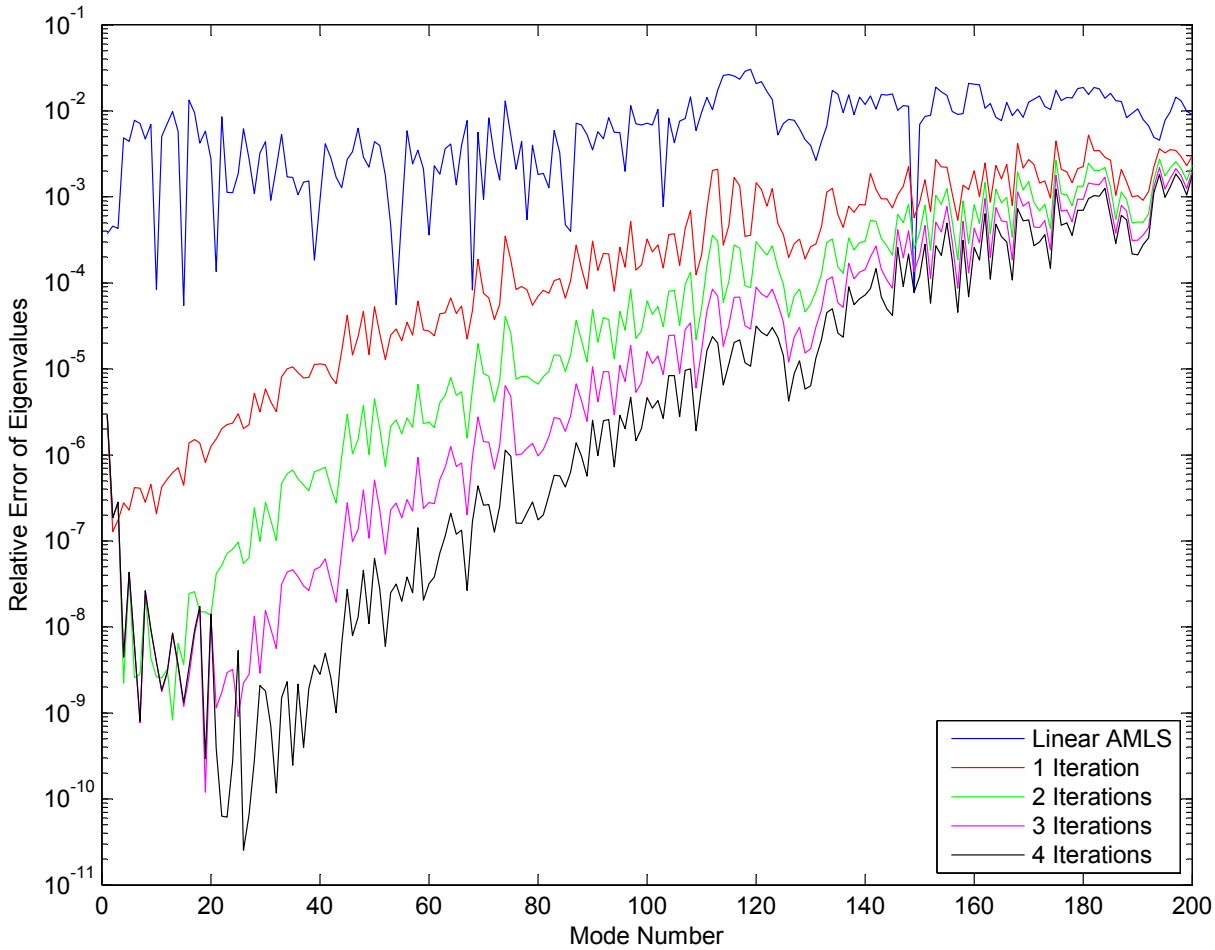


Figure: Relative errors of eigenvalues computed with subspace iteration with AMLS utilizing 208 iteration vectors

Numerical example

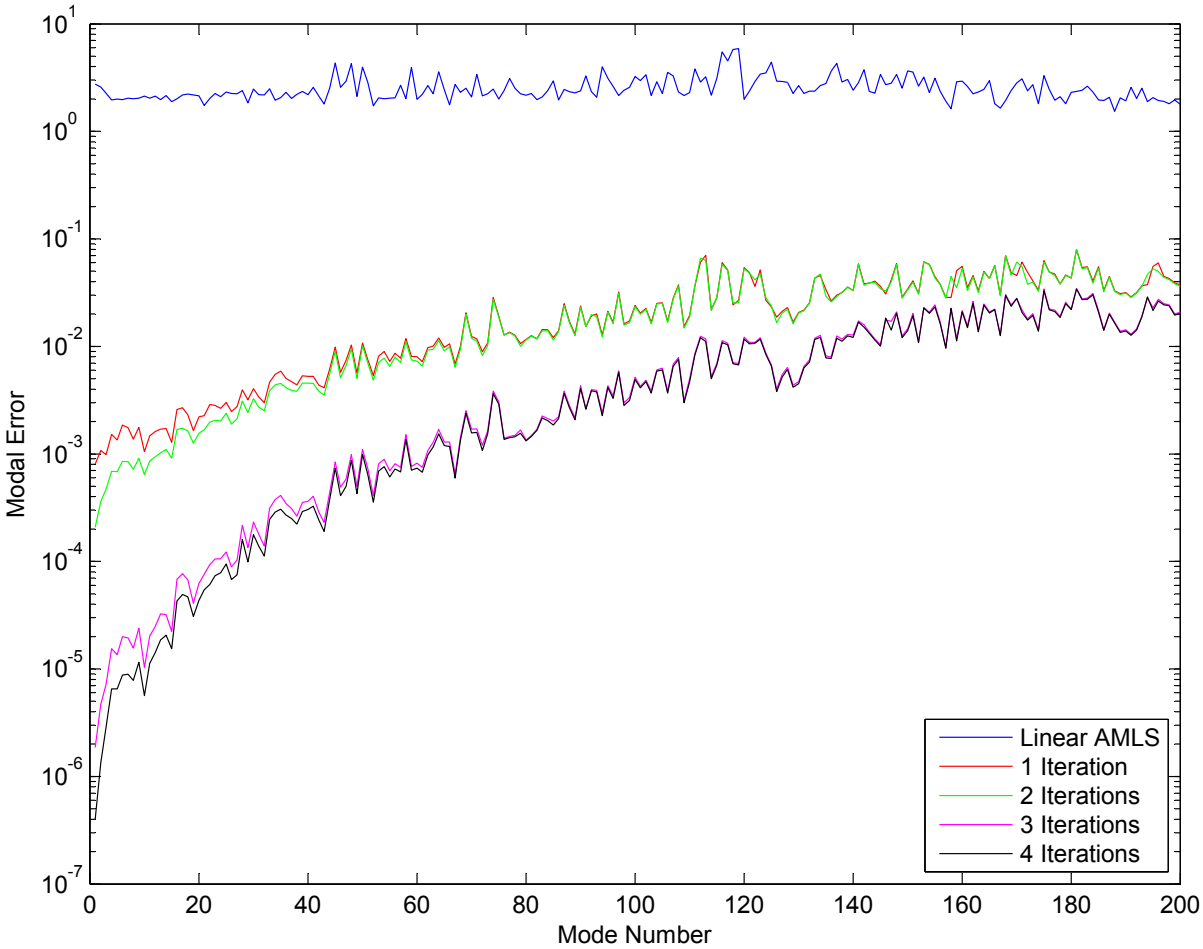


Figure: Modal errors computed with subspace iteration with AMLS utilizing 208 iteration vectors

Conclusions

AMLS is an efficient condensation method for computing a huge number of eigenmodes and frequency responses for large complex structures.

It usually provides approximate solutions which are less accurate than the ones obtained with standard Krylov type methods.

However, in many applications the underlying algebraic eigenproblem is a FE model of a continuous structure, and so the required level of accuracy is no more than what is furnished by the FE model.

Numerical examples demonstrate that the approximations to eigenvalues computed with AMLS are often of this limited but sufficient accuracy, whereas the modal errors of eigenvectors are usually still quite large.

In a recent paper we proposed a combination of AMLS with subspace iteration taking advantage of the block structure of the transformed stiffness matrix, but avoiding the use of the highly populated transformed mass matrix.

In this presentation we generalized this approach to gyroscopic eigenvalue problems to improve the interesting eigenpairs corresponding to the smallest positive eigenvalues although these are in the interior of the spectrum of the linearization.