A minimization problem for an elliptic eigenvalue problem with nonlinear dependence on the eigenparameter

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Outline

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Problem definition

Let Ω be a bounded, connected, open set in $\mathbb{R}^N$ with smooth boundary, and let $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, positive functions with $\alpha(\lambda) > \beta(\lambda)$ for every $\lambda \geq 0$.
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Find a measurable set $D \subset \Omega$ with $|D| = A$ such that the principle eigenvalue of

$$-\operatorname{div}\big((\alpha(\lambda)\chi_D + \beta(\lambda)\chi_{D^c})\nabla u\big) = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

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Such nonlinear eigenvalue problems appear as the Hamiltonian equation governing some quantum dot nanostructures, where $\alpha(\lambda)$ and $\beta(\lambda)$ correspond to the effective mass of the carrier (electron or hole) and the surrounding matrix, respectively, and λ is the ground state energy.

Linear case

Problem (1) is in fact a generalization of the linear case where α and β are two positive constants, $\alpha > \beta$. In this case the optimization problem is the problem of optimal design where two material phases are to be distributed inside a fixed region Ω.

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A simpler existence proof based on rearrangement techniques was given by Conca, Mahavedon & Sanz (2009).

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Based on the properties of Bessel functions, it has been proved by Mohammadi & Yousefnezhad (2014) that the conjecture is also not true for all dimensions $N \geq 2$.

Outline

Nonlinear problem

$$-\operatorname{div}\big(G(\lambda, x)\nabla u\big) = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{2}$$

where

$$G(\lambda, \cdot) \text{ is bounded for every } \lambda \geq 0, \tag{3}$$

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Multiplying (2) by $\varphi \in H_0^1(\Omega)$ and integrating by parts, one gets the following variational formulation of (1):

Find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$, $u \neq 0$ such that

$$\int_\Omega G(\lambda, x)\nabla u \cdot \nabla \varphi dx = \lambda \int_\Omega u\varphi dx, \tag{5}$$

for all φ in $H_0^1(\Omega)$.

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Thanks to the Riesz representation theorem, (2) is equivalent to the nonlinear eigenvalue problem

$$\mathcal{F}(\lambda)u=0, \tag{6}$$

where $\mathcal{F} : H_0^1(\Omega) \to H_0^1(\Omega)$, is a family of self-adjoint and bounded operators for $\lambda \geq 0$.

Variational characterization

Theorem: Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Consider the nonlinear eigenvalue problem $\mathcal{F}(\lambda)u = 0$, where $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is a family of self-adjoint and bounded operators on $\mathcal{H}$ depending continuously on a parameter $\lambda \in J$, and J is an open real interval.

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Assume that

(A_1) for every fixed $u \in \mathcal{H}$, $u \neq 0$ the real equation

$$f(\lambda; u) := \langle \mathcal{F}(\lambda)u, u \rangle = 0 \tag{7}$$

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Then problem (6) has a countable set of eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$, and it holds that

$$\lambda_j = \min_{\dim V = j} \max_{0 \neq u \in V} \mathcal{P}(u). \tag{9}$$

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$f(\lambda; u) = 0$ defines a functional $\mathcal{P}$ on $\mathcal{H} \setminus \{0\}$, which generalizes the Rayleigh quotient for linear eigenvalue problems with $\mathcal{F}(\lambda) = \lambda I - A$, and therefore it is called Rayleigh functional of (6), and (A_2) generalizes the definiteness requirement for linear pencils $\mathcal{F}(\lambda) = \lambda B - A$.

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The minmax characterization applies to G if we further assume

$$G(0, \cdot) > 0 \text{ a.e. in } \Omega, \tag{10}$$

and that $G(\lambda, x)$ is a decreasing function with respect to λ for every $x \in \Omega$, i.e.

$$G(\lambda_1, \cdot) \geq G(\lambda_2, \cdot) \text{ a.e. in } \Omega \text{ for } \lambda_1, \lambda_2 \geq 0 \text{ with } \lambda_1 < \lambda_2. \tag{11}$$

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Then obviously

$$f(\lambda, u) := \langle \mathcal{F}(\lambda)u, u\rangle = \lambda \|u\|^2_{L^2(\Omega)} - \int_\Omega G(\lambda, x)|\nabla u|^2 dx, \tag{12}$$

for $u \neq 0$ is strictly monotonically increasing and $f(0, u) < 0$.

Variational characterization

Theorem: Suppose that conditions (3), (4), (10) and (11) hold. Then the principal eigenvalue of (1) allows for a variational formulation

$$\lambda = \min_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^2(\Omega)}=1}} \mathcal{P}(v) = \int_\Omega G(\lambda, x)|\nabla u|^2 dx, \tag{13}$$

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Note that

$$G(\lambda, x) = \alpha(\lambda)\chi_D + \beta(\lambda)\chi_{D^c}$$

satisfies conditions (3), (4), (10) and (11) if the continuous functions $\alpha(\cdot)$ and $\beta(\cdot)$ are positive and decreasing, and $\alpha(\lambda) \geq \beta(\lambda)$ for every $\lambda \geq 0$.

Existencetheorem

Theorem: Let $\Omega = \mathcal{B}(0, R)$, and assume that the conditions (3), (4), (10) and (11) are satisfied. Then the minimization problem (1) is solvable, i.e. there exists $\widetilde{D} \subset \Omega$ with $|\widetilde{D}| = A$, such that

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Sketch of proof: For fixed $\lambda \in J = (0, +\infty)$ we consider the linear eigenvalue problem

$$-\mathrm{div}(G(\lambda, x)\nabla u) = \mu(\lambda, D)u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\mu(\lambda, D)$ is the principal eigenvalue that depends upon both λ and D.

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Applying results of Alvino et al. and Conca et al., the minimization problem

$$\zeta(\lambda) = \inf_{\substack{D \subset \Omega \\ |D| = A}} \mu(\lambda, D),$$

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Hence, the function $\zeta : J \to \mathbb{R}$ is well defined and

$$\zeta(\lambda) = \inf_{\substack{D\subset\Omega;\ |D|=A \\ u\in H_0^1(\Omega);\ \|u\|_{L^2(\Omega)}=1}} \int_\Omega G(\lambda, x)|\nabla u|^2 dx.$$

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The proof is completed showing that $\zeta(\cdot)$ has a fixed point.

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Modified bathtub principle

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Theorem: Let $f \in L^1(\Omega)$ be a nonnegative function and

$$\mathcal{M} = \{\eta \in L^\infty(\Omega) \, : \, \beta \le \eta(x) \le \alpha \text{ a.e. in } \Omega, \int_\Omega \eta(x)dx = \alpha A + \beta(|\Omega| - A)\}.$$

Then the minimization problem

$$\inf_{\eta \in \mathcal{M}} \int_\Omega f(x)\eta(x)dx,$$

is solvable by some $\widehat{\eta}(x) = \alpha\chi_{\widehat{D}}(x) + \beta\chi_{\widehat{D}^c}(x)$.

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With

$$t = \inf\{s \in \mathbb{R} : |\{x : \ f(x) \le s\}| \ge A\}$$

it holds that

$$|\widehat{D}| = A \text{ and } \{x : \ f(x) < t\} \subseteq \widehat{D} \subseteq \{x : \ f(x) \le t\}.$$

Decent approach

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Denote by u_n a normalized eigenfunction of problem (1) with $D = D_n$, let $f(x) := |\nabla u_n(x)|^2$ and fix $\lambda > 0$.

By the last theorem there exists $D_{n+1} \subset \Omega$ with $|D_{n+1}| = A$ such that

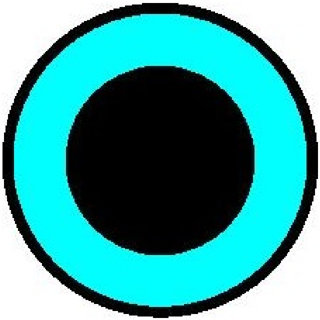
$$\int_\Omega (\alpha(\lambda)\chi_{D_n} + \beta(\lambda)\chi_{D_n{}^c})|\nabla u_n|^2 dx \geq \int_\Omega (\alpha(\lambda)\chi_{D_{n+1}} + \beta(\lambda)\chi_{D_{n+1}{}^c})\,|\nabla u_n|^2 dx,$$

and D_{n+1} can be obtained from the level set of $f(x) = |\nabla u_n|^2$.

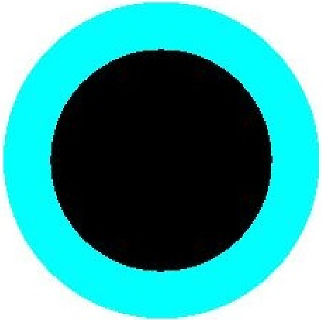
Outline

Example 1

$$\Omega = \mathcal{B}(0,1) \subset \mathbb{R}^2, \quad A/|\Omega| = 0.5$$
$$\alpha(\lambda) = 1/(1+\lambda), \quad \beta(\lambda) = 1/(1+\theta+\lambda), \quad \theta > 0$$



$\lambda_{\text{min}} = 1.95, \theta = 0.01$



$\lambda_{\text{min}} = 0.71, \theta = 10$

Example 2

$$\Omega = \mathcal{B}(0,1) \subset \mathbb{R}^3, \quad A/|\Omega| = 0.4$$

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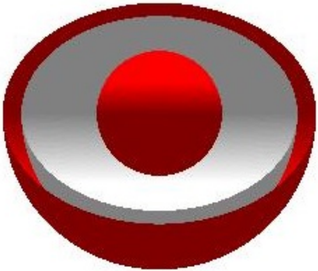
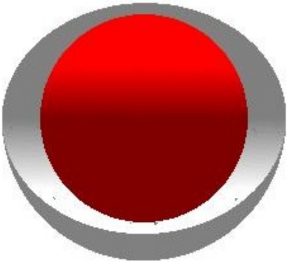


$\lambda_{\text{min}} = 2.67, \theta = 0.01$

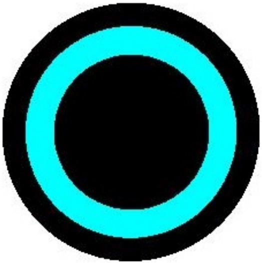


$\lambda_{\text{min}} = 1.16, \theta = 10$

Example 3

$$\alpha(\lambda) = \exp(-\lambda), \quad \beta(\lambda) = 3 - 1/\cos(\lambda)$$

$\Omega = \mathcal{B}(0,1) \subset \mathbb{R}^2,\ A/|\Omega| = 0.7$



$\lambda_{\text{min}} = 1.20$

$\Omega = \mathcal{B}(0,1) \subset \mathbb{R}^3,\ A/|\Omega| = 0.8$

$\lambda_{\text{min}} = 1.22$