A minimization problem for an elliptic eigenvalue problem with nonlinear dependence on the eigenparameter

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Introduction

Problem definition

Let $\Omega$ be a bounded, connected, open set in $\mathbb{R}^N$ with smooth boundary, and let $\alpha, \beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous, positive functions with $\alpha(\lambda) > \beta(\lambda)$ for every $\lambda \geq 0$.

Assume that $A$ is a positive number, $0 < A < |\Omega|$, where $|\cdot|$ denotes the Lebesgue measure.

Find a measurable set $D \subset \Omega$ with $|D| = A$ such that the principal eigenvalue of

$$-\text{div} \left( \alpha(\lambda) \chi_D + \beta(\lambda) \chi_{D^c} \right) \nabla u = \lambda u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega \quad (1)$$

is as small as possible.

Such nonlinear eigenvalue problems appear as the Hamiltonian equation governing some quantum dot nanostructures, where $\alpha(\lambda)$ and $\beta(\lambda)$ correspond to the effective mass of the carrier (electron or hole) and the surrounding matrix, respectively, and $\lambda$ is the ground state energy.
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\]

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Problem (1) is in fact a generalization of the linear case where $\alpha$ and $\beta$ are two positive constants, $\alpha > \beta$. In this case the optimization problem is the problem of optimal design where two material phases are to be distributed inside a fixed region $\Omega$. 
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A simpler existence proof based on rearrangement techniques was given by Conca, Mahavedon & Sanz (2009).
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This suggests for higher dimensions that $B(0, R^*)$ is a natural candidate to be the optimal domain, and this conjecture has been supported by numerical tests by Conca, Mahavedon & Sanz (2009a).
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In spite of the above evidences, it was established in Conca, Lurian & Mahadevan (2012) that the conjecture is not true at least in two- or three-dimensional spaces when $\alpha$ and $\beta$ are close to each other (low contrast regime) and $A$ is sufficiently large. This makes clear that the optimal domain can not be a ball.
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Based on the properties of Bessel functions, it has been proved by Mohammadi & Yousefnezhad (2014) that the conjecture is also not true for all dimensions $N \geq 2$. 
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Nonlinear problem

\[- \text{div}(G(\lambda, x) \nabla u) = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \]  

where

\[G(\lambda, \cdot) \text{ is bounded for every } \lambda \geq 0, \]
\[G(\cdot, x) \text{ is a continuous function.} \]
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\[- \text{div} (G(\lambda, x) \nabla u) = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \quad (2)\]

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\[G(\lambda, \cdot) \text{ is bounded for every } \lambda \geq 0, \quad (3)\]
\[G(\cdot, x) \text{ is a continuous function.} \quad (4)\]

Multiplying (2) by \(\varphi \in H^1_0(\Omega)\) and integrating by parts, one gets the following variational formulation of (1):

Find \(\lambda \in \mathbb{R}\) and \(u \in H^1_0(\Omega), u \neq 0\) such that

\[
\int_{\Omega} G(\lambda, x) \nabla u \cdot \nabla \varphi \, dx = \lambda \int_{\Omega} u \varphi \, dx, \quad (5)
\]

for all \(\varphi \in H^1_0(\Omega)\).
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for all \( \varphi \) in \( H^1_0(\Omega) \).

Thanks to the Riesz representation theorem, (2) is equivalent to the nonlinear eigenvalue problem

\[\mathcal{F}(\lambda) u = 0, \quad (6)\]

where \( \mathcal{F} : H^1_0(\Omega) \rightarrow H^1_0(\Omega) \), is a family of self-adjoint and bounded operators for \( \lambda \geq 0 \).
Variational characterization

Theorem: Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Consider the nonlinear eigenvalue problem $\mathcal{F}(\lambda)u = 0$, where $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is a family of self-adjoint and bounded operators on $\mathcal{H}$ depending continuously on a parameter $\lambda \in J$, and $J$ is an open real interval.
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Assume that

$(A_1)$ for every fixed $u \in \mathcal{H}$, $u \neq 0$ the real equation

$$f(\lambda; u) := \langle \mathcal{F}(\lambda)u, u \rangle = 0$$

has exactly one solution $\lambda := P(u) \in J$. 

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(A2) for every $u \neq 0$ and every $\lambda \in J$ with $\lambda \neq \mathcal{P}(u)$ it holds that

$$(\lambda - \mathcal{P}(u))f(\lambda, u) > 0,$$
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Then problem (6) has a countable set of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$, and it holds that

$$\lambda_j = \min_{\dim V=j} \max_{0 \neq u \in V} \mathcal{P}(u).$$
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\(f(\lambda; u) = 0\) defines a functional \(P\) on \(H \setminus \{0\}\), which generalizes the Rayleigh quotient for linear eigenvalue problems with \(F(\lambda) = \lambda I - A\), and therefore it is called Rayleigh functional of (6), and \((A_2)\) generalizes the definiteness requirement for linear pencils \(F(\lambda) = \lambda B - A\).
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The minmax characterization applies to \(G\) if we further assume

\[ G(0, \cdot) > 0 \text{ a.e. in } \Omega, \quad (10) \]

and that \(G(\lambda, x)\) is a decreasing function with respect to \(\lambda\) for every \(x \in \Omega\), i.e.

\[ G(\lambda_1, \cdot) \geq G(\lambda_2, \cdot) \text{ a.e. in } \Omega \text{ for } \lambda_1, \lambda_2 \geq 0 \text{ with } \lambda_1 < \lambda_2. \quad (11) \]
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Then obviously

\[ f(\lambda, u) := \langle F(\lambda)u, u \rangle = \lambda \|u\|^2_{L^2(\Omega)} - \int_{\Omega} G(\lambda, x)|\nabla u|^2 \, dx, \]  

(12)

for \(u \neq 0\) is strictly monotonically increasing and \(f(0, u) < 0\).
Theorem: Suppose that conditions (3), (4), (10) and (11) hold. Then the principal eigenvalue of (1) allows for a variational formulation

$$\lambda = \min_{v \in H^1_0(\Omega)} P(v) = \int_{\Omega} G(\lambda, x) |\nabla u|^2 dx,$$

with $u$ as the associated eigenfunction.

From the variational characterization for the first eigenvalue we obtain that the optimization problem (1) has a solution if $\Omega = B(0, R)$. Note that $G(\lambda, x) = \alpha(\lambda) \chi_D + \beta(\lambda) \chi_{D^c}$ satisfies conditions (3), (4), (10) and (11) if the continuous functions $\alpha(\cdot)$ and $\beta(\cdot)$ are positive and decreasing, and $\alpha(\lambda) \geq \beta(\lambda)$ for every $\lambda \geq 0$. 
Theorem: Suppose that conditions (3), (4), (10) and (11) hold. Then the principal eigenvalue of (1) allows for a variational formulation

\[
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\[ \lambda = \min_{v \in H_0^1(\Omega)} \mathcal{P}(v) = \int_{\Omega} G(\lambda, x) |\nabla u|^2 \, dx, \]  
\[ \|v\|_{L^2(\Omega)} = 1 \]

with \( u \) as the associated eigenfunction.

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Note that

\[ G(\lambda, x) = \alpha(\lambda) \chi_D + \beta(\lambda) \chi_{D^c} \]

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Theorem: Let $\Omega = B(0, R)$, and assume that the conditions (3), (4), (10) and (11) are satisfied. Then the minimization problem (1) is solvable, i.e. there exists $\tilde{D} \subset \Omega$ with $|\tilde{D}| = A$, such that

$$\hat{\lambda} = \lambda(\tilde{D}) = \inf_{D \subset \Omega} \lambda(D).$$
Existence result for nonlinear problem

Theorem: Let \( \Omega = B(0, R) \), and assume that the conditions (3), (4), (10) and (11) are satisfied. Then the minimization problem (1) is solvable, i.e. there exists \( \tilde{D} \subset \Omega \) with \( |\tilde{D}| = A \), such that

\[
\hat{\lambda} = \lambda(\tilde{D}) = \inf_{D \subset \Omega, |D| = A} \lambda(D).
\]

Sketch of proof: For fixed \( \lambda \in J = (0, +\infty) \) we consider the linear eigenvalue problem

\[
-\text{div}(G(\lambda, x) \nabla u) = \mu(\lambda, D)u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

where \( \mu(\lambda, D) \) is the principal eigenvalue that depends upon both \( \lambda \) and \( D \).
Applying results of Alvino et al. and Conca et al., the minimization problem

$$\zeta(\lambda) = \inf_{D \subset \Omega, |D| = A} \mu(\lambda, D),$$

admits a radially symmetric solution $u$ for every $\lambda \geq 0$. 
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admits a radially symmetric solution \( u \) for every \( \lambda \geq 0 \).

Hence, the function \( \zeta : J \rightarrow \mathbb{R} \) is well defined and

\[ \zeta(\lambda) = \inf_{D \subset \Omega; \ |D| = A} \int_\Omega G(\lambda, x) |\nabla u|^2 dx. \]

\( u \in H^1_0(\Omega); \ |u|_{L^2(\Omega)} = 1 \)
Applying results of Alvino et al. and Conca et al., the minimization problem

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\[ \zeta(\lambda) = \inf_{D \subset \Omega, |D|=A} \int_\Omega G(\lambda, x) |\nabla u|^2 \, dx. \]

The proof is completed showing that \( \zeta(\cdot) \) has a fixed point.
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**Theorem:** Let $f \in L^1(\Omega)$ be a nonnegative function and $\mathcal{M} = \{\eta \in L^\infty(\Omega) : \beta \leq \eta(x) \leq \alpha \text{ a.e. in } \Omega, \int_\Omega \eta(x)dx = \alpha A + \beta(|\Omega| - A)\}$. Then the minimization problem

$$\inf_{\eta \in \mathcal{M}} \int_\Omega f(x)\eta(x)dx,$$

is solvable by some $\widehat{\eta}(x) = \alpha \chi_{\widehat{\mathcal{D}}}(x) + \beta \chi_{\widehat{\mathcal{D}}^c}(x)$. 
Modified bathtub principle

Based on a modification of the bathtub principle we used a descent approach to numerically determine a minimizing set.

Theorem: Let $f \in L^1(\Omega)$ be a nonnegative function and

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Then the minimization problem

$$\inf_{\eta \in \mathcal{M}} \int_\Omega f(x)\eta(x)dx,$$

is solvable by some $\widehat{\eta}(x) = \alpha \chi_{\widehat{D}}(x) + \beta \chi_{\widehat{D}^c}(x)$.

With

$$t = \inf\{s \in \mathbb{R} : |\{x : f(x) \leq s\}| \geq A\}$$

it holds that

$$|\widehat{D}| = A \text{ and } \{x : f(x) < t\} \subseteq \widehat{D} \subseteq \{x : f(x) \leq t\}.$$
The bathtub principle is the basis for constructing a sequence of domains $D_n$ such that $|D_n| = A$ for every $n$ and

$$\lambda(D_{n+1}) \leq \lambda(D_n).$$
Decent approach

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Denote by $u_n$ a normalized eigenfunction of problem (1) with $D = D_n$, let $f(x) := |\nabla u_n(x)|^2$ and fix $\lambda > 0$. 
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$$\lambda(D_{n+1}) \leq \lambda(D_n).$$

Denote by $u_n$ a normalized eigenfunction of problem (1) with $D = D_n$, let $f(x) := |\nabla u_n(x)|^2$ and fix $\lambda > 0$.

By the last theorem there exists $D_{n+1} \subset \Omega$ with $|D_{n+1}| = A$ such that

$$\int_{\Omega} (\alpha(\lambda) \chi_{D_n} + \beta(\lambda) \chi_{D_n^c}) |\nabla u_n|^2 dx \geq \int_{\Omega} (\alpha(\lambda) \chi_{D_{n+1}} + \beta(\lambda) \chi_{D_{n+1}^c}) |\nabla u_n|^2 dx,$$

and $D_{n+1}$ can be obtained from the level set of $f(x) = |\nabla u_n|^2$. 
Example 1

\[ \Omega = B(0, 1) \subset \mathbb{R}^2, \quad A/|\Omega| = 0.5 \]
\[ \alpha(\lambda) = \frac{1}{1 + \lambda}, \quad \beta(\lambda) = \frac{1}{1 + \theta + \lambda}, \quad \theta > 0 \]

\[ \lambda_{\text{min}} = 1.95, \theta = 0.01 \]
\[ \lambda_{\text{min}} = 0.71, \theta = 10 \]
Example 2

\[ \Omega = B(0, 1) \subset \mathbb{R}^3, \quad A/|\Omega| = 0.4 \]

\[ \alpha(\lambda) = \frac{1}{1 + \lambda}, \quad \beta(\lambda) = \frac{1}{1 + \theta + \lambda}, \quad \theta > 0 \]

\[ \lambda_{\text{min}} = 2.67, \theta = 0.01 \]

\[ \lambda_{\text{min}} = 1.16, \theta = 10 \]
Example 3

\[ \alpha(\lambda) = \exp(-\lambda), \quad \beta(\lambda) = 3 - 1/\cos(\lambda) \]

\[ \Omega = B(0, 1) \subset \mathbb{R}^2, \quad A/|\Omega| = 0.7 \]

\[ \Omega = B(0, 1) \subset \mathbb{R}^3, \quad A/|\Omega| = 0.8 \]

\[ \lambda_{\min} = 1.20 \]

\[ \lambda_{\min} = 1.22 \]