Variational characterization of eigenvalues of a non–symmetric eigenvalue problem governing elastoacoustic vibrations

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FE approximations based on third type of modeling are favored today, since one obtains symmetric matrix eigenvalue problems: easier to solve, and variational characterizations of eigenvalues allow to use standard spectral approximation theory for convergence analysis.

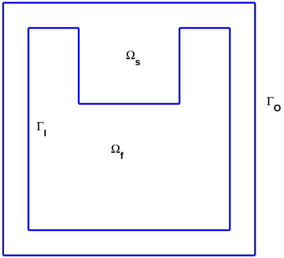
Outline

1. Problem definition and properties
2. Variational characterization
3. Discretized problem
4. Structure preserving iterative projection methods
5. Numerical Results
6. Conclusions

Outline

Problem definition

$$\text{div}\,[\sigma(u)] + \omega^2 \rho_s u = 0 \text{ in } \Omega_s,$$
$$\Delta p + \frac{\omega^2}{c^2} p = 0 \text{ in } \Omega_f,$$
$$\sigma(u)\cdot n - pn = 0 \text{ on } \Gamma_I,$$
$$\nabla p \cdot n + \omega^2 \rho_f u \cdot n = 0 \text{ on } \Gamma_I,$$
$$u = 0 \text{ on } \Gamma_D,$$
$$\nabla p \cdot n = 0 \text{ on } \Gamma_N,$$



where
- u: solid displacement
- p: fluid pressure
- $\lambda = \omega^2$: eigenparameter
- $\sigma(u)$: linearized stress tensor
- ρ_s, ρ_f: densities of solid and fluid

The first interface condition results from the action of the pressure forces exerted by the fluid on the structure,
and the second one expresses the continuity of the fluid and structural normal displacement at the interface.

Variational formulation

To write the eigenproblem in variational form let

$$a_s : \quad H^1_{\Gamma_D}(\Omega_s)^d \times H^1_{\Gamma_D}(\Omega_s)^d \to \mathbb{R}, \qquad a_s(v,u) = \int_{\Omega_s} \sigma(u) : \nabla v dx$$

$$c : \quad H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \to \mathbb{R}, \qquad c(v,p) = \int_{\Gamma_I} -p\, n \cdot v ds$$

$$a_f : \quad H^1(\Omega_f) \times H^1(\Omega_f) \to \mathbb{R}, \qquad a_f(q,p) = \int_{\Omega_f} \frac{1}{\rho_f} \nabla p \cdot \nabla q\, dx$$

$$b_s : \quad H^1_{\Gamma_D}(\Omega_s)^d \times H^1_{\Gamma_D}(\Omega_s)^d \to \mathbb{R}, \qquad b_s(v,u) = \int_{\Omega_s} \rho_s u v dx$$

$$b_f : \quad H^1(\Omega_f) \times H^1(\Omega_f) \to \mathbb{R}, \qquad b_f(q,p) = \int_{\Omega_f} \frac{1}{\rho_f c^2} p\, q\, dx$$

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Find $\lambda \in \mathbb{C}$ and nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ such that

$$a_s(v,u) + c(v,p) = \lambda b_s(v,u) \text{ and} \tag{1a}$$

$$a_f(q,p) = \lambda(-c(u,q) + b_f(q,p)). \tag{1b}$$

for all $(v,q) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$.

Properties

For the linearized strain tensor ϵ in the solid we assume that the strain-stress relationship satisfies

$$\sigma(v) : \nabla v \geq C_1 \epsilon(v) : \epsilon(v)$$

for some constant $C_1 > 0$, such that Korn's second inequality implies that a_s is a coercive bilinear form.

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(i) The eigenvalue problem (1) and its adjoint problem have a zero eigenvalue with corresponding one dimensional eigenspaces (u_0, p_0) and $(0, p_0)$ where $p_0 \equiv 1$ and u_0 is the solution of equation (1a).

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(iii) Eigenfunctions (u_1, p_1) and (u_2, p_2) corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal with respect to the inner product

$$\langle (u, p), (v, q) \rangle := a_s(u, v) + b_f(p, q).$$

Properties ctn.

(iv) Assume that (u_1, p_1) is an eigensolution of problem (1) and $(\hat{u}_2, \hat{p}_2)$ an eigensolution of the adjoint problem corresponding to the eigenvalues λ_1 and λ_2, respectively.
If $\lambda_1 \neq \lambda_2$ then it holds that

$$a_s(\hat{u}_2, u_1) + c(\hat{u}_2, p_1) + a_f(\hat{p}_2, p_1) = b_s(\hat{u}_2, u_1) - c(u_1, \hat{p}_2) + b_f(p_1, \hat{p}_2) = 0.$$

If $\lambda_1 = \lambda_2$ and $(\hat{u}_2, \hat{p}_2) = (\lambda_1 u_1, p_1)$ then it holds that

$$a_s(\hat{u}_2, u_1) + c(\hat{u}_2, p_1) + a_f(\hat{p}_2, p_1) \geq 0 \text{ and}$$
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(v) The eigenvalue problem (1) has a countable number of eigenvalue $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$ the eigenspace of which are all finite dimensional.

Outline

Rayleigh functional

Let (u, p) be an eigenelement of (1) corresponding to λ. Using the solution $(\lambda u, p)$ of the adjoint problem as a test function in equation (1) we obtain

$$\lambda a_s(u,u) + \lambda c(u,p) + a_f(p,p) = \lambda^2 b_s(u,u) - \lambda c(u,p) + \lambda b_f(p,p),$$

i.e. λ it is a zero of the function

$$g(\lambda;(u,p)) := \lambda^2 b_s(u,u) + \lambda\left(b_f(p,p) - a_s(u,u) - 2c(u,p)\right) - a_f(p,p). \quad (2)$$

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If $b_s(u,u) > 0$, this equation is quadratic in λ and the question arises which of its roots is the eigenvalue λ of (1).

Lemma Let (u, p) be an eigenfunction of problem (1). Then the maximal root of $g(\lambda; (u,p))$ is an eigenvalue of (1) corresponding to (u, p).

Rayleigh functional

This suggests to introduce an eigenvalue approximation for some general nonzero $(u, p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ by g and we define the nonlinear Rayleigh functional as the maximal root of $g(\cdot; (u, p))$.

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Definition

The functional $r : H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)\backslash\{0\} \to \mathbb{R}$, where any nonzero $(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f)$ is mapped to the maximal root of $g(\cdot,(u,p))$ is called the nonlinear Rayleigh functional, i.e.

$$r(u,p) = \begin{cases} \Delta + \sqrt{\Delta^2 + \frac{a_f(p,p)}{b_s(u,u)}} & \text{if } b_s(u,u) \neq 0, \\ \frac{a_f(p,p)}{b_f(p,p)} & \text{if } b_s(u,u) = 0, \end{cases}$$

where

$$\Delta = \frac{1}{2}\frac{-b_f(p,p) + a_s(u,u) + 2c(u,p)}{b_s(u,u)}.$$

Rayleigh functional

Lemma

Let $I = \mathbb{N}$ or $I = \{1, \dots, m\}$ an index set, $(u_i, p_i)_{i \in I}$ linearly independent eigenfunctions of (1) corresponding to distinct eigenvalues $(\lambda_i)_{i \in I}$ enumerated in ascendending order, $\lambda_i < \lambda_j$ if $i < j$, and let

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(ii) It holds that

$$\lambda_1 \le r(u, p) \le \sup_{i \in I} \lambda_i.$$

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$$\lambda_k = \min_{\substack{S_k \subset H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \\ \dim S_k = k}} \; \max_{0 \neq (u,p) \in S_k} r(u,p).$$

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(iii) (maxmin characterization)

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where

$$S^\perp := \{(u,p) \in H^1_{\Gamma_D}(\Omega_s)^d \times H^1(\Omega_f) \ : \ a_s(u,v) + b_f(p,q) = 0 \text{ for } (v,q) \in S\}.$$

Outline

Discretization

Discretizing the elastoacoustic problem with finite elements where the triangulation obeys the geometric partition into the fluid and the structure domain one obtains a non–symmetric matrix eigenvalue problem

$$Kx := \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} =: \lambda Mx, \tag{3}$$

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The standard spectral approximation theory applies to prove convergence results for Galerkin type methods.

Common numerical approach

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One first determines the eigenpairs of the symmetric and definite eigenvalue problems

$$K_s x_s = \omega_s M_s x_s \quad \text{and} \quad K_f x_f = \omega_f M_f x_f \tag{4}$$

by the Lanczos method or automated multi-level sub-structuring, and then projects problem (3) to diag$\{X_s, X_f\}$, where the columns of X_s and X_f are the eigenmodes of problem (4) not exceeding a given cut-off level.

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The projected problem

$$\begin{bmatrix} X_s^T K_s X_s & X_s^T C X_f \\ 0 & X_f^T K_f X_f \end{bmatrix} \begin{bmatrix} y_s \\ y_f \end{bmatrix} = \lambda \begin{bmatrix} X_s^T M_s X_s & 0 \\ -X_f^T C^T X_s & X_f^T M_f X_f \end{bmatrix} \begin{bmatrix} y_s \\ y_f \end{bmatrix} \tag{5}$$

has the same structure as the original problem but is of much smaller dimension.

Example

Consider a two dimensional coupled structure consisting of steel and air portions discretized by finite elements. The resulting problem has 120473 degrees of freedom, 67616 of which are located in the solid region and 52857 in the fluid part.

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Table: STEEL-AIR-STRUCTURE

coupled	steel	air	rel.dev. [%]	proj. [%]
0.00		0.00		
41.25		41.32	0.16	2.5e-4
48.67		48.71	0.08	2.7e-4
56.96	56.90		0.11	2.2e-3
75.55	75.51		0.06	3.3e-3
93.18		93.19	0.01	1.0e-4
129.99		130.04	0.05	6.1e-4
150.94	151.03		0.06	3.5e-3
158.16		158.18	0.01	1.8e-4
186.64	186.66		0.12	4.2e-3

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Table: STEEL-WATER-STRUCTURE

coupled	steel	water	proj.	rel.error [%]
0.00	56.90	0.00	0.00	
28.01	75.51	178.63	28.33	1.2
41.54	151.03	210.64	43.01	3.5
92.73	186.66	402.93	101.98	10.0
124.70	225.54		133.60	7.1
138.26	451.76		141.87	2.6
270.40	472.45		285.18	5.5
321.79			343.80	6.8
388.73			416.87	7.2
402.77			439.83	9.2

Structure preserving projection

Proposition

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and let $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \dots \le \tilde{\lambda}_k$ *be the eigenvalues of the projected problem*

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Assume that $V = \begin{pmatrix} V_s & 0 \\ 0 & V_f \end{pmatrix} \in \mathbb{R}^{s+f \times k}$ *has maximal rank* k*.*

Let

$$K_V := V^T K V = \begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix}, M_V := V^T M V = \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix};$$

and let $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \dots \le \tilde{\lambda}_k$ *be the eigenvalues of the projected problem*

$$K_V y = \tilde{\lambda} M_V y.$$

Then it holds that

$$\lambda_j \le \tilde{\lambda}_j, \quad j = 1, \dots, k.$$

In particular, the jth eigenvalue λ_j does not exceed the jth eigenvalue of the structure eigenproblems $K_s x_s = \lambda M_s x_s$ and of the fluid eigenvalue problem $K_f x_f = \lambda M_f x_f$.

Outline

Iterative projection method

Require: Initial basis V with $V^H V = I$; set $m = 1$
while $m \leq$ number of wanted eigenvalues **do**
compute eigenpair (μ, y) of projected problem $V^T(K - \mu M)Vy = 0$.
determine Ritz vector $u = Vy$, $\|u\| = 1$,and residual $r = (K - \mu M)u$
if $\|r\| < \varepsilon$ **then**
accept approximate eigenpair $\lambda_m = \mu$, $x_m = u$; increase $m \leftarrow m + 1$
reduce search space V if necessary
choose approximation (λ_m, u) to next eigenpair, and compute $r = (K - \lambda_m M)u$
end if
expand search space $V = [V, v_{\text{new}}]$
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Main tasks

- expand search space
- choose eigenpair of projected problem (locking, purging)

Jacobi-Davidson method

Expanding the current subspace V_k by

$$x^{(k+1)} := (K - \rho_k M)^{-1} M x^{(k)}$$

is equivalent to expanding it by $t := x^{(k+1)} + \alpha x^{(k)}$ for every $\alpha \in \mathbb{R}$.

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The most robust expansion of this type with respect to inexact solves of $(K - \rho_k M)x^{(k+1)} = Mx^{(k)}$ satisfies ${x^{(k)}}^T \tilde{M} t = 0$, $\tilde{M} := \text{diag}\{K_s, M_f\}$ (cf. V. 2007)

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which leads to the equivalent so called correction equation ($\hat{M} := \mathrm{diag}\{M_s, K_f\}$)

$$(I - \frac{\hat{M} x x^T}{x^T \hat{M} x})(K - \rho_k M)(I - \frac{x x^T \tilde{M}}{x^T \tilde{M} x})t = -(K - \rho_k M)x, \quad t^T \tilde{M} x = 0$$

for a given eigenpair approximation (ρ_k, x).

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Iterative projection methods of this type are known as Jacobi-Davidson method and were introduced by Sleijpen and van der Vorst (1996).

structure preserving Jacobi-Davidson method

The standard approach of the Jacobi-Davidson method, i.e. considering

$$V^T KVy = \lambda V^T MVy \tag{6}$$

where V is expanded by the solution of the correction equation destroys the structure of problem (1).

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(6) often has non-real eigenvalues and eigenvectors.

To preserve the structure of the fluid-solid eigenproblems and to ensure real eigenvalues of the projected eigenproblem we expand the ansatz space in every step by 2 vectors $\begin{pmatrix} t_s \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ t_f \end{pmatrix}$ where $t = \begin{pmatrix} t_s \\ t_f \end{pmatrix}$ approximately solves the correction equation. Hence, we use structure preserving projection matrices

$$V = \begin{pmatrix} V_s & 0 \\ 0 & V_f \end{pmatrix}$$

and obtain

$$V^TKV = \begin{pmatrix} V_s^TK_sV_s & V_s^TCV_f \\ 0 & V_f^TK_fV_f \end{pmatrix} \text{ and } V^TMV = \begin{pmatrix} V_s^TM_sV_s & 0 \\ -V_f^TC^TV_s & V_f^TM_fV_f \end{pmatrix}.$$

Structure preserving Jacobi-Davidson method

From the minmax characterization we obtain the following monotonicity result:

Structure preserving Jacobi-Davidson method

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Proposition

Let $\lambda_j^{(k)}$ be the j smallest eigenvalue of the kth projected eigenproblem of the structure preserving Jacobi-Davidson method.

Then it holds that

$$\lambda_j \leq \lambda_j^{(k+1)} \leq \lambda_j^{(k)} \text{ for } k = 1, \dots, \dim(V_k).$$

Structure preserving Jacobi-Davidson method

REQUIRE Initial basis $V = \text{diag}\{V_s, V_f\}$, $V_s^T K_s V_s = I$, $V_f^T M_f V_f = I$,
$m = 1$; θ_m=0;

determine preconditioner $L \approx (K - \sigma M)^{-1}$, $\sigma \approx \lambda_{\min}$
while $\theta_m \le$ maxeig **do**
  solve the projected problem

$$\begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix} = \theta \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix}$$

  choose m smallest eigval. θ_m and corresp. eigvec. $(y_s^T, y_f^T)^T$
  determine Ritz vector $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and residual $r = (K - \theta_m M)x$
  if $\|r\|/\|x\| < \epsilon$
    while $\|r\|/\|x\| < \epsilon$
      accept approximate mth eigenpair (θ_m, x); increase $m \leftarrow m + 1$;
      choose m smallest eigval. θ_m and corresp. eigvec. $(y_s^T, y_f^T)^T$
      determ. Ritz vec. $x = \begin{pmatrix} V_s y_s \\ V_f y_f \end{pmatrix}$ and res. $r = (K - \theta_m M)x$
    endwhile

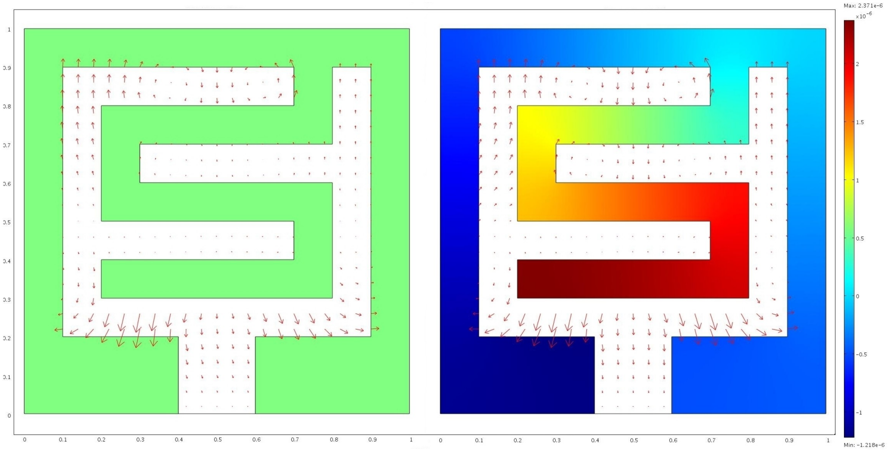
    reduce search space V if indicated
    determine new preconditioner $L \approx (K - \theta M)^{-1}$ if necessary
    endif
compute approximate solution $t = (t_s^T, t_f^T)^T$ of the correction equation

$$(I - \frac{\hat{M}xx^T}{x^T\hat{M}x})(K - \rho_m M)(I - \frac{xx^T\tilde{M}}{x^T\tilde{M}x})t = r, x^T\tilde{M}t = 0$$

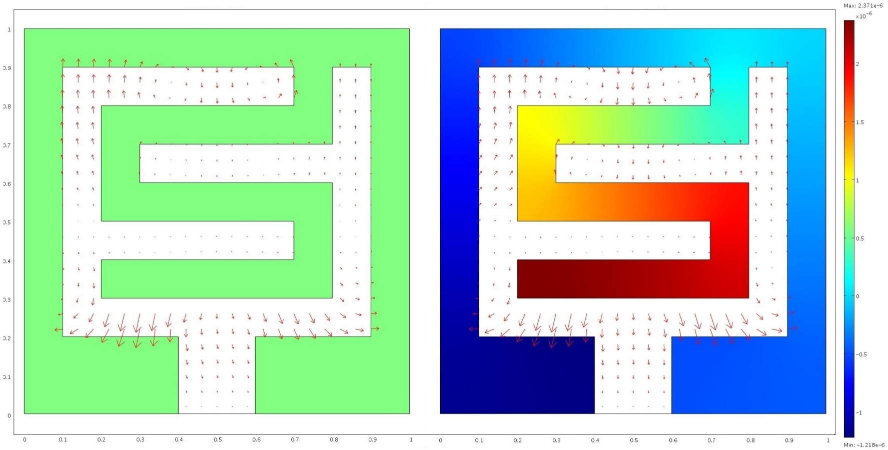
orthogonalize $v_s = t_s - K_s V_s V_s^T K_s t_s$, $v_f = t_f - M_f V_f V_f^T M_f t_f$
if $\|v_s\|_{K_s} >$ tol expand $V_s \leftarrow [V_s, v_s/\|v_s\|_{K_s}]$
if $\|v_f\|_{M_f} >$ tol expand $V_f \leftarrow [V_f, v_f/\|v_f\|_{M_f}]$
update projected problem
endwhile

Outline

Numerical Example



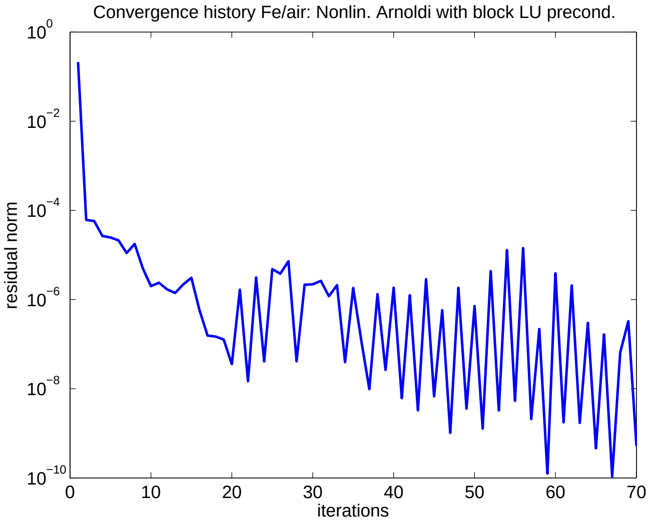
Numerical Example



Consider a finite element model of a coupled steel/air system with 120473 DoFs (67616 Fe, 52857 air).

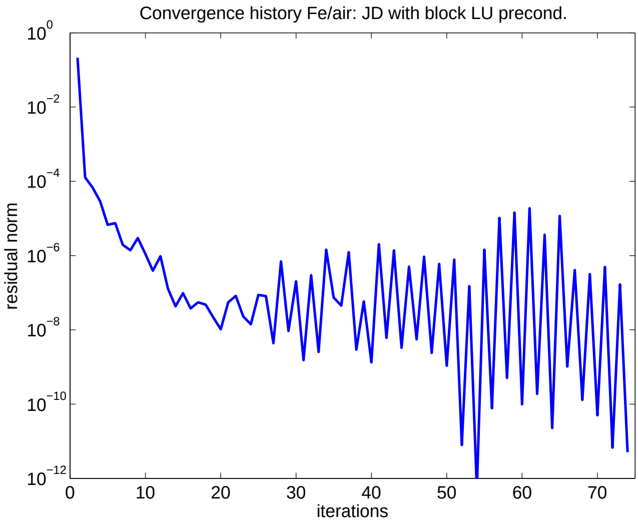
Determine all 23 eigenfrequencies less than 500 Hz.

Numerical Example; Fe-air



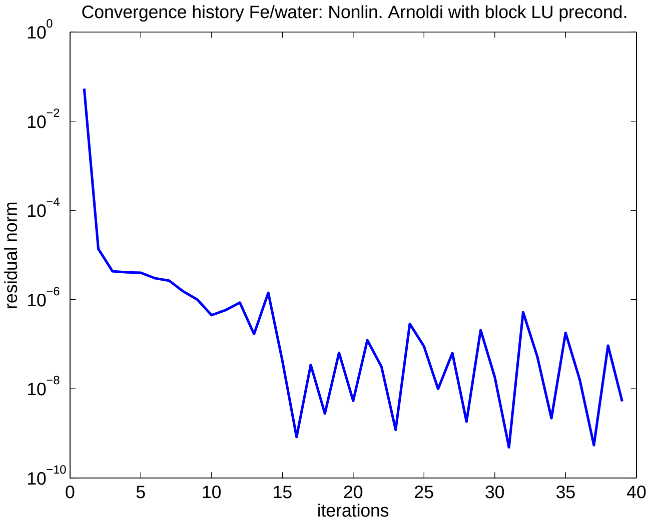
CPU time (Pentium D, 3.4 GHz, 4GB) : 19.5 sec.

Numerical Example; Fe-air



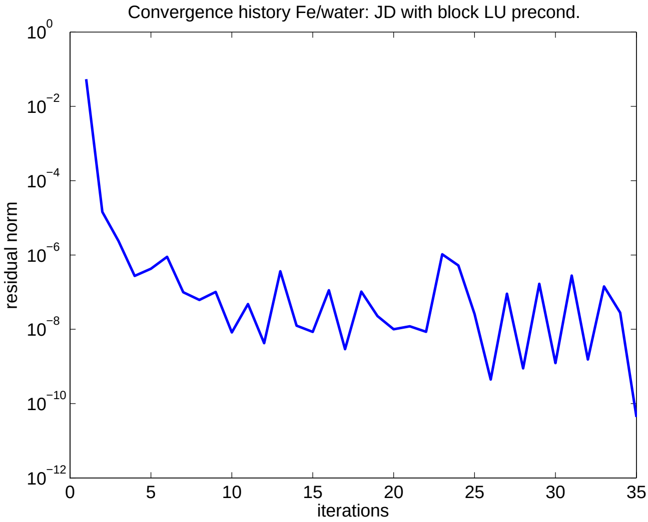
CPU time : 77.9 sec.

Numerical Example; Fe-H_2O



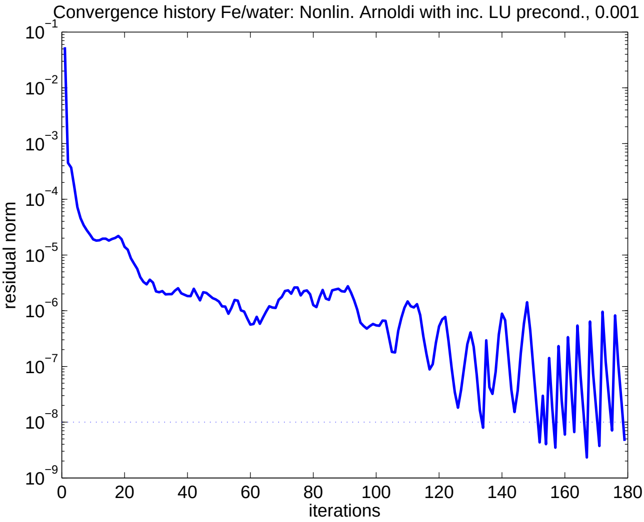
CPU time : 10.4 sec.

Numerical Example; Fe-H_2O



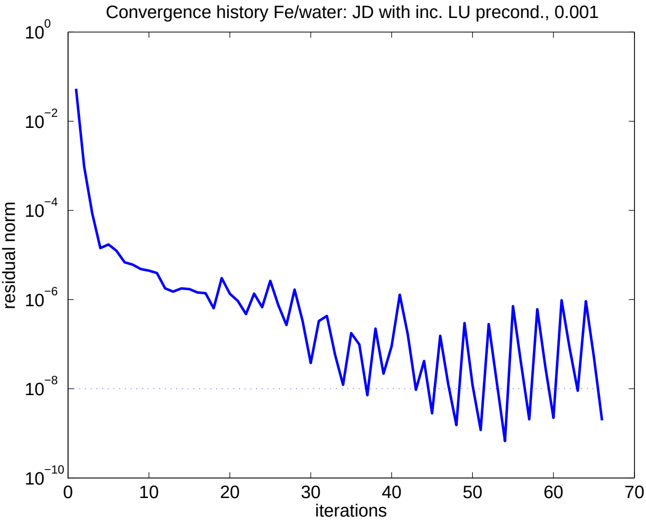
CPU time : 30.0 sec.

Numerical Example; Fe-H_2O



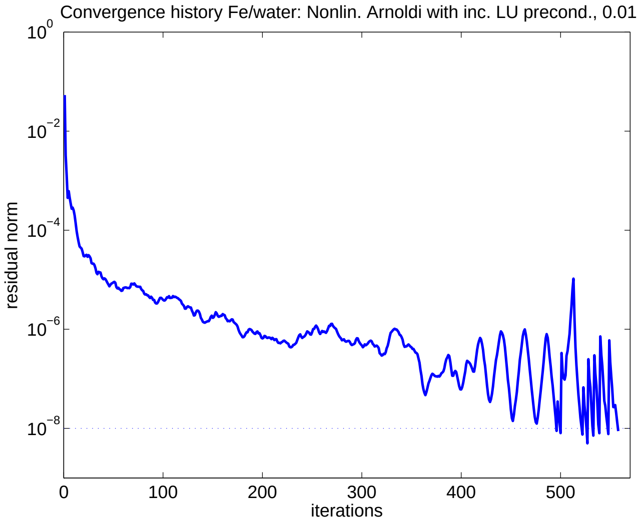
CPU time : 113.4 sec.

Numerical Example; Fe-H_2O



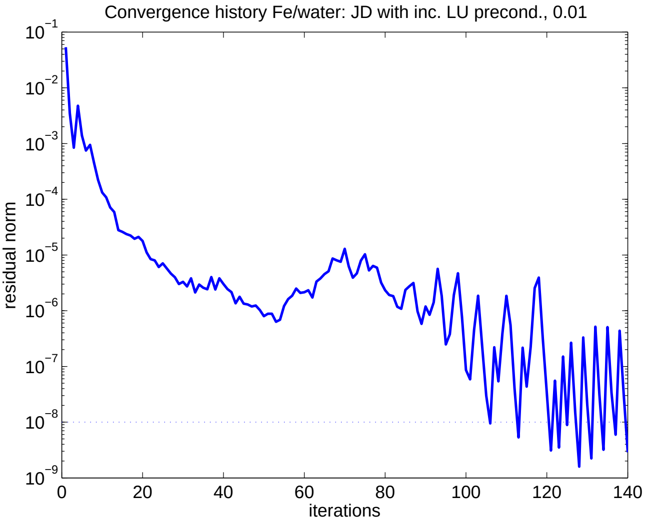
CPU time : 60.5 sec.

Numerical Example; Fe-H_2O



CPU time : 3411.1 sec.

Numerical Example; Fe-H_2O



CPU time : 125.6 sec.

Outline

Conclusions

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Conclusions

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- Rayleigh functional iteration converges locally and cubically to eigenvalues and eigenvectors.
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- For accurate preconditioners the Nonlinear Arnoldi method is superior to the Jacobi–Davidson method, but JD is much more robust with respect to coarse preconditioners.
- An equivalent symmetric problem of doubled size allows for a generalization of AMLS which can be implemented such that it requires the same effort as a linear definite eigenproblem of the original size.
  An a priori bound for AMLS can be given

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Thanks for your attention!