Measure-perturbed one-dimensional Schrödinger operators

A continuum model for quasicrystals

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Introduction

In quantum mechanics the time evolution of a system, for example an electron moving in some media, is described by the time-dependent Schrödinger equation

$$i\partial_t u = (-\Delta + V)u, \quad u(0,\cdot) = u_0,$$

on some L_2-space, where the initial state u_0 is a normalized L_2-element. The self-adjoint Hamiltonian $H := -\Delta + V$ on the right-hand side is composed of two parts: the Laplacian $-\Delta$ describing the kinetic energy and the potential V related to the classical potential energy of the media. Therefore, many material properties such as positions of atoms in a model can be (more or less directly) transferred to properties of the potential. The solution u of the Schrödinger equation is given by the unitary group $(e^{-itH})_{t\in\mathbb{R}}$ generated by H, i.e.,

$$u(t,\cdot) = e^{-itH}u_0 \quad (t \in \mathbb{R}).$$

As u_0 is normalized, also $u(t,\cdot)$ is normalized for all $t \in \mathbb{R}$. The function $|u(t,\cdot)|^2$ is interpreted as the probability density for the position of the electron at time t.

The celebrated RAGE-Theorem (see for example [57]) connects dynamical properties of the solution $u(t,\cdot) = e^{-itH}u_0$ of the Schrödinger equation with spectral properties of the Hamiltonian H. Different transport properties of the media correspond to different spectral types. Loosely speaking, absolutely continuous spectrum corresponds to good transport, i.e., the electron may easily move through the material, while pure point spectrum corresponds to bad transport—the particle will (with high probability) stay in some bounded region in space for all times. Thus, in order to derive qualitative results on the time evolution of the initial state u_0 one can investigate the spectral types of H.

Let us have a closer look on quasicrystalline media, first discussed by Shechtman et al. in [53]. From the physical point of view these media are on the borderline between perfectly ordered and amorphous materials. They share properties with both of them: on the one hand quasicrystals exhibit a long-range order which is a typical phenomenon for crystalline materials. On the other hand they are globally aperiodic, a feature they share with amorphous media. That is the reason for saying that quasicrystals are *aperiodically ordered*. Hence, potentials modeling quasicrystals should be aperiodically ordered. Then, such models have a tendency for "strange" spectral properties: the Hamiltonians are likely to have Cantor sets as spectra. Furthermore the spectrum typically is purely singular continuous. This resembles the fact that quasicrystals are between order and disorder. The arrangement of atoms should be close to the periodic case of crystalline materials and therefore the pure point spectrum should be absent. Moreover, the aperiodicity breaks symmetry, although one may still have—locally—a finite complexity of the material. Since amorphous media can be described by random operators, these facts should rule out absolutely continuous spectra.

Many of the properties stated above were proven in the discrete one-dimensional setting, see for example [34, 4, 35, 12, 32, 36]. The aim of this thesis is to prove the analogous spectral behaviour for continuum one-dimensional models with very singular potentials. There already exist some results concerning Cantor spectra for almost periodic potentials, see for example [24, 25]. For quasicrystalline $L_{2,\mathrm{loc}}$-potentials many results can be found in [28, 15]. In this thesis, we want to allow measures as potentials in order to cover point interactions as well. Such a general setup was studied in [6, 47, 52, 29].

Let us now introduce the model we are interested in and then give an outline of the thesis. We consider continuum one-dimensional models, i.e., our Hilbert space will be $L_2(\mathbb{R})$. The big advantage of this setting is that we can apply the theory of dynamical systems and ordinary differential equations to study the Hamiltonian H. The disadvantage is obvious: the world (and hence a real quasicrystal in nature) is hardly one-dimensional. Our Hamiltonian will be of the form

$$-\Delta + \mu$$

in $L_2(\mathbb{R})$, where μ is a measure. Since also point interactions are allowed the model exhibits quite interesting mathematical phenomena. As motivated above, we are interested in spectral properties of this operator.

In Chapter 1 we will define the Hamiltonian such that it becomes self-adjoint and investigate basic notions such as (generalized) solutions of the eigenvalue equation. There exist two different methods to define the operator in the literature ([29, 47, 6]), and we will show that both lead to the same realization. The theory of Sturm-Liouville differential expressions (see for example [62, 16]) is well-developed and we will apply parts of this theory throughout the thesis. One of the main objects in our analysis are transfer matrices which we will also define in this chapter.

The Chapters 2 and 3 are devoted to connections between the geometric properties of the material (and, therefore, the potential) and spectral properties of the Hamiltonian: we show that being close to periodic potentials results in the fact that the pure point spectrum of H is empty. Such a result is called a Gordon type theorem ([52, 12, 15, 21]). The second connection concerns the absolutely continuous spectrum. If the potential is not periodic and satisfies a certain local complexity condition then the Hamiltonian does not have absolutely continuous spectrum at all ([29]). With these two chapters in hand we can—deterministically—prove purely singular continuous spectra for a large class of operators.

Amorphous materials typically are described by random operators, i.e., a whole family of operators. The remaining chapters 4, 5 and 6 will focus on this aspect.

In Chapter 4 we introduce such a family of operators. The question how to measure the common properties of the family will be answered: either one can use a probability measure and prove statements for almost all realizations or one can try to show statements for all operators in the family. We explain various connections between dynamics on the space of potentials and spectral properties of the corresponding family of Schrödinger operators. For example, we prove that minimality of the dynamical system of potentials implies that all Schrödinger operators generated by such potentials have the same spectrum as a set. Besides this, several preliminary properties of the transfer matrices are stated.

Chapter 5 provides abstract results on cocycles. All these results are motivated by the transfer matrices, which form a cocycle. First, we prove (semi)uniform ergodic theorems which will then be applied to cocycles (see [20] for the discrete case). We introduce the notion of (uniform) hyperbolicity and characterize it by means of exponential splittings (see [24, 25] and [37] in the discrete case). We also prove that uniform hyperbolicity is stable under small perturbations (in the version of [25]). Although some of the results are folklore we will give full proofs in order to supply a complete picture of the theory.

Chapter 6 finally collects many main results of the thesis. We characterize the common spectrum of the operator family by means of the Lyapunov exponent, as was done in [34] for the discrete case. After generalizing Ishii-Pastur-Kotani theory (see [8, 31, 9]) we conclude Cantor spectra. We also prove almost surely purely singular continuous spectra for quasicrystalline models in the random case (see also [29]).

The Appendix provides some well-known results needed for the thesis. We will state and prove a measure version of Gronwall's inequality. This is followed by a short introduction to sesquilinear forms and associated operators, and also to perturbations of closed forms. Afterwards, we collect some facts in connection with forms concerning solutions of the eigenvalue equations and results on the spectrum of the associated operator. Herglotz functions and representations of such functions are also briefly mentioned. Since the thesis mainly concerns spectral theoretic aspects we also state some facts concerning the spectral theorem and spectral theory for Sturm-Liouville operators.

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Chapter 1

Schrödinger operators with measures

In this chapter we provide a precise definition of the operator $-\Delta + \mu$ in $L_2(\mathbb{R})$, where μ is a uniformly locally bounded signed local Radon measure on $\mathbb{R}$.

This can be done in (at least) two different ways. One can use the form method to interpret μ as a form small perturbation of the classical Dirichlet form. One then obtains a self-adjoint operator H_μ representing this form by general theory. Since this method is quite general—for example, it does not make use of the one-dimensional space $\mathbb{R}$ we have—we will follow this approach. However, we only get a rather abstract characterization of the operator.

The other way to define $-\Delta + \mu$ follows along the lines of classical Sturm-Liouville theory by defining a so-called quasi-derivative. There is a big advantage in doing so: one obtains a direct description of how the operator actually acts on functions. Therefore, we will also describe this way a little bit, showing in the end that both ways lead to the same operator.

Since we need to develop some tools beforehand, we will define the notion of generalized solutions of the corresponding eigenvalue equation and prove various properties of these solutions. Then we will define the notions of limit point case and limit circle case which are well-known in the theory of Sturm-Liouville operators. We show that H_μ will be in the limit point case at both endpoints, thus yielding the equality of both realizations of the operator $\Delta + \mu$. We conclude this chapter with a section on transfer matrices since our methods in the next chapters heavily rely on these objects. We will prove the cocycle property of the transfer matrices as well as holomorphic dependence on the spectral parameter.

For the whole thesis let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. All function spaces will then be $\mathbb{K}$-valued unless otherwise stated.

1.1. Measure perturbed Schrödinger operators

We start by defining Radon measures on $\mathbb{R}$ and the suitable space of uniformly locally bounded (signed local Radon) measures. Then we define a self-adjoint realization of the operator $-\Delta + \mu$ via form methods.

Definition. A measure $\mu\colon \mathcal{B}(\mathbb{R}) \to [0, \infty]$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-field, is called a *Radon measure* if $\mu(K) < \infty$ for all $K \subseteq \mathbb{R}$ compact and μ is inner regular, i.e.,

$$\mu(A) = \sup \{\mu(K);\ K \subseteq \mathbb{R}\,\text{compact},\ K \subseteq A\} \quad (A \in \mathcal{B}(\mathbb{R})).$$

Let $\mathcal{M}_+(\mathbb{R})$ be the set of Radon measures on $\mathbb{R}$. A Radon measure is *finite* if $\mu(\mathbb{R}) < \infty$. We call μ a *signed Radon measure* if there exist $\mu_\pm \in \mathcal{M}_+(\mathbb{R})$, where at least one of them is finite, such that $\mu = \mu_+ - \mu_-$. A signed Radon measure μ is *finite* if $\mu_\pm(\mathbb{R}) < \infty$. A mapping $\mu\colon \{B \in \mathcal{B}(\mathbb{R});\ B\,\text{bounded}\} \to \mathbb{R}$ is called a *signed local Radon measure* if $\mathbb{1}_K\mu := \mu(\cdot \cap K)$ is a finite signed Radon measure for all $K \subseteq \mathbb{R}$ compact. Let $\mathcal{M}_{\mathrm{loc}}(\mathbb{R})$ be the space of signed local Radon measures.

For a signed local Radon measure μ there exist $\mu_\pm \in \mathcal{M}_+(\mathbb{R})$ such that $\mathbb{1}_K\mu = \mathbb{1}_K\mu_+ - \mathbb{1}_K\mu_-$ for all $K \subseteq \mathbb{R}$ compact. Then $|\mu| := \mu_+ + \mu_-$ is called the *total variation* of μ.

A signed local Radon measure μ is said to be *uniformly locally bounded* if

$$\|\mu\|_{\mathrm{loc}} := \sup_{t\in\mathbb{R}} |\mu|\,([t, t+1]) < \infty.$$

Let $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be the space of all uniformly locally bounded (signed local Radon) measures on $\mathbb{R}$.

Note that $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ generalizes the class of $L_{1,\mathrm{loc,unif}}(\mathbb{R})$-functions.

For a signed local Radon measure μ, a measurable mapping $f\colon \mathbb{R} \to \mathbb{K}$ and a measurable set $A \subseteq \mathbb{R}$ we define

$$\int_A f\,d\mu := \lim_{\substack{T\to\infty\\ S\to-\infty}} \int_A f\,d(\mathbb{1}_{[S,T]}\mu),$$

if the right-hand side exists finitely. Note that if f is bounded and A is compact this is always the case. Also, if f is bounded and has compact support, then the right-hand side exists finitely. Furthermore, we have the well-known inequality

$$\left|\int_A f\,d\mu\right| \le \int_A |f|\,d|\mu|\,.$$

We will mainly deal with bounded sets A. However, for the definition of the operator we will need $A = \mathbb{R}$.

Let

$$\begin{aligned} D(\tau_0) &:= W_2^1(\mathbb{R}),\\ \tau_0(u,v) &:= \int u'\overline{v}', \end{aligned}$$

be the classical Dirichlet form associated with $-\Delta$ in $L_2(\mathbb{R})$, where $W_2^1(\mathbb{R})$ is the Sobolev space of $L_2(\mathbb{R})$-functions with (distributional) derivative in $L_2(\mathbb{R})$. Note that for an integral with respect to Lebesgue-measure λ we drop the measure.

At first we show that $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ can be used as a perturbation of τ_0.

1.1.1 Lemma. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Then μ is infinitesimally form small with respect to τ_0, i.e., for any $a > 0$ there exists $C_a > 0$ such that*

$$|\mu(u,u)| \le a\tau_0(u,u) + C_a \|u\|^2_{L_2(\mathbb{R})} \quad (u \in D(\tau_0)),$$

where

$$\mu(u,u) := \int |u|^2 \, d\mu.$$

Proof. By means of Sobolev's imbedding theorem, every $u \in W_2^1(\mathbb{R})$ can be considered to be continuous (i.e., possesses a continuous representative).

If $\mu = 0$ there is nothing to prove. Let $\mu \neq 0$. For $\delta := \min\left\{\frac{a}{2\|\mu\|_{\mathrm{loc}}}, 1\right\} \in (0,1]$ and $n \in \mathbb{Z}$ we have

$$\|u\|^2_{L_\infty(n\delta,(n+1)\delta)} \le 2\delta \left\|u'\right\|^2_{L_2(n\delta,(n+1)\delta)} + \frac{2}{\delta}\|u\|^2_{L_2(n\delta,(n+1)\delta)}$$

by a direct computation using the fundamental theorem of calculus and the Cauchy-Schwarz inequality. Now, we estimate

$$\begin{aligned}
\int_{\mathbb{R}} |u|^2 \, d|\mu| &\le \sum_{n\in\mathbb{Z}} \int_{[n\delta,(n+1)\delta]} |u|^2 \, d|\mu| \\
&\le \sum_{n\in\mathbb{Z}} \|u\|^2_{L_\infty(n\delta,(n+1)\delta)} \|\mu\|_{\mathrm{loc}} \\
&\le \|\mu\|_{\mathrm{loc}} \sum_{n\in\mathbb{Z}} \left(2\delta \left\|u'\right\|^2_{L_2(n\delta,(n+1)\delta)} + \frac{2}{\delta}\|u\|^2_{L_2(n\delta,(n+1)\delta)}\right) \\
&= 2\delta \|\mu\|_{\mathrm{loc}} \left\|u'\right\|^2_{L_2(\mathbb{R})} + \frac{2\|\mu\|_{\mathrm{loc}}}{\delta}\|u\|^2_{L_2(\mathbb{R})} \\
&\le a\left\|u'\right\|^2_{L_2(\mathbb{R})} + \max\left\{\frac{4\|\mu\|^2_{\mathrm{loc}}}{a}, 2\|\mu\|_{\mathrm{loc}}\right\} \|u\|^2_{L_2(\mathbb{R})}.
\end{aligned}$$

Hence, $\mu(u,u)$ exists for all $u \in D(\tau_0)$ and the assertion follows. //

Since $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ is a form small perturbation of the classical Dirichlet form τ_0, we can define the form sum $\tau_\mu := \tau_0 + \mu$ and τ_μ will have good properties. Although the lemma follows from Theorem A.2.4 we state the proof for convenience.

1.1.2 Lemma. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. The form $\tau_\mu = \tau_0 + \mu$ defined by*

$$\begin{aligned}
D(\tau_\mu) &:= W_2^1(\mathbb{R}), \\
\tau_\mu(u,v) &:= \int u'\overline{v}' + \int u\overline{v}\, d\mu,
\end{aligned}$$

is densely defined, semibounded from below, symmetric and closed in $L_2(\mathbb{R})$.

Proof. The form τ_μ is densely defined as $W_2^1(\mathbb{R})$ is dense in $L_2(\mathbb{R})$. Symmetry of τ_μ is obvious since μ is a real measure. Let $u \in W_2^1(\mathbb{R}) \subseteq C(\mathbb{R})$. Then, using Lemma 1.1.1

$$\begin{aligned}
\tau_\mu(u,u) &= \tau_0(u,u) + \mu(u,u) \ge \tau_0(u,u) - |\mu(u,u)| \\
&\ge \tau_0(u,u) - \frac{1}{2}\tau_0(u,u) - C_{1/2}\|u\|_2^2 \ge -C_{1/2}\|u\|_2^2.
\end{aligned}$$

Hence, τ_μ is semibounded. Furthermore, the mapping $I\colon D_{\tau_\mu} \to W_2^1(\mathbb{R})$, $u \mapsto u$ is continuous, yielding that τ_μ is closed. //

For every $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ the first representation theorem (see Theorem A.2.3) gives rise to a unique self-adjoint operator H_μ in $L_2(\mathbb{R})$ associated with τ_μ, i.e.,

$$\tau_\mu(u,v) = (H_\mu u \,|\, v) \quad (u \in D(H_\mu), v \in D(\tau_\mu))$$

and $D(H_\mu)$ is dense in D_{τ_μ}. Here, $(\cdot\,|\,\cdot)$ denotes the inner product in $L_2(\mathbb{R})$ which is linear in the first component.

The operator H_μ is a self-adjoint realization of $-\Delta + \mu$ in $L_2(\mathbb{R})$.

1.2. Generalized solutions

Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $z \in \mathbb{C}$. We will define solutions of $H_\mu u = zu$ in a weak form. Beforehand, the direct approach due to Ben Amor and Remling (see [6]) for defining the operator $-\Delta + \mu$ is described. We then prove various properties of solutions u of the Schrödinger equation $H_\mu u = zu$, such as continuity and holomorphic dependence on z, and also show a uniqueness result: given an initial condition for u and u' at some fixed point, say $t = 0$, there is a unique solution of the equation satisfying these conditions. Note that $W^1_{1,\mathrm{loc}}(\mathbb{R}) = \{u \in L_{1,\mathrm{loc}}(\mathbb{R});\ u' \in L_{1,\mathrm{loc}}(\mathbb{R})\}$ is the space of locally absolutely continuous functions. More precisely, every $u \in W^1_{1,\mathrm{loc}}(\mathbb{R})$ has an locally absolutely continuous representative, and the equivalence class of every locally absolutely continuous function lies in $W^1_{1,\mathrm{loc}}(\mathbb{R})$.

Definition. For $u \in W^1_{1,\mathrm{loc}}(\mathbb{R})$ define $A_\mu u \in L_{1,\mathrm{loc}}(\mathbb{R})$ by

$$A_\mu u(t) := u'(t) - \int_0^t u(s)\, d\mu(s)$$

for λ-almost all $t \in \mathbb{R}$, where λ denotes the Lebesgue measure on $\mathbb{R}$. Here,

$$\int_0^t = \begin{cases} \int_{[0,t]} & t \geq 0, \\ -\int_{(t,0)} & t < 0. \end{cases}$$

The function $A_\mu u$ plays the role of a quasi-derivative of u. It takes into account the effect of the potential μ.

Now, the operator T_μ is defined as the maximal operator associated with $-\Delta + \mu$ via a Sturm-Liouville differential expression, cf. [16, 62], as follows

$$\begin{aligned} D(T_\mu) &:= \left\{u \in L_2(\mathbb{R});\ u, A_\mu u \in W^1_{1,\mathrm{loc}}(\mathbb{R}),\ (A_\mu u)' \in L_2(\mathbb{R})\right\}, \\ T_\mu u &:= -(A_\mu u)', \end{aligned}$$

cf. [6, 16].

We now ask for connections between H_μ and T_μ, since both operators realize $-\Delta+\mu$ in a certain sense (H_μ as the form sum, T_μ via Sturm-Liouville theory). For now, we will show that T_μ extends H_μ. Later in this chapter we actually prove equality. Note that it is not obvious that $u \in D(T_\mu)$ satisfies $u' \in L_2(\mathbb{R})$.

1.2.1 Lemma. *Let* $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. *Then we have* $H_\mu \subseteq T_\mu$.

Proof. Let $u \in D(H_\mu)$. Then $u \in W_2^1(\mathbb{R}) \subseteq W_{1,\mathrm{loc}}^1(\mathbb{R})$ and $A_\mu u \in L_{1,\mathrm{loc}}(\mathbb{R})$. Let $\varphi \in C_c^\infty(\mathbb{R}) \subseteq D(\tau_\mu)$, where $C_c^\infty(\mathbb{R})$ denotes the space of infinitely differentiable functions with compact support. We compute

$$\begin{aligned}
&\int_{\mathbb{R}} (A_\mu u)(t)\varphi'(t)\,dt \\
&= \int_{\mathbb{R}} \left(u'(t) - \int_0^t u(s)\,d\mu(s) \right) \varphi'(t)\,dt \\
&= \int_{\mathbb{R}} u'(t)\varphi'(t)\,dt - \int_{\mathbb{R}} \int_0^t u(s)\,d\mu(s)\varphi'(t)\,dt \\
&= \int_{\mathbb{R}} u'(t)\varphi'(t)\,dt + \int_{(-\infty,0)} \int_{(t,0)} u(s)\,d\mu(s)\varphi'(t)\,dt - \int_{[0,\infty)} \int_{[0,t]} u(s)\,d\mu(s)\varphi'(t)\,dt.
\end{aligned}$$

Using Fubini's Theorem, we further obtain

$$\begin{aligned}
&= \int_{\mathbb{R}} u'(t)\varphi'(t)\,dt + \int_{(-\infty,0)} \int_{(-\infty,s)} \varphi'(t)\,dt\,u(s)\,d\mu(s) - \int_{[0,\infty)} \int_{[s,\infty)} \varphi'(t)\,dt\,u(s)\,d\mu(s) \\
&= \int_{\mathbb{R}} u'(t)\varphi'(t)\,dt + \int_{(-\infty,0)} u(s)\varphi(s)\,d\mu(s) + \int_{[0,\infty)} u(s)\varphi(s)\,d\mu(s) \\
&= \int_{\mathbb{R}} u'(t)\varphi'(t)\,dt + \int_{\mathbb{R}} u(t)\varphi(t)\,d\mu(t) = \tau_\mu(u,\overline{\varphi}) = (H_\mu u \,|\, \overline{\varphi}) = \int_{\mathbb{R}} H_\mu u(t)\varphi(t)\,dt.
\end{aligned}$$

Hence, $(A_\mu u)' = -H_\mu u \in L_2(\mathbb{R})$. We conclude that $A_\mu u \in W_{1,\mathrm{loc}}^1(\mathbb{R})$ and therefore $u \in D(T_\mu)$, $T_\mu u = -(A_\mu u)' = H_\mu u$. //

Later we will prove that $H_\mu = T_\mu$. But before we can actually do this, we need to introduce the notion of (generalized) solutions to the eigenvalue equation of H_μ and T_μ.

Definition. A function $u \in L_{1,\mathrm{loc}}(\mathbb{R})$ is called a *solution* of the equation $H_\mu u = zu$ (or $T_\mu u = zu$) if $u \in W_{1,\mathrm{loc}}^1(\mathbb{R})$ and

$$-(A_\mu u)' = zu \tag{1.1}$$

in the sense of distributions.

1.2.2 Remark. Let u be a solution of (1.1). Since $u \in W_{1,\mathrm{loc}}^1(\mathbb{R})$, u can considered to be continuous and $A_\mu u \in W_{1,\mathrm{loc}}^2(\mathbb{R})$, so we have

$$-(A_\mu u)' = zu$$

almost everywhere. Since the functions on both sides have continuous representatives the equation may hold everywhere. Moreover, as $A_\mu u$ can considered to be continuous and $t \mapsto \int_0^t u(s)\, d\mu(s)$ is right continuous by definition, we may assume that u' is right continuous. Furthermore, equation (1.1) is equivalent to $u'' = u\mu - zu$ in the sense of distributions.

We end this section by stating some properties of solutions of $H_\mu u = zu$. The next lemma is well-known. Note that $BV_{\mathrm{loc}}(\mathbb{R})$, the space of functions which are locally of bounded variation, consists of all $u \in L_{1,\mathrm{loc}}(\mathbb{R})$ such that for all $U \subseteq \mathbb{R}$ open and bounded the distributional derivative of $u|_U$ on U is a finite complex Radon measure.

1.2.3 Lemma. *Let $u \in BV_{\mathrm{loc}}(\mathbb{R})$.*

(a) For all $t \in \mathbb{R}$: $u(t+) := \lim_{\substack{r\to t\\ r>t}} u(r)$ and $u(t-) := \lim_{\substack{r\to t\\ r<t}} u(r)$ exist.

(b) $t \mapsto u(t+)$ is right continuous.

Proof. Since $u \in BV_{\mathrm{loc}}(\mathbb{R})$, $|u'|$ is a Radon measure.

(a) Let $t \in \mathbb{R}$, $r' > r > t$. Then

$$\left|u(r') - u(r)\right| \leq \int_r^{r'} d\left|u'\right|(s) \leq \left|u'\right|([r,r']) \leq \left|u'\right|((t,r']) \to 0 \quad (r' \to t, r' > t).$$

Since $\mathbb{K}$ is complete, $u(t+) := \lim_{\substack{r\to t\\ r>t}} u(r)$ exists. Analogously, $u(t-)$ exists.

(b) Let $t \in \mathbb{R}$, $\varepsilon > 0$. There exists $\delta > 0$ such that for all $r > t$ with $r < t + \delta$ we have

$$|u(r) - u(t+)| \leq \frac{\varepsilon}{2}.$$

Let $t < s < t + \delta$. There exists $\delta' > 0$ such that for all $r > s$ with $r < s + \delta'$ we have

$$|u(r) - u(s+)| \leq \frac{\varepsilon}{2}.$$

For $s < r < \min\{s + \delta', t + \delta\}$ we obtain

$$|u(s+) - u(t+)| \leq |u(s+) - u(r)| + |u(r) - u(t+)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $t \mapsto u(t+)$ is right continuous. //

1.2.4 Lemma. *Let $z \in \mathbb{C}$, $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, u a solution of $H_\mu u = zu$. Then $u \in C(\mathbb{R})$, $u' \in BV_{\mathrm{loc}}(\mathbb{R})$, $t \mapsto u'(t+)$ is right continuous, and $u'(t) = u'(t+)$ and $u'(t+) - u'(t-) = u(t)\mu(\{t\})$ for all $t \in \mathbb{R}$.*

Proof. As $W^1_{1,\mathrm{loc}}(\mathbb{R}) \subseteq C(\mathbb{R})$, solutions are continuous. As $u'' = u\mu - zu$ in the sense of distributions we have $u' \in BV_{\mathrm{loc}}(\mathbb{R})$. By Lemma 1.2.3, for each $t \in \mathbb{R}$, the left and right limits $u'(t-)$ and $u'(t+)$ exist and $t \mapsto u'(t+)$ is right continuous.

Integration of (1.1) yields

$$u'(t) = A_\mu u(0) - z\int_0^t u(s)\, ds + \int_0^t u(s)\, d\mu(s)$$

for all $t \in \mathbb{R}$, where we chose the right continuous representative of u'. Hence, $u'(t) = u'(t+)$ and $u'(t+) - u'(t-) = u(t)\mu(\{t\})$ for all $t \in \mathbb{R}$. //

1.2.5 Lemma. *Let $z \in \mathbb{C}$, $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, u a solution of $H_\mu u = zu$. Define the right derivative of u by $D^+u(t) := \lim_{h\to 0+} \frac{u(t+h)-u(t)}{h}$. Then $D^+u(t)$ exists and equals $u'(t+)$ for all $t \in \mathbb{R}$, $D^+u \in BV_{\mathrm{loc}}(\mathbb{R})$ and $D^+u(t+) = D^+u(t)$ $(t \in \mathbb{R})$.*

Proof. Let $t \in \mathbb{R}$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $|u'(t+s) - u(t+)| < \varepsilon$ for $s \in [0, \delta)$. For $0 < h < \delta$ we obtain

$$\left| \frac{u(t+h) - u(t)}{h} - u'(t+) \right| \le \frac{1}{h} \int_0^h \left| u'(t+s) - u'(t+) \right| \, ds \le \varepsilon.$$

Therefore, $D^+u(t)$ exists and $D^+u(t) = u'(t+)$. Now, the remaining assertions follow from Lemma 1.2.4. //

We continue with a uniqueness result obtained in [6]: given initial data at 0 the solution will be unique. The striking consequence will be that the space of solutions will be two-dimensional (as it is in the case for linear second order ordinary differential equations).

1.2.6 Lemma (see also [6, Theorem 2.3])**.** *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, $z \in \mathbb{C}$ and $a, b \in \mathbb{C}$. Then there exists a unique solution $u(\cdot, z)$ of the equation $H_\mu u = zu$ such that $u(0, z) = a$ and $u'(0+, z) = b$. Furthermore, for all $t \in \mathbb{R}$ the function $\mathbb{C} \ni z \mapsto u(t, z)$ is holomorphic.*

Proof. Integrating (1.1) we obtain

$$u'(t) = A_\mu u(0) + \int_0^t u(s) \, d\mu(s) - z \int_0^t u(s) \, ds.$$

Integrating once again and using Fubini's Theorem, we arrive at

$$u(t) = u(0) + (A_\mu u(0)) \, t + \int_0^t (t-s) u(s) \, d(\mu - z\lambda)(s).$$

Plugging in the initial conditions we obtain, using $A_\mu u(0) = u'(0+) - u(0)\mu(\{0\})$,

$$u(t) = a + (b - a\mu(\{0\})) \, t + \int_0^t (t-s) u(s) \, d(\mu - z\lambda)(s).$$

Choosing $\eta > 0$ sufficiently small, the right hand side defines a contractive mapping on $C[0, \eta]$ for u. A fixed point argument yields existence and uniqueness on $[0, \eta]$. Now, the same argument with $(u(\eta), u'(\eta+))$ yields a unique solution on $[\eta, 2\eta]$ (the η can be chosen independent of the initial condition). Repeating this procedure finally gives the unique solution.

Holomorphic dependence on z also follows from this method, since the fixed point argument is applied on a space with supremum norm. //

1.2.7 Remark. If $E \in \mathbb{R}$ and $a, b \in \mathbb{R}$, then the solution $u(\cdot, E)$ is real.

We now ask if also the (right) derivative of a solution depends holomorphically on the spectral parameter z.

1.2.8 Lemma. *Let $\mu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R})$, $u(\cdot, z)$ the solution of $H_\mu u = zu$ subject to some fixed initial conditions at 0. Let $t \in \mathbb{R}$. Then $\mathbb{C} \ni z \mapsto u'(t+, z)$ is holomorphic.*

Proof. Without loss of generality, let $t \geq 0$. We have

$$u(t, z) = u(0, z) + u'(0+, z)t + \int_{(0,t]} (t - s)u(s, z)\, d(\mu - z\lambda)(s).$$

By Gronwall's inequality (see Lemma A.1) we obtain

$$\sup_{t\in[0,T]} \sup_{z\in K} |u(t, z)| < \infty$$

for all $T \geq 0$, $K \subseteq \mathbb{C}$ compact, see also the proof of Lemma 4.3.1 and Remark 4.3.2. Let $T := t + 1$, $K \subseteq \mathbb{C}$ be compact. Let $s \in [t, T]$. We compute

$$\begin{aligned}\sup_{z\in K} |A_\mu u(s, z) - A_\mu u(t, z)| &\leq \sup_{z\in K} \left| \int_t^s (A_\mu u)'(r, z)\, dr \right| \\ &\leq \sup_{z\in K} \int_t^s |zu(r, z)|\, dr \leq \sup_{z\in K} \sup_{r\in[t,T]} |zu(r, z)|\, (s - t).\end{aligned}$$

The right-hand side tends to zero as $s \to t$. Furthermore, using this result,

$$\begin{aligned}&\sup_{z\in K} \left|u'(s+, z) - u'(t+, z)\right| \\ &\leq \sup_{z\in K} \left| A_\mu u(s, z) + \int_0^s u(r, z)\, d\mu(r) - A_\mu u(t, z) - \int_0^t u(r, z)\, d\mu(r) \right| \\ &\leq \sup_{z\in K} \left(|A_\mu u(s, z) - A_\mu u(t, z)| + \int_{(t,s]} |u(r, z)|\, d\,|\mu|\,(r) \right) \\ &\to 0 \quad (s \to t).\end{aligned}$$

Let (h_n) in $(0, 1)$, $h_n \to 0$. Then

$$\begin{aligned}\sup_{z\in K} \left| \frac{u(t + h_n, z) - u(t, z)}{h_n} - u'(t+, z) \right| &\leq \sup_{z\in K} \frac{1}{h_n} \int_t^{t+h_n} \left|u'(s, z) - u'(t+, z)\right|\, ds \\ &\to 0 \quad (s \to t).\end{aligned}$$

Since, $z \mapsto \frac{u(t+h_n,z)-u(t,z)}{h_n}$ is holomorphic by Lemma 1.2.6, $z \mapsto u'(t+, z)$ is holomorphic. //

1.3. Limit point case

This section is devoted to Sturm-Liouville theory and essentially allows us to describe the operator T_μ, where $\mu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R})$. First, we show that T_μ is in the limit point case at both $\pm\infty$. Loosely speaking, limit point case means that no additional boundary conditions have to be imposed for getting self-adjoint realizations of the operator. After having proved limit point case we easily get self-adjointness of T_μ which leads to the equality $H_\mu = T_\mu$.

Definition. We say that T_μ is in *limit circle case* at ∞ (or $-\infty$) if there exists $z \in \mathbb{C}$ such that every solution u of $T_\mu u = zu$ satisfies $u \in L_2(0,\infty)$ (or $u \in L_2(-\infty, 0)$). Otherwise, T_μ is said to be in *limit point case* at ∞ (or $-\infty$).

Definition. Let u, v be two solutions of the equation $H_\mu u = zu$. Then we define their *Wronskian* by $W(u,v)(t) := u(t)v'(t+) - u'(t+)v(t)$ $(t \in \mathbb{R})$.

1.3.1 Remark. The Wronskian of two solutions to the same equation is constant, see [6]. Furthermore, u and v are linearly independent if and only if $W(u,v) \neq 0$.

The next proposition states that limit point/limit circle case is independent of z.

1.3.2 Proposition (compare [10, Theorem 9.2.1] and [16, Lemma 5.1])**.** *Let $z_0 \in \mathbb{C}$ and assume that every solution u of $T_\mu u = z_0 u$ satisfies $u \in L_2(0,\infty)$. Then, for every $z \in \mathbb{C}$, every solution of $T_\mu u = zu$ is in $L_2(0,\infty)$.*

Proof. For the proof see [16, Lemma 5.1]. //

1.3.3 Lemma. *Let u be a solution of $T_\mu u = 0$, $a, b \in \mathbb{R}$, $a < b$, $v \in W_1^1(a,b)$. Then*

$$\int_a^b v(s)u(s)\,d\mu(s) = v(b)u'(b+) - v(a)u'(a-) - \int_a^b v'(s)u'(s)\,ds.$$

Proof. Since u is a solution, $u' \in BV_{\text{loc}}(a,b)$, and $u'' = u\mu$ in the sense of distributions.

By [19, Theorem 5.3.1], for $v \in C^1[a,b]$ we have

$$\int_{(a,b)} v(s)u(s)\,d\mu(s) = v(b)u'(b-) - v(a)u'(a+) - \int_a^b v'(s)u'(s)\,ds.$$

For $v \in W_1^1(a,b)$ there exists (v_n) in $C^1[a,b]$ such that $v_n \to v$ in $W_1^1(a,b)$. Since $W_1^1(a,b)$ is continuously embedded into $C[a,b]$ by Sobolev's inequality (see [1, Theorem 4.12]), $v_n \to v$ uniformly. Furthermore, u is continuous. For $n \in \mathbb{N}$ we have

$$\int_{(a,b)} v_n(s)u(s)\,d\mu(s) = v_n(b)u'(b-) - v_n(a)u'(a+) - \int_a^b v_n'(s)u'(s)\,ds.$$

Since all four terms converge, we end up with

$$\int_{(a,b)} v(s)u(s)\,d\mu(s) = v(b)u'(b-) - v(a)u'(a+) - \int_a^b v'(s)u'(s)\,ds.$$

Note that

$$\begin{aligned} v(a)u(a)\mu(\{a\}) &= v(a)u'(a+) - v(a)u'(a-), \\ v(b)u(b)\mu(\{b\}) &= v(b)u'(b+) - v(b)u'(b-). \end{aligned}$$

Hence, we finally arrive at

$$\begin{aligned} \int_a^b v(s)u(s)\,d\mu(s) &= \int_{(a,b)} v(s)u(s)\,d\mu(s) + v(a)u(a)\mu(\{a\}) + v(b)u(b)\mu(\{b\}) \\ &= v(b)u'(b+) - v(a)u'(a-) - \int_a^b v'(s)u'(s)\,ds. \qquad // \end{aligned}$$

1.3.4 Lemma (see [9, Lemma III.1.4])**.** *Let $E \in \mathbb{R}$ and let u be a real solution of $T_\mu u = Eu$. Suppose that $u \in L_2(1,\infty)$. Then*

$$\int_1^\infty \frac{|u'(t)|^2}{t^2}\,dt < \infty.$$

A similar result holds true for solutions being square integrable at $-\infty$.

Proof. The previous lemma implies

$$\begin{aligned} &\int_1^t \frac{u(s)}{s^2}u(s)\,d\mu(s) \\ &= E\int_1^t \frac{u(s)}{s^2}u(s)\,ds + \frac{u(t)u'(t+)}{t^2} - u(1)u'(1-) - \int_1^t \frac{u'(s)^2}{s^2}\,ds + 2\int_1^t \frac{u(s)u'(s)}{s^3}\,ds. \end{aligned}$$

Define $h(t) := \int_1^t \frac{|u'(s)|^2}{s^2}\,ds$.

Since $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, for all $a \in (0, \frac{1}{2})$ there is $C_a \geq 0$ such that

$$\int_1^t |v(s)|^2\,d|\mu|(s) \leq a\int_1^t |v'(s)|^2\,ds + C_a\int_1^t |v(s)|^2\,ds \quad (v \in W_2^1(1,t)),$$

compare Lemma 1.1.1. Since $[1,t] \ni s \mapsto \frac{u(s)}{s}$ is in $W_2^1(1,t)$, we obtain

$$\begin{aligned} \left|\int_1^t \frac{|u(s)|^2}{s^2}\,d\mu(s)\right| &\leq a\int_1^t \left|\frac{u'(s)}{s} - \frac{u(s)}{s^2}\right|^2 ds + C_a\int_1^t \frac{|u(s)|^2}{s^2}\,ds \\ &\leq 2ah(t) + (2a + C_a)\int_1^\infty |u(s)|^2\,ds. \end{aligned}$$

Hence, using the first identity we see that there exists $c_1 \geq 0$ such that

$$-\frac{u(t)u'(t+)}{t^2} + h(t) - 2\int_1^t \frac{u(s)u'(s)}{s^3}\, ds \leq c_1 + 2ah(t).$$

The Cauchy-Schwarz inequality implies

$$\int_1^t \frac{u(s)u'(s)}{s^3}\, ds \leq \left(\int_1^t \frac{|u'(s)|^2}{s^2}\, ds\right)^{1/2} \left(\int_1^t \frac{|u(s)|^2}{s^4}\, ds\right)^{1/2} \leq \sqrt{h(t)} \left(\int_1^\infty |u(s)|^2\, ds\right)^{1/2}.$$

Therefore, for some $c_2 \geq 0$ we have

$$-\frac{u(t)u'(t+)}{t^2} + (1-2a)h(t) - c_2\sqrt{h(t)} \leq c_1.$$

If $h(t) \to \infty$ as $t \to \infty$ we would obtain $u(t)u'(t+) \geq \frac{t^2 h(t)}{2}$ for large t, i.e., u and u' have the same sign and therefore u cannot be square integrable near ∞. //

Now we can state the first main result on measure-perturbed Schrödinger operators.

1.3.5 Proposition (see also [9, Corollary III.1.5])**.** *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Then T_μ is in limit point case at $\pm\infty$.*

Proof. Let $z \in \mathbb{R}$, u, v be linearly independent solutions of $H_\mu u = zu$ such that $W(u,v) = 1$. Then

$$\frac{1}{t} = u(t)\frac{v'(t+)}{t} - v(t)\frac{u'(t+)}{t} \quad (t \in \mathbb{R}).$$

Since the left hand side is not square integrable, also the right hand side cannot be square integrable at $\pm\infty$. Note that we can choose the repesentatives such that $u'(t) = u'(t+)$ and $v'(t) = v'(t+)$ for all $t \in \mathbb{R}$. The Cauchy-Schwarz inequality and the previous lemma imply that u and v cannot both be square integrable at $\pm\infty$. //

Limit point case quite easily leads to self-adjointness of T_μ. We will state this as a theorem, however referring to the literature for the proof.

1.3.6 Theorem. *The operator T_μ is self-adjoint.*

Proof. Since T_μ is in limit point case at both $\pm\infty$, [16, Theorem 6.2] yields self-adjointness of T_μ. //

The main result of this section (and in fact of this chapter) will now be an easy corollary.

1.3.7 Corollary. $H_\mu = T_\mu$.

Proof. By Lemma 1.2.1, $H_\mu \subseteq T_\mu$. Since both are self-adjoint, we obtain

$$H_\mu \subseteq T_\mu = T_\mu^* \subseteq H_\mu^* = H_\mu.$$

Hence, $H_\mu = T_\mu$. //

We end this section with a brief remark on the terminology of limit point case.

1.3.8 Remark. Let $z \in \mathbb{C}$, $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Let $l > 0$ and consider the operator $T_\mu|_{[0,l]}$ defined by

$$\begin{aligned} D(T_\mu|_{[0,l]}) &:= \left\{u \in L_2(0,l);\ u, A_\mu u \in W^1_{1,\mathrm{loc}}([0,l]), (A_\mu u)' \in L_2(0,l)\right\}, \\ T_\mu|_{[0,l]}u &:= -(A_\mu u)'. \end{aligned}$$

Let u_N and u_D be the solutions of $T_\mu|_{[0,l]}u = zu$ such that

$$\begin{pmatrix} u_N(0) \\ u_N'(0+) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_D(0) \\ u_D'(0+) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $\beta \in (0, \pi)$. Then there exists a unique $m(z, l, \beta) \in \mathbb{C}$ such that

$$u = u_N + m(z, l, \beta)u_D$$

is a solution and satisfies the boundary condition

$$u(l)\cos\beta + \big(\mu(\{l\})u(l) - u'(l-)\big)\sin\beta = 0.$$

One can deduce that the image of $\beta \mapsto m(z, l, \beta)$ forms a circle in the complex plane and that the radius of this circle becomes smaller when l increases (the larger circle contains the smaller one). One now asks whether the limit object as $l \to \infty$ is still a circle (then we are in limit circle case) or if the circles shrink to some point (then we are in limit point case). Assume now that we are in the limit point case. We call this limit point

$$m_+(z) := \lim_{l\to\infty} m(z, l, \beta).$$

Let $K \subseteq \mathbb{C}^+ := \{z \in \mathbb{C};\ \mathrm{Im}\, z > 0\}$ be compact. We fix $\beta \in (0, \pi)$. For each $l > 1$ one can show that the meromorphic functions $K \ni z \mapsto m(z, l, \beta)$ are bounded. Hence, they are holomorphic. Furthermore, they are equicontinuous. Hence, they converge uniformly on K and the limit m_+ is holomorphic on K.

Note that $m(z, l, \frac{\pi}{2})$ can be written as

$$m(z, l, \frac{\pi}{2}) = -\frac{u_N(l)}{u_D(l)},$$

if we investigate the boundary condition at l. We conclude that the limit point can also be written as

$$m_+(z) = -\lim_{l\to\infty} \frac{u_N(l)}{u_D(l)}.$$

We will exploit this fact in more detail in Chapter 6.

1.4. Transfer matrices

Fix $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $z \in \mathbb{C}$. We consider the solutions of $H_\mu u = zu$. For $t \in \mathbb{R}$ define $T_z(t,\mu)\colon \mathbb{K}^2 \to \mathbb{K}^2$ such that $T_z(t,\mu)$ maps $(u(0)u'(0+))$ to $(u(t), u'(t+))$ for all solutions u (we suppress the dependence of u on z and μ for the sake of an easier notation). These matrices are called *transfer matrices*. Let u_N and u_D be the solutions of $H_\mu u = zu$ such that

$$\begin{pmatrix} u_N(0) \\ u_N'(0+) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_D(0) \\ u_D'(0+) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $T_z(t,\mu)$ has the matrix representation

$$T_z(t,\mu) = \begin{pmatrix} u_N(t) & u_D(t) \\ u_N'(t+) & u_D'(t+) \end{pmatrix}.$$

Since $W(u_N, u_D)(t) = W(u_N, u_D)(0) = 1$ for all $t \in \mathbb{R}$, we obtain $\det T_z(t,\mu) = 1$ $(t \in \mathbb{R})$.

Exploiting the uniqueness of solutions we obtain

$$T_z(s+t,\mu) = T_z(s, \mu(\cdot + t))T_z(t,\mu) \quad (s,t \in \mathbb{R}).$$

In fact, let $a, b \in \mathbb{K}$. Then

$$\begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} = T_z(t,\mu) \begin{pmatrix} a \\ b \end{pmatrix}$$

yields the solution u of the equation $H_\mu u = zu$ at t subject to the initial condition $u(0) = a$, $u'(0+) = b$. Now, fixing $t \in \mathbb{R}$ and shifting everything we see that

$$T_z(s, \mu(\cdot + t)) \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} = \begin{pmatrix} u(s+t) \\ u'((s+t)+) \end{pmatrix}.$$

Hence,

$$T_z(s+t,\mu) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u(s+t) \\ u'((s+t)+) \end{pmatrix} = T_z(s, \mu(\cdot + t))T_z(t,\mu) \begin{pmatrix} a \\ b \end{pmatrix}.$$

1.4.1 Lemma. *Let $t \in \mathbb{R}$. Then $z \mapsto T_z(t,\mu)$ is holomorphic.*

Proof. This is a direct consequence of Lemma 1.2.6 and Lemma 1.2.8. //

Chapter 2

Gordon's Theorem

The main goal of this chapter is to show absence of eigenvalues of H_μ when $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ can be very well approximated by periodic measures. The argument is due to Gordon (see [21]), who first proved such a result for bounded potentials. In the end, we can exclude point spectrum for such models.

The results in this chapter are already published in [52].

Definition. We call $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ a *Gordon measure* if there exists a sequence $(\mu_m)_{m\in\mathbb{N}}$ of periodic signed local Radon measures in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ with period sequence (p_m) such that $p_m \to \infty$ and for all $C \in \mathbb{R}$ we have

$$\lim_{m\to\infty} e^{Cp_m} \left|\mu - \mu_m\right| ([-p_m, 2p_m]) = 0,$$

i.e., (μ_m) approximates μ on increasing intervals. Here, a measure is p-periodic, if $\mu = \mu(\cdot + p)$.

For $t \in \mathbb{R}$ we abbreviate $I_t := [\min\{t, 0\}, \max\{t, 0\}]$ and $I_t(s) := I_t \cap ([s, t] \cup [t, s])$ for all $s \in \mathbb{R}$.

Let μ be uniformly locally bounded. Then

$$\left|\mu\right| (I_t) \leq (|t| + 1) \left\|\mu\right\|_{\mathrm{loc}} \quad (t \in \mathbb{R}).$$

Furthermore, if μ is periodic and locally bounded, μ is uniformly locally bounded.

The proof of the main result in this chapter lasts on basically three ingredients. First, we need a stability (or continuity) statement, locally estimating solutions for different measures by the difference of the measures. Secondly, we seek for estimates of the solution of the eigenvalue equation with a periodic measure. Finally, we show that for functions $u \in D(H_\mu)$, the value of the function and of the derivative tends to zero at $\pm\infty$. Note that the last fact is not that obvious since in general $D(H_\mu) \not\subseteq W_2^2(\mathbb{R})$. Nevertheless, $u' \in BV_{\mathrm{loc}}(\mathbb{R})$ for solutions $u \in D(H_\mu)$ and this fact is sufficient for vanishing at $\pm\infty$.

2.1. A stability result for solutions

We need some lemmas to prove the stability estimate in Proposition 2.1.4. For a vector $v \in \mathbb{K}^2$ let $\|v\| := \sqrt{|v_1|^2 + |v_2|^2}$ be the euclidean norm of v.

2.1.1 Lemma. *Let* $\mu_1, \mu_2 \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, $E \in \mathbb{R}$ *and* u_1 *and* u_2 *solutions of*

$$H_{\mu_1} u_1 = E u_1, \qquad H_{\mu_2} u_2 = E u_2$$

subject to the same initial conditions at 0, *i.e.,*

$$u_1(0) = u_2(0), \quad u_1'(0+) = u_2'(0+).$$

Then, for all $t \in \mathbb{R}$,

$$\begin{aligned}&\left\| \begin{pmatrix} u_1(t) \\ u_1'(t+) \end{pmatrix} - \begin{pmatrix} u_2(t) \\ u_2'(t+) \end{pmatrix} \right\| \\ &\leq \int_{I_t} |u_2(s)|\, d\,|\mu_1 - \mu_2|\,(s) \\ &+ \int_{I_t} \Big(\int_{I_s} |u_2|\, d\,|\mu_1 - \mu_2| \Big) e^{(\lambda + |\mu_1 - E\lambda|)(I_t(s))}\, d(\lambda + |\mu_1 - E\lambda|)(s).\end{aligned}$$

Proof. Without loss of generality, let $t \geq 0$ (the case $t < 0$ is even simpler). Write

$$u_1(t) - u_2(t) = \int_0^t (u_1'(s+) - u_2'(s+))\, ds = \int_0^t (u_1'(s-) - u_2'(s-))\, ds$$

and (integrating (1.1))

$$u_1'(t-) - u_2'(t-) = -\int_{[0,t)} u_2(s)\, d(\mu_1 - \mu_2)(s) - \int_{[0,t)} (u_1(s) - u_2(s))\, d(\mu_1 - E\lambda)(s).$$

We conclude that

$$\begin{aligned}&\left\| \begin{pmatrix} u_1(t) \\ u_1'(t-) \end{pmatrix} - \begin{pmatrix} u_2(t) \\ u_2'(t-) \end{pmatrix} \right\| \\ &\leq \int_{[0,t)} |u_2(s)|\, d\,|\mu_1 - \mu_2|\,(s) + \int_{[0,t)} \left\| \begin{pmatrix} u_1(s) \\ u_1'(s-) \end{pmatrix} - \begin{pmatrix} u_2(s) \\ u_2'(s-) \end{pmatrix} \right\| d(\lambda + |\mu_1 - E\lambda|)(s).\end{aligned}$$

An application of Lemma A.1 yields

$$\begin{aligned}&\left\| \begin{pmatrix} u_1(t) \\ u_1'(t-) \end{pmatrix} - \begin{pmatrix} u_2(t) \\ u_2'(t-) \end{pmatrix} \right\| \\ &\leq \int_{[0,t)} |u_2(s)|\, d\,|\mu_1 - \mu_2|\,(s) \\ &+ \int_{[0,t)} \Big(\int_{[0,s)} |u_2|\, d\,|\mu_1 - \mu_2| \Big) e^{(\lambda + |\mu_1 - E\lambda|)((s,t))}\, d(\lambda + |\mu_1 - E\lambda|)(s).\end{aligned}$$

Since

$$u_1'(t+) - u_2(t+) = u_1'(t-) - u_2(t-) + u_2(t)(\mu_1 - \mu_2)(\{t\}) + (u_1 - u_2)(t)\mu_1(\{t\}),$$

we arrive at

$$\begin{aligned}
&\left\| \begin{pmatrix} u_1(t) \\ u_1'(t+) \end{pmatrix} - \begin{pmatrix} u_2(t) \\ u_2'(t+) \end{pmatrix} \right\| \\
&\le \int\limits_{[0,t]} |u_2(s)| \, d|\mu_1 - \mu_2|(s) \\
&+ \int\limits_{[0,t]} \Big(\int\limits_{[0,s]} |u_2| \, d|\mu_1 - \mu_2| \Big) e^{(\lambda + |\mu_1 - E\lambda|)([s,t])} \, d(\lambda + |\mu_1 - E\lambda|)(s).
\end{aligned}$$

//

2.1.2 Lemma. *Let $E \in \mathbb{R}$ and u_0 be a solution of $-\Delta u_0 = E u_0$. Then there is $C \ge 0$ such that $|u_0(t)| \le C e^{C|t|}$ for all $t \in \mathbb{R}$.*

Proof. Since $u_0(t) = C_1 e^{\sqrt{-E}t} + C_2 e^{-\sqrt{-E}t}$ ($t \in \mathbb{R}$) for $E \neq 0$ the assertion follows in this case. In case $E = 0$ we have $u_0(t) = C_1 + C_2 t$ and the assertion follows as well. //

In the following lemmas and proofs the constant C may change (to be more precise: increase) from line to line, but we will always state the dependence on the important quantities.

2.1.3 Lemma. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, $E \in \mathbb{R}$, u a solution of $H_\mu u = Eu$. Then there is $C \ge 0$ such that*

$$|u(t)| \le C e^{C|t|} \quad (t \in \mathbb{R}).$$

Proof. Let u_0 be the solution of $-\Delta u_0 = E u_0$ subject to

$$(u_0(0), u_0'(0+)) = (u(0), u'(0+)).$$

By Lemma 2.1.1 we have

$$\begin{aligned}
&|u(t) - u_0(t)| \\
&\le \int\limits_{I_t} |u_0(s)| \, d|\mu|(s) \\
&+ \int\limits_{I_t} \left(\int\limits_{I_s} |u_0(r)| \, d|\mu|(r) \right) e^{(\lambda + |\mu - E\lambda|)(I_t(s))} \, d(\lambda + |\mu - E\lambda|)(s) \\
&\le |\mu|(I_t) C e^{C|t|} \\
&+ \int\limits_{I_t} \left(C |\mu|(I_s) e^{C|s|} \right) e^{(\lambda + |\mu - E\lambda|)(I_t(s))} \, d(\lambda + |\mu - E\lambda|)(s) \\
&\le \left(C |\mu|(I_t) e^{C|t|} \right) \left(1 + e^{(\lambda + |\mu - E\lambda|)(I_t)} (\lambda + |\mu - E\lambda|)(I_t) \right).
\end{aligned}$$

Since $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, also $\mu - E\lambda \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and we have

$$|\mu - E\lambda|(I_t) \le (|t| + 1) \|\mu - E\lambda\|_{\mathrm{loc}}.$$

We conclude that

$$\begin{aligned}
&|u(t) - u_0(t)| \\
&\leq \left(C(|t|+1) \|\mu\|_{\mathrm{loc}} e^{C|t|} \right) \left(1 + e^{(|t|+1)(1+\|\mu - E\lambda\|_{\mathrm{loc}})} (|t|+1)(1 + \|\mu - E\lambda\|_{\mathrm{loc}}) \right) \\
&\leq Ce^{C|t|},
\end{aligned}$$

where C is depending on E, $\|\mu\|_{\mathrm{loc}}$ and $\|\mu - E\lambda\|_{\mathrm{loc}}$. Hence,

$$|u(t)| \leq |u(t) - u_0(t)| + |u_0(t)| \leq Ce^{C|t|}. \qquad //$$

Now, we are in the position to prove the stability estimate. We show that locally the solutions of the eigenvalue equation continuously depend on the potentials.

2.1.4 Proposition. *Let $\mu, \nu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, u a solution of $H_\mu u = Eu$, v a solution of $H_\nu v = Ev$ with the same initial conditions at 0. Then there is $C \geq 0$ such that*

$$\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} - \begin{pmatrix} v(t) \\ v'(t+) \end{pmatrix} \right\| \leq Ce^{C|t|} |\mu - \nu| (I_t) \quad (t \in \mathbb{R}).$$

Proof. By Lemma 2.1.1 we know that

$$\begin{aligned}
&\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} - \begin{pmatrix} v(t) \\ v'(t+) \end{pmatrix} \right\| \\
&\leq \int_{I_t} |v(s)| \, d\,|\mu - \nu| (s) \\
&+ \int_{I_t} \Big(\int_{I_s} |v| \, d\,|\mu - \nu| \Big) e^{(\lambda + |\mu - E\lambda|)(I_t(s))} \, d(\lambda + |\mu - E\lambda|)(s).
\end{aligned}$$

Lemma 2.1.3 yields

$$|v(t)| \leq Ce^{C|t|}.$$

Therefore,

$$\begin{aligned}
&\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} - \begin{pmatrix} v(t) \\ v'(t+) \end{pmatrix} \right\| \\
&\leq Ce^{C|t|} |\mu - \nu| (I_t) \left(1 + e^{(\lambda + |\mu - E\lambda|)(I_t)} (\lambda + |\mu - E\lambda|)(I_t) \right).
\end{aligned}$$

Since

$$|\mu - E\lambda| (I_t) \leq (|t| + 1) \|\mu - E\lambda\|_{\mathrm{loc}},$$

we further estimate

$$\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} - \begin{pmatrix} v(t) \\ v'(t+) \end{pmatrix} \right\| \leq Ce^{C|t|} |\mu - \nu| (I_t),$$

where C is depending on $\|\mu - E\lambda\|_{\mathrm{loc}}$ (and of course on M, $\|\mu\|_{\mathrm{loc}}$ and E). //

2.1.5 Remark. One would like to prove a similar result, where one uses the vague topology on the measures instead of the topology induced by the total variation. Since point measures as potentials imply discontinuities of the derivative of the solutions and point measure potentials can easily be approximated vaguely by L_1-potentials, we do not expect that to be achievable.

The next corollary states the variant of the preceding proposition which we will need in the sequel.

2.1.6 Corollary. *Let μ be a Gordon measure and (μ_m) the periodic approximations with period sequence (p_m). Let u be a solution of $H_\mu u = Eu$ with* normalized *initial condition at* 0*, i.e., $|u(0)|^2 + |u'(0+)|^2 = 1$, and u_m the solution of $H_{\mu_m} u_m = E u_m$ for $m \in \mathbb{N}$, obeying the same initial condition as u at* 0*. Then there is $C \geq 0$ such that*

$$\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} - \begin{pmatrix} u_m(t) \\ u_m'(t+) \end{pmatrix} \right\| \leq C e^{C|t|} \left|\mu - \mu_m\right|(I_t) \quad (t \in \mathbb{R}).$$

Proof. Note that

$$M := \sup_{m \in \mathbb{N}} \|\mu_m\|_{\mathrm{loc}} < \infty,$$

since (μ_m) approximates μ. Hence, also

$$\sup_{m \in \mathbb{N}} \|\mu_m - E\lambda\|_{\mathrm{loc}} < \infty$$

and Lemma 2.1.3 yields

$$|u_m(t)| \leq C e^{C|t|},$$

where C can be chosen independently of m. Hence, as the proof of Proposition 2.1.4 shows, the constant C in Proposition 2.1.4 can be chosen independently of m. //

2.2. Solutions to periodic measures

By Proposition 2.1.4 we have an estimate on the difference of two solutions to two measures. Since we know that one of these measures is periodic, we obtain estimates of the solutions to a Gordon measure by estimating the solutions to periodic measures.

2.2.1 Lemma. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be p-periodic and $E \in \mathbb{R}$. Let u be a solution of $H_\mu u = Eu$ subject to*

$$|u(0)|^2 + \left|u'(0+)\right|^2 = 1.$$

Then

$$\max\left\{ \left\| \begin{pmatrix} u(-p) \\ u'((-p)+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(2p) \\ u'(2p+) \end{pmatrix} \right\| \right\} \geq \frac{1}{2}.$$

Proof. For $s, t \in \mathbb{R}$ define the mapping $T_E(s,t) : \begin{pmatrix} u(s) \\ u'(s+) \end{pmatrix} \mapsto \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix}$. Then $T_E(0,t)$ is the transfer matrix at t. By periodicity of μ we have that

$$T := T_E(-p, 0) = T_E(0, p) = T_E(p, 2p).$$

Since $\det T = 1$ the Cayley-Hamilton Theorem yields

$$T^2 - \operatorname{tr}(T)T + I = 0. \tag{2.1}$$

Now, consider two cases. First, assume that $|\operatorname{tr}(T)| \leq 1$. Applying equation (2.1) to $\begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix}$ yields

$$\begin{pmatrix} u(2p) \\ u'(2p+) \end{pmatrix} - \operatorname{tr}(T) \begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} = - \begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix},$$

and by the triangle inequality,

$$1 = \left\| \begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} u(2p) \\ u'(2p+) \end{pmatrix} \right\| + \left\| \begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} \right\|.$$

Hence,

$$\max \left\{ \left\| \begin{pmatrix} u(2p) \\ u'(2p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} \right\| \right\} \geq \frac{1}{2}.$$

On the other hand if $|\operatorname{tr}(T)| > 1$ we apply equation (2.1) to $\begin{pmatrix} u(-p) \\ u'((-p)+) \end{pmatrix}$. This gives

$$\begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} + \begin{pmatrix} u(-p) \\ u'((-p)+) \end{pmatrix} = \operatorname{tr}(T) \begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix}.$$

Now, the triangle inequality yields

$$1 < |\operatorname{tr}(T)| \left\| \begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} \right\| + \left\| \begin{pmatrix} u(-p) \\ u'((-p)+) \end{pmatrix} \right\|$$

and therefore

$$\max \left\{ \left\| \begin{pmatrix} u(p) \\ u'(p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(-p) \\ u'((-p)+) \end{pmatrix} \right\| \right\} \geq \frac{1}{2}.$$ //

The lemma essentially states that solutions u of $H_\mu u = Eu$ to periodic measures μ cannot decay too fast.

2.3. Absence of eigenvalues

Before proving the main theorem of this chapter, we show that for functions u in the domain of H_μ we necessarily have

$$\lim_{|t| \to \infty} u(t) = \lim_{|t| \to \infty} u'(t) = 0.$$

2.3.1 Lemma. *Let $v \in L_2(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$ and assume that for all $r > 0$ we have*

$$|v(t) - v(t+r)| \to 0 \quad (|t| \to \infty).$$

Then $|v(t)| \to 0$ as $|t| \to \infty$.

Proof. Without restriction, we can assume that $v \geq 0$. We prove this lemma by contradiction. Assume that $v(t) \to 0$ does not hold for $t \to \infty$. Then we can find $\delta > 0$ and (t_k) in $\mathbb{R}$ with $t_k \to \infty$ such that $v(t_k) \geq \delta$ for all $k \in \mathbb{N}$. By square integrability of v we have $\left\|v\mathbb{1}_{[t_k,t_k+1]}\right\|_{L_2(\mathbb{R})} \to 0$. Therefore, we can find a subsequence $(t_{k_n})_n$ of (t_k) satisfying

$$\left\|v\mathbb{1}_{[t_{k_n},t_{k_n}+1]}\right\|_{L_2(\mathbb{R})} \leq 2^{-\frac{3}{2}n} \quad (n \in \mathbb{N}).$$

Now, Chebyshev's inequality implies

$$\lambda(\{t \in [t_{k_n}, t_{k_n}+1];\ v(t) \geq 2^{-n}\}) \leq 2^{2n} \left\|v\mathbb{1}_{[t_{k_n},t_{k_n}+1]}\right\|^2_{L_2(\mathbb{R})} \leq 2^{-n} \quad (n \in \mathbb{N}).$$

Denote $A_n := \{t \in [t_{k_n}, t_{k_n}+1];\ v(t) \geq 2^{-n}\} - t_{k_n} \subseteq [0,1]$. Then $\lambda(A_n) \leq 2^{-n}$ and

$$\lambda\left(\bigcup_{n\geq 3} A_n\right) \leq \sum_{n\geq 3} \lambda(A_n) \leq 2^{-2} < 1.$$

Hence, $G := [0,1] \setminus (\bigcup_{n\geq 3} A_n)$ has positive measure. For $r \in G$, $r > 0$ it follows

$$v(t_{k_n} + r) \leq 2^{-n} \quad (n \geq 3).$$

Therefore,

$$\liminf_{n\to\infty} |v(t_{k_n}) - v(t_{k_n} + r)| \geq \delta > 0,$$

a contradiction. //

2.3.2 Lemma. *Let μ be a Gordon measure, $E \in \mathbb{R}$, $u \in D(H_\mu)$ a solution of $H_\mu u = Eu$. Then $u(t) \to 0$ as $|t| \to \infty$ and $u'(t) \to 0$ as $|t| \to \infty$.*

Proof. Since $u \in D(H_\mu) \subseteq D(\tau_\mu) = W_2^1(\mathbb{R})$ we have $u(t) \to 0$ as $|t| \to \infty$. Let $r > 0$. Then, for almost all $t \in \mathbb{R}$,

$$\begin{aligned} u'(t+r) - u'(t) &= A_\mu u(t+r) - A_\mu u(t) + \int\limits_{(t,t+r]} u(s)\,d\mu(s) \\ &= \int\limits_t^{t+r} (A_\mu u)'(s)\,ds + \int\limits_{(t,t+r]} u(s)\,d\mu(s). \end{aligned}$$

Hence,

$$\begin{aligned}\left|u'(t+r)-u'(t)\right| &\leq |E| \int_t^{t+r} |u(s)|\, ds + \int_{(t,t+r]} |u(s)|\, d\,|\mu|\,(s) \\ &\leq |E|\, r\, \|u\|_{L_\infty(t,t+r)} + \|u\|_{L_\infty(t,t+r)}\, |\mu|\,([t,t+r]) \\ &\leq \|u\|_{L_\infty(t,t+r)} \left(|E|\, r + (r+1)\, \|\mu\|_{\text{loc}}\right).\end{aligned}$$

By Sobolev's inequality, there is $C \in \mathbb{R}$ such that

$$\|u\|_{L_\infty(t,t+r)} \leq C\, \|u\|_{W_2^1(t,t+r)} \to 0 \quad (|t| \to \infty).$$

Thus,

$$\left|u'(t+r)-u'(t)\right| \to 0 \quad (|t| \to \infty).$$

An application of Lemma 2.3.1 with $v := u'$ yields $u'(t) \to 0$ as $|t| \to \infty$. //

Now, we can state the main result of this chapter.

2.3.3 Theorem. *Let μ be a Gordon measure. Then H_μ has no eigenvalues.*

Proof. Let (μ_m) be the periodic approximations of μ, $E \in \mathbb{R}$ and u be a solution of $H_\mu u = Eu$ and let (u_m) be the sequence of solutions for the measures (μ_m) with the same normalized initial conditions at 0 as u. By Corollary 2.1.6 we find $m_0 \in \mathbb{N}$ such that

$$\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} - \begin{pmatrix} u_m(t) \\ u_m'(t+) \end{pmatrix} \right\| \leq \frac{1}{4}$$

for $m \geq m_0$ and $t \in [-p_m, 2p_m]$. By Lemma 2.2.1 we have

$$\limsup_{|t|\to\infty} \left(|u(t)|^2 + \left|u'(t)\right|^2 \right) \geq \left(\frac{1}{4}\right)^2 > 0.$$

Hence, u cannot be in $D(H_\mu)$ by Lemma 2.3.2. Therefore, there is no solution of the equation $H_\mu u = Eu$ which also satisfies $u \in D(H_\mu)$. //

Some examples of Gordon measures may be found in [15] and [52].

It may be quite hard to prove that a given measure is actually a Gordon measure (since one has to find the periodic approximations). However, one can easily construct quasicrystalline potentials which are Gordon-measures. One of the well-established methods to construct such potentials is based on substitution rules. This construction is done by an iteration procedure. We will give an easy example, where also the idea of such substitutions should become clear.

2.3.4 Example. Let $\alpha := 1 + \sqrt{2}$. Choose a signed Radon measure ν_1 on $\mathrm{S} = [0,1]$ and a signed Radon measure ν_α on $\mathrm{L} = [0,\alpha]$ and suppose that

$$\nu_1(\{0\}) = \nu_1(\{1\}) = \nu_\alpha(\{0\}) = \nu_\alpha(\{\alpha\}).$$

Furthermore, we define the following substitution rules: replace S by L and L by LLS (this may be done symbolically). Then we obtain the following iteration scheme, where the vertical line indicates the position $0 \in \mathbb{R}$.

$$\begin{array}{r|l}
\text{S} & \text{L} \\
\text{L} & \text{LLS} \\
\text{LLS} & \text{LLSLLSL} \\
\text{LLSLLSL} & \text{LLSLLSLLLSLLSLLLS} \\
\text{LLSLLSLLLSLLSLLLS} & \text{LLSLLSLLLSLLSLLLSLLSLLSLLLSLLSLLLSLLSLLSL} \\
\multicolumn{1}{r}{} & \hspace{-0.6em}0
\end{array}$$

Choosing either only the even or the odd iteration steps one sees that one always obtains an extension of the previous ones. In this way we divide the whole real line into intervals of length either 1 or α (in fact, one may think of the endpoints of these intervals forming a grid on $\mathbb{R}$). Now, put on each interval represented by S (a translate of) the measure ν_1 and on each L (a translate of) ν_α. We end up with a signed local Radon measure $\mu \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R})$ which is easily be seen to be a Gordon measure (the periodic approximants can be read off from the scheme above; they are the periodic extensions of the parts on the left of the line indicating 0).

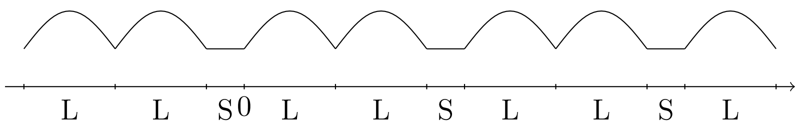

Figure 2.1.: Part of the measure μ corresponding to the third line in the iteration scheme.

Chapter 3

Measures of finite local complexity

In this chapter we will focus on the second main property of quasicrystalline potentials: they are globally aperiodic. Furthermore, these potentials are likely to attain only "finitely many values", i.e., locally they are of finite complexity.

We will show that if the measure has a certain finite local complexity and is aperiodic, then the corresponding operator does not have absolutely continuous spectrum.

We also introduce the notion of Delone measures, since they provide an appropriate class of potentials.

Most of the results in this chapter were obtained in [29]. However, we included this chapter in the thesis since it sheds another light on quasicrystalline potentials.

3.1. Measures of finite local complexity

Let us recall some definitions from [29].

Definition. A *piece* is a pair (ν, I) consisting of a closed interval $I \subseteq \mathbb{R}$ with positive length $\lambda(I) > 0$ (which is then called the *length* of the piece) and a signed (local) measure ν on $\mathbb{R}$ supported on I. We abbreviate pieces by ν^I. A *finite piece* is a piece of finite length. We say ν^I *occurs* in a signed (local) measure μ at $x \in \mathbb{R}$, if $\mathbb{1}_{[x,x+\lambda(I)]}\mu$ is a translate of ν.

The *concatenation* $\nu^I = \nu_1^{I_1} \mid \nu_2^{I_2} \mid \dots$ of a finite or countable family $(\nu_j^{I_j})_{j\in N}$, with $N = \{1, 2, \dots, |N|\}$ (for N finite) or $N = \mathbb{N}$ (for N infinite), of finite pieces is defined by

$$I = \left[\min I_1, \min I_1 + \sum_{j\in N} \lambda(I_j)\right],$$

$$\nu = \nu_1 + \sum_{j\in N, j\geq 2} \nu_j\Big(\,\cdot - \Big(\min I_1 + \sum_{k=1}^{j-1} \lambda(I_k) - \min I_j\Big)\Big).$$

We also say that ν^I is *decomposed* by $(\nu_j^{I_j})_{j\in N}$.

Definition. Let μ be a signed (local) measure on $\mathbb{R}$. We say that μ has the *finite decomposition property* (f.d.p.), if there exist a finite set $\mathcal{P}$ of finite pieces (called the

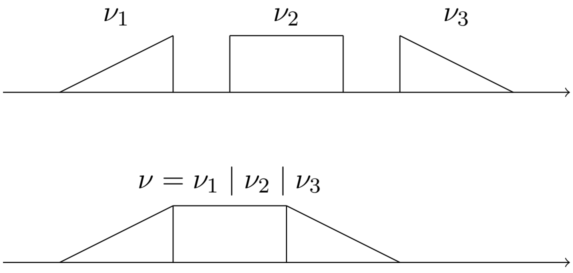

Figure 3.1.: Concatenation of three pieces.

local pieces) and $x_0 \in \mathbb{R}$, such that $\mathbb{1}_{[x_0,\infty)}\mu^{[x_0,\infty)}$ is a translate of a concatenation $\nu_1^{I_1} \mid \nu_2^{I_2} \mid \ldots$ with $\nu_j^{I_j} \in \mathcal{P}$ for all $j \in \mathbb{N}$. Without restriction, we may assume that $\min I = 0$ for all $\nu^I \in \mathcal{P}$.

A signed (local) measure μ has the *simple finite decomposition property* (s.f.d.p.), if it has the f.d.p. with a decomposition such that there is $\ell > 0$ with the following property: Assume that the two pieces

$$\nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0} \mid \nu_1^{I_1} \mid \ldots \mid \nu_{m_1}^{I_{m_1}} \quad \text{and} \quad \nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0} \mid \mu_1^{J_1} \mid \ldots \mid \mu_{m_2}^{J_{m_2}}$$

occur in the decomposition of μ with a common first part $\nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0}$ of length at least ℓ and such that

$$\mathbb{1}_{[0,\ell)}(\nu_1^{I_1} \mid \ldots \mid \nu_{m_1}^{I_{m_1}}) = \mathbb{1}_{[0,\ell)}(\mu_1^{J_1} \mid \ldots \mid \mu_{m_2}^{J_{m_2}}),$$

where $\nu_j^{I_j}$, $\mu_k^{J_k}$ are pieces from the decomposition (in particular, all belong to $\mathcal{P}$ and start at 0) and the latter two concatenations are of lengths at least ℓ. Then

$$\nu_1^{I_1} = \mu_1^{J_1}.$$

Having the s.f.d.p. can be interpreted as some sort of predictability of the measure. If a sufficiently long piece occurs twice in such a measure, then we know that the same shorter piece will follow at both occurrences.

3.2. Absence of absolutely continuous spectrum

We now prove the following fact. If the measure μ has the s.f.d.p. in both directions, i.e., μ and $\mu(-(\cdot))$ have the s.f.d.p., then either H_μ has empty absolutely continuous spectrum, or μ or $\mu(-(\cdot))$ are eventually periodic. Note that μ is called *eventually periodic* if there exists $x \in \mathbb{R}$ and $p > 0$ such that $\mu(A) = \mu(A + p)$ for all $A \subseteq [x, \infty)$ measurable.

3.2.1 Theorem ([29, Theorem 4.1]). *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be a measure that has the s.f.d.p. and assume that μ is not eventually periodic. Then the absolutely continuous spectrum of the half line operator $H_\mu|_{[0,\infty)}$ is empty, where $H_\mu|_{[0,\infty)}$ denotes the self-adjoint restriction of H_μ to $[0, \infty)$ with Dirichlet boundary conditions at 0.*

3.2.2 Theorem. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ such that μ and the reflected measure $\mu(-(\cdot))$ have the s.f.d.p. and assume that neither μ nor $\mu(-(\cdot))$ are eventually periodic. Then H_μ does not have absolutely continuous spectrum.*

Proof. By Theorem 3.2.1, $H_\mu|_{[0,\infty)}$ and $H_{\mu(-(\cdot))}|_{[0,\infty)}$ do not have absolutely continuous spectrum. Let $U\colon L_2((-\infty,0]) \to L_2([0,\infty))$, $Uf(t) := f(-t)$. Then U is unitary and

$$U^* H_{\mu(-(\cdot))}|_{[0,\infty)} U = H_\mu|_{(-\infty,0]}.$$

Hence, both half line operators $H_\mu|_{[0,\infty)}$ and $H_\mu|_{(-\infty,0]}$ do not have absolutely continuous spectrum. Therefore, also H_μ cannot have any absolutely continuous spectrum. //

3.3. Delone measures of finite local complexity

In this section we describe a device to construct potentials having the s.f.d.p. in both directions.

Definition. Let (X,d) be a metric space, $D \subseteq X$. Then D is called *uniformly discrete* if there exists $r > 0$ such that $B(x,r) \cap B(y,r) = \varnothing$ for all $x, y \in D$, $x \neq y$, where $B(x,r) := \{y \in X;\ d(x,y) < r\}$ denotes the open ball around x with radius r (in the metric space $\mathbb{R}$). We call D *relatively dense* if there exists $R > 0$ such that

$$\bigcup_{x\in D} B(x,R) = X.$$

Finally, D is called a *Delone set* if D is uniformly discrete and relatively dense.

Definition. Let $A \subseteq \mathbb{R}$ be a discrete set. Then A is of *finite local complexity* if for any $L \geq 0$

$$\{B[x,L] \cap (A - x);\ x \in A\}$$

is a finite set of subsets of $\mathbb{R}$. Here, $B[x,L] := \{y \in \mathbb{R};\ |x-y| \leq L\}$ is the closed ball (in the metric space $\mathbb{R}$).

3.3.1 Remark. A set $D \subseteq \mathbb{R}$ is a Delone set if and only if $D = \{x_n;\ n \in \mathbb{Z}\}$ with (x_n) increasing and there exist $r, R > 0$ such that $x_{n+1} - x_n \in [2r, R]$ for all $n \in \mathbb{N}$. Furthermore, if $\{x_{n+1} - x_n;\ n \in \mathbb{Z}\}$ is finite, then D is of finite local complexity.

Definition. We say that $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ is a *Delone measure of finite local complexity* if there exist finitely many signed measures $\nu_1, \ldots, \nu_N \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ supported on a compact interval starting at 0 such that with the sets D_j of occurrences of ν_j in μ ($j \in \{1,\ldots,N\}$) the following holds: $D := \bigcup_{j=1}^N D_j$ is a Delone set of finite local complexity and for any $x \in \operatorname{spt}\mu$, the support of μ, there exist $j \in \{1,\ldots,N\}$ and $p \in D_j$ such that $x \in p + [0, \sup \operatorname{spt}\nu_j)$ and $\mathbb{1}_{p+[0,\sup\operatorname{spt}\nu_j)}\mu$ is a translate of ν_j.

3.3.2 Lemma. *Let A be a finite set, $D \subseteq \mathbb{R}$ be a Delone set of finite local complexity. Let $f\colon D \to A$. For $a \in A$ let $\nu_a \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ have compact support. Define*

$$\mu := \sum_{x\in D} \delta_x * \nu_{f(x)} = \sum_{x\in D} \nu_{f(x)}(\cdot - x).$$

Then μ is a Delone measure of finite local complexity.

Proof. For $a \in A$ let $D_a := f^{-1}(\{a\})$, $D_a^{ini} := \{x + \inf \operatorname{spt} \nu_a;\ x \in D_a\}$ and $D_a^{end} := \{x + \sup \operatorname{spt} \nu_a;\ x \in D_a\}$. Let

$$\tilde{D} := \bigcup_{a \in A} (D_a^{ini} \cup D_a^{end}).$$

Then $\tilde{D}$ is a Delone set of finite local complexity, since D is such a set and A is finite. Let (x_n) be an increasing enumeration of $\tilde{D}$ and $X := \{x_{n+1} - x_n;\ n \in \mathbb{Z}\}$. Then X is a finite set. We now decompose μ with respect to the grid (x_n). For $n \in \mathbb{Z}$ let $\nu_n := (\mathbb{1}_{[x_n, x_{n+1})}\mu)(\cdot + x_n)$. Then μ is decomposed by $(\nu_n^{[0, x_{n+1} - x_n]})_{n \in \mathbb{Z}}$. Due to finiteness of X and the compact supports of the ν_a the set $\{\nu_n;\ n \in \mathbb{Z}\}$ is finite. Let $\tilde{\nu}_1, \ldots, \tilde{\nu}_N$ be an enumeration of this set, $\tilde{D}_j$ be the set of occurrences of $\tilde{\nu}_j$ $(j \in \{1, \ldots, N\})$. Then $\tilde{D} = \bigcup_{j=1}^N \tilde{D}_j$. Furthermore, for each $x \in \operatorname{spt} \mu$ there exist $j \in \{1, \ldots, N\}$ and $p \in \tilde{D}_j$ such that $x \in p + [0, \sup \operatorname{spt} \tilde{\nu}_j)$ and $\mathbb{1}_{p + [0, \sup \operatorname{spt} \tilde{\nu}_j)}\mu$ is a translate of $\tilde{\nu}_j$. //

3.3.3 Remark. The proof of the preceding lemma also shows, that the decomposition of μ is very simple. The finitely many pieces $\tilde{\nu}_j$ fit to the grid defined by the Delone set D in such a way that each piece is supported on exactly one (closed) interval defined by the grid. Furthermore, all pieces start at 0.

3.3.4 Lemma. *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be a Delone measure of finite local complexity with pieces $\nu_1, \ldots, \nu_N$ such that for all $j, j' \in \{1, \ldots, N\}$, $j \neq j'$ we have*

$$\mathbb{1}_{[0, \min\{\sup \operatorname{spt} \nu_j, \sup \operatorname{spt} \nu_{j'}\}]}\nu_j \neq \mathbb{1}_{[0, \min\{\sup \operatorname{spt} \nu_j, \sup \operatorname{spt} \nu_{j'}\}]}\nu_{j'}.$$

Then μ and $\mu(-(\cdot))$ have the s.f.d.p.

Proof. Let $\nu_1, \ldots, \nu_N$ be the pieces for the decomposition of μ according to the definition. Note that without loss of generality all pieces start at 0. Let s be the maximum of the lengths of the pieces $\nu_1, \ldots, \nu_N$. Let R be the parameter of D for being relatively dense. Choose $\ell > \max\{s, R\}$. Assume that

$$\nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0} \mid \nu_1^{I_1} \mid \ldots \mid \nu_{m_1}^{I_{m_1}} \quad \text{and} \quad \nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0} \mid \mu_1^{J_1} \mid \ldots \mid \mu_{m_2}^{J_{m_2}}$$

occur in the decomposition of μ with a common first part $\nu_{-m}^{I_{-m}} \mid \ldots \mid \nu_0^{I_0}$ of length at least ℓ and such that

$$\mathbb{1}_{[0,\ell)}(\nu_1^{I_1} \mid \ldots \mid \nu_{m_1}^{I_{m_1}}) = \mathbb{1}_{[0,\ell)}(\mu_1^{J_1} \mid \ldots \mid \mu_{m_2}^{J_{m_2}}),$$

where $\nu_j^{I_j}$, $\mu_k^{J_k}$ are pieces from the decomposition (in particular, all belong to $\mathcal{P}$ and start at 0) and the latter two concatenations are of length at least ℓ. Let p be the point where $\nu_1^{I_1}$ starts and p' be the point where $\mu_1^{J_1}$ starts. For $x \geq p$ such that x is covered by $\nu_1^{I_1}$ there exists j such that $\mathbb{1}_{p+[0, \sup \operatorname{spt} \nu_j)}\mu = \nu_1^{I_1}(\cdot - p)$ is a translate of ν_j. Also, for $x' \geq p'$ such that x' is covered by $\mu_1^{J_1}$ there exists j' such that $\mathbb{1}_{p'+[0, \sup \operatorname{spt} \nu_{j'})}\mu = \mu_1^{J_1}(\cdot - p')$ is a translate of $\nu_{j'}$. Since, by assumption,

$$\mathbb{1}_{p+[0,s)}\mu = (\mathbb{1}_{p'+[0,s)}\mu)(\cdot - (p - p')),$$

we conclude that ν_j is a translate of $\nu_{j'}$. So,

$$\nu_1^{I_1} = \mu_1^{J_1}.$$

The same argument applies to $\mu(-(\cdot))$. //

With the two lemmas at hand we can construct various measures having the s.f.d.p. (in both directions).

3.3.5 Example. Recall Example 2.3.4. Let $\alpha := 1 + \sqrt{2}$, $A := \{1, \alpha\}$, ν_a a signed Radon measure on $[0, a]$ $(a \in A)$ such that $\nu_1(\{0\}) = \nu_1(\{1\}) = \nu_\alpha(\{0\}) = \nu_\alpha(\{\alpha\})$ and $\nu_1 \neq \mathbb{1}_{[0,1]}\nu_\alpha$. Let μ be (one of the two) measure(s) constructed by the substitution rule given in Example 2.3.4. Let D be the corresponding grid and $f\colon D \to A$ such that $f(x)$ equals the length of the interval starting from x to the next point larger than x in the grid. Note that D is a Delone set of finite local complexity (since there are only two possible interval lengths). By Lemma 3.3.2 μ is a Delone measure of finite local complexity and by Lemma 3.3.4 μ and $\mu(-(\cdot))$ have the s.f.d.p. Thus, H_μ does not have absolutely continuous spectrum by Theorem 3.2.2 (since, obviously, neither μ nor $\mu(-(\cdot))$ are eventually periodic by construction).

Since μ is also a Gordon measure, Theorem 2.3.3 yields that H_μ does not have any pure point spectrum. Thus, we obtain purely singular continuous spectrum for H_μ.

Chapter 4

Random Schrödinger Operators 1

In this chapter we "randomize" the operator. That is, instead of choosing one particular measure (and hence operator) we investigate a whole family of measures (and hence operators). This leads to the notion of (random) operator families $(H_\omega)_{\omega\in\Omega}$. The general aim is then to prove spectral properties for the whole family instead of just one operator. There are basically two different ways to proceed. One can try to prove properties for all operators in that family. Unfortunately, that can hardly be done in general. Instead, one tries to obtain the properties for a large subset of the family. This will be implemented by means of a probability measure (and one then asks for the properties to hold on a set of full measure).

We will construct the operator family in a way such that Ω (the index set parametrizing the family) will be a compact metric space and $\mathbb{R}$ will act continuously on Ω. In other words, we impose a continuous flow $\alpha\colon \mathbb{R}\times\Omega\to\Omega$ on Ω and so obtain a dynamical system (Ω,α). Now, the question arises whether dynamical properties of the system (Ω,α) will lead to spectral properties of the operator family $(H_\omega)_{\omega\in\Omega}$. We will prove several theorems of this kind later in Chapter 6. For now (i.e., for this chapter) we aim to set the stage. We define the operator family and show first connections between dynamics on Ω and spectral properties of $(H_\omega)_{\omega\in\Omega}$: If (Ω,α) is minimal then the spectrum (as a set) of H_ω does not depend on ω. If (Ω,α) is ergodic, then $(H_\omega)_{\omega\in\Omega}$ is ergodic and by well-known arguments the spectrum and the spectral parts are almost surely constant as sets.

The remaining two sections are then devoted to continuity properties of solutions of the Schrödinger equation and to the transfer matrices of the family $(H_\omega)_{\omega\in\Omega}$. This motivates the objects studied in the next chapter.

4.1. The family of operators

In this section we introduce the suitable space of potentials. In order to obtain "nice" dependence of the operator H_μ on the potential μ we have to introduce the right topology on the space $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ of uniformly locally bounded signed local Radon measures. We then prove a uniform lower bound on the operators if the measures are $\|\cdot\|_{\mathrm{loc}}$-bounded. Since we also want to apply ideas from the theory of dynamical systems, we investigate the (natural) group action of $\mathbb{R}$ on the space of measures, i.e., show continuity of the group action with respect to the introduced vague topology on (a $\|\cdot\|_{\mathrm{loc}}$-bounded subset of) $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$.

4.1.1 Remark. (a) The *vague topology* on $C_c(\mathbb{R})'$ is defined to be the weak*-topology $\sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))$, where $C_c(\mathbb{R})$ is considered to be the inductive limit of the spaces $((C_0(-N,N), \|\cdot\|_\infty))_{N\in\mathbb{N}}$ (equipped with the inductive topology), where $C_0(-N,N)$ denotes the space of continuous function on $(-N,N)$ vanishing at the boundary. In fact, with $j_N\colon C_0(-N,N) \to C_c(\mathbb{R})$ defined by

$$j_N(f)(t) := \begin{cases} f(t) & t \in (-N,N), \\ 0 & t \notin (-N,N) \end{cases}$$

for $N \in \mathbb{N}$, we have $\bigcup_{N\in\mathbb{N}} j_N(C_0(-N,N)) = C_c(\mathbb{R})$. Furthermore, for $f \in C_c(\mathbb{R})$, $f \neq 0$ there exists $t \in \mathbb{R}$ with $f(t) \neq 0$, i.e., $\langle f, \delta_t\rangle \neq 0$ (where $\langle\cdot,\cdot\rangle$ denotes the dual pairing), and $\delta_t \circ j_N\colon C_0(-N,N) \to \mathbb{K}$ is continuous ($N \in \mathbb{N}$). So, by [40, Lemma 24.6], $C_c(\mathbb{R})$ can be equipped with the inductive topology of $((C_0(-N,N), \|\cdot\|_\infty))_{N\in\mathbb{N}}$. Since $C_0(-N,N)$ is separable for all $N \in \mathbb{N}$, also $C_c(\mathbb{R})$ as inductive limit is separable. Indeed, for $N \in \mathbb{N}$ let $\{f_n^N;\ n \in \mathbb{N}\}$ be a countable dense subset of $C_0(-N,N)$. Then $\{j_N(f_n^N);\ n, N \in \mathbb{N}\}$ is countable. Let $\mu \in C_c(\mathbb{R})'$, $\langle j_N(f_n^N), \mu\rangle = 0$ for all $n, N \in \mathbb{N}$. Then $\mu \circ j_N \in C_0(-N,N)'$, so $\mu \circ j_N = 0$ for all $N \in \mathbb{N}$. Hence, $\mu = 0$ and therefore $\{j_N(f_n^N);\ n, N \in \mathbb{N}\}$ is dense in $C_c(\mathbb{R})$.

(b) Note that (by the above considerations) $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}) \subseteq C_c(\mathbb{R})'$. Hence, the vague topology on $\mathcal{M}_{\mathrm{loc}}(\mathbb{R})$ is defined to be the restriction of the vague topology of $C_c(\mathbb{R})'$ to $\mathcal{M}_{\mathrm{loc}}(\mathbb{R})$.

The next proposition is probably well-known. However, we could not find a good reference for it.

4.1.2 Proposition. *Let $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be $\|\cdot\|_{\mathrm{loc}}$-bounded and closed with respect to the vague topology. Then Ω is $\sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))$-compact. Furthermore, the vague topology on Ω is induced by some metric, i.e., Ω is metrizable.*

Proof. (i) For $A > 0$ let $U_A := \{f \in C_c(\mathbb{R});\ |f(t)| \leq Ae^{-|t|}\ (t \in \mathbb{R})\}$. Then U_A is a neighborhood of 0 in $C_c(\mathbb{R})$, since $j_N^{-1}(U_A)$ is a neighborhood of 0 in $(C_0(-N,N), \|\cdot\|_\infty)$ for all $N \in \mathbb{N}$; cf. [40, Lemma 24.6].

(ii) For $U \subseteq C_c(\mathbb{R})$ we define the (absolute) polar set

$$U^\circ := \{\mu \in C_c(\mathbb{R})';\ |\langle f, \mu\rangle| \leq 1 \quad (f \in U)\}.$$

There exists $C \geq 0$, such that $\|\omega\|_{\mathrm{loc}} \leq C$ for all $\omega \in \Omega$. For $\mu \in \Omega$, $A > 0$ and $f \in U_A$ we have

$$\begin{aligned} |\langle f, \mu\rangle| &\leq \int |f|\, d|\mu| \leq \sum_{k=0}^{-\infty} \int_{k-1}^{k} |f|\, d|\mu| + \sum_{k=0}^{\infty} \int_{k}^{k+1} |f|\, d|\mu| \\ &\leq \sum_{k=0}^{-\infty} \|f\|_{L_\infty(k-1,k)} \|\mu\|_{\mathrm{loc}} + \sum_{k=0}^{\infty} \|f\|_{L_\infty(k-1,k)} \|\mu\|_{\mathrm{loc}} \\ &\leq C\left(\sum_{k=0}^{-\infty} Ae^{-|k|} + \sum_{k=0}^{\infty} Ae^{-|k|}\right) = CA\left(\sum_{k=0}^{\infty} e^{-k} + \sum_{k=0}^{\infty} e^{-k}\right) = CA\frac{2e}{e-1}. \end{aligned}$$

For $A \le \frac{e-1}{2Ce}$ we obtain

$$|\langle f,\mu\rangle| \le 1 \quad (\mu\in\Omega, f\in U_A).$$

Hence, $\Omega \subseteq U_A^\circ$.

(iii) The Theorem of Alaoglu-Bourbaki (see [40, Satz 23.5]) assures that U_A° is compact with respect to $\sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))$. For $A \le \frac{e-1}{2Ce}$ we have $\Omega \subseteq U_A^\circ$, and since Ω is $\sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))$-closed, Ω is $\sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))$-compact.

(iv) Since Ω is $\sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))$-compact, the topology $\sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))$ on Ω is induced by some metric d. Indeed, let $\mathcal{T}$ be the initial topology on $C_c(\mathbb{R})'$ induced by $(|\langle j_N(f_n^N), \cdot\rangle|\,;\ n,N\in\mathbb{N})$. Then $\mathcal{T}$ is semimetrizable by some semimetric d and $\mathcal{T}$ separates the points in $C_c(\mathbb{R})'$, i.e., $\langle j_N(f_n^N), \mu\rangle = 0$ for all $n,N\in\mathbb{N}$ implies $\mu = 0$. Hence, d is even a metric. Since the identity $I\colon (\Omega, \sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))\cap\Omega) \to (\Omega, \mathcal{T}\cap\Omega)$ is continuous, Ω is $\sigma(C_c(\mathbb{R})', C_c(\mathbb{R}))$-compact and $\mathcal{T}$ is separated, I is a homeomorphism. So, the vague topology on Ω is metrizable. //

From now on assume that $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ is $\|\cdot\|_{\mathrm{loc}}$-bounded and closed with respect to the vague topology. In this setting we always equip Ω with the vague topology such that Ω becomes a compact metric space. Furthermore, assume Ω to be *translation invariant*, i.e., for $\omega\in\Omega$ let also $\omega(\cdot+t)\in\Omega$ $(t\in\mathbb{R})$.

For $\omega\in\Omega$ the operator H_ω can be defined as above by means of the form

$$D(\tau_\omega) := W_2^1(\mathbb{R}), \quad \tau_\omega(u,v) := \tau_0(u,v) + \int u\bar{v}\,d\omega,$$

see Chapter 1.

4.1.3 Lemma. *There exists $\gamma\in\mathbb{R}$ such that $H_\omega \ge -\gamma$ $(\omega\in\Omega)$.*

Proof. Since Ω is $\|\cdot\|_{\mathrm{loc}}$-bounded there exists $C\ge 0$ such that

$$\|\omega\|_{\mathrm{loc}} \le C \quad (\omega\in\Omega).$$

By Lemma 1.1.1 we have ($a=\frac12$)

$$C_{1/2}(\omega) = \max\left\{8\,\|\omega\|_{\mathrm{loc}}^2, 2\,\|\omega\|_{\mathrm{loc}}\right\} \le \max\left\{8C^2, 2C\right\} =: \gamma \quad (\omega\in\Omega).$$

For $u\in W_2^1(\mathbb{R})$ we conclude

$$\tau_\omega(u) \ge \left\|u'\right\|_{L_2(\mathbb{R})}^2 - |\omega(u,u)| \ge \frac12\left\|u'\right\|_{L_2(\mathbb{R})}^2 - \gamma\,\|u\|_{L_2(\mathbb{R})}^2 \ge -\gamma\,\|u\|_{L_2(\mathbb{R})}^2$$

and hence $H_\omega \ge -\gamma$ for all $\omega\in\Omega$. //

The additive group $\mathbb{R}$ induces a group action of translations on Ω via $\alpha\colon \mathbb{R}\times\Omega\to\Omega$, $\alpha_t(\omega) := \alpha(t,\omega) := \omega(\cdot+t)$. Then α_t is bijective for all $t\in\mathbb{R}$.

4.1.4 Lemma. *α is continuous.*

Proof. Let (t_n) in $\mathbb{R}$, $t_n \to t$ and (ω_n) in Ω, $\omega_n \to \omega$. Then

$$\alpha_t(\omega_n) = \omega_n(\cdot + t) \to \omega(\cdot + t) = \alpha_t(\omega).$$

Let $f \in C_c(\mathbb{R})$, $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\left| \int f \, d(\alpha_t(\omega_n) - \alpha_t(\omega)) \right| \leq \varepsilon \quad (n \geq N).$$

Furthermore, by uniform continuity of f and convergence of (t_n), there exists $N' \geq N$ such that $|t_n - t| \leq 1$ and

$$|f(\cdot - t_n) - f(\cdot - t)| \leq \varepsilon$$

for $n \geq N'$. Hence, for $n \geq N'$, we obtain

$$\begin{aligned} &\left| \int f \, d(\alpha_{t_n}(\omega_n) - \alpha_t(\omega)) \right| \\ &\leq \int\limits_{\operatorname{spt} f + B(t,1)} |f(\cdot - t_n) - f(\cdot - t)| \; d\,|\omega_n| + \left| \int f(\cdot - t) \, d(\omega_n - \omega) \right| \\ &\leq \varepsilon \, |\omega_n| \, (\operatorname{spt} f + B(t,1)) + \varepsilon = (|\omega_n| \, (\operatorname{spt} f + B(t,1)) + 1) \, \varepsilon. \end{aligned}$$

As $\|\omega_n\|_{\mathrm{loc}} \leq C$ for all $n \in \mathbb{N}$ and $\operatorname{spt} f$ is compact, we arrive at $\alpha_{t_n}(\omega_n) \to \alpha_t(\omega)$. //

4.2. Constancy of the spectrum

This section deals with the mapping $\omega \mapsto H_\omega$. To show continuity of this mapping we have to choose the suitable topology on the space of operators. We will obtain continuity in the strong resolvent sense. Thus, also measurability of the mapping follows. With this at hand we can start to investigate the connection between dynamical properties of (Ω, α) and spectral properties of $(H_\omega)_{\omega \in \Omega}$.

It will be helpful to collect some prerequisites. We loosely follow [5]. Note that every finite signed Radon measure ν on $\mathbb{R}$ induces a continuous linear functional on $C_b(\mathbb{R})$, the space of bounded and continuous functions, via

$$\langle f, \nu \rangle := \int f \, d\nu \quad (f \in C_b(\mathbb{R})).$$

4.2.1 Lemma. *Let $\mu_n, \mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R})$ $(n \in \mathbb{N})$, $\mu_n \to \mu$ vaguely, $u \in C_c(\mathbb{R})$. Then $u\mu_n \to u\mu$ weakly (i.e., $\langle v, u\mu_n \rangle \to \langle v, u\mu \rangle$ for all $v \in C_b(\mathbb{R})$).*

Proof. Let $v \in C_b(\mathbb{R})$. Then $vu \in C_c(\mathbb{R})$ and

$$\int v \, d(u\mu_n) = \int vu \, d\mu_n \to \int vu \, d\mu = \int v \, d(u\mu). \qquad //$$

4.2.2 Remark. (a) For $f \in L_1(\mathbb{R})$ define the Fourier transform by

$$\hat{f}(p) := \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ipx} \, dx \quad (p \in \mathbb{R}).$$

It is well-known that $\hat{f} \in L_2(\mathbb{R})$ for $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, and that the Fourier transform extends to a unitary map on $L_2(\mathbb{R})$.

(b) For a finite signed measure ν on $\mathbb{R}$ define the Fourier transform by

$$\hat{\nu}(p) := \frac{1}{\sqrt{2\pi}} \int e^{-ipx}\, d\nu(x) \quad (p \in \mathbb{R}).$$

(c) For $f \in L_2(\mathbb{R})$ and a finite signed measure ν on $\mathbb{R}$ define

$$f * \nu(x) := \int f(x-y)\, d\nu(y)$$

for almost all $x \in \mathbb{R}$. Then $f * \nu \in L_2(\mathbb{R})$ and we have

$$\|f * \nu\|_{L_2(\mathbb{R})} = \sqrt{2\pi} \left\| \hat{f} \cdot \hat{\nu} \right\|_{L_2(\mathbb{R})}.$$

4.2.3 Lemma. *Let ν be a finite signed Radon measure on $\mathbb{R}$. Then $\nu \in W_2^1(\mathbb{R})'$ and*

$$\|\nu\|_{W_2^1(\mathbb{R})'} \leq \left\| \widehat{J(-(\cdot))} \cdot \hat{\nu} \right\|_{L_2(\mathbb{R})},$$

where $\hat{J}(p) = \frac{1}{\sqrt{1+p^2}}$ $(p \in \mathbb{R})$ and the hat indicates the Fourier transform.

Proof. We follow the ideas of [5, proof of Lemma 2]. There exists a unique $J \in L_2(\mathbb{R})$ such that $\hat{J}(p) = \frac{1}{\sqrt{1+p^2}}$ $(p \in \mathbb{R})$ and

$$J * f = \sqrt{2\pi}(-\Delta + 1)^{-1/2} f \quad (f \in L_2(\mathbb{R})).$$

Let $v \in C_c^\infty(\mathbb{R})$. Then we have

$$\begin{aligned} \int\int |J(x-y)|\, |v(y)|\, dy\, d|\nu|(x) &\leq \int \|J(x-\cdot)\|_{L_2(\mathbb{R})} \|v\|_{L_2(\mathbb{R})}\, d|\nu|(x) \\ &= \|J\|_{L_2(\mathbb{R})} \|v\|_{L_2(\mathbb{R})} |\nu|(\mathbb{R}) < \infty. \end{aligned}$$

Hence, Fubini's Theorem applies and we obtain

$$\begin{aligned} \left| \int v\, d\nu \right| &= \left| \int (-\Delta+1)^{-1/2}(-\Delta+1)^{1/2} v\, d\nu \right| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int\int J(x-y)(-\Delta+1)^{1/2} v(y)\, dy\, d\nu(x) \right| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int\int J(x-y)\, d\nu(x)(-\Delta+1)^{1/2} v(y)\, dy \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \|J(-(\cdot)) * \nu\|_{L_2(\mathbb{R})} \left\| (-\Delta+1)^{1/2} v \right\|_{L_2(\mathbb{R})} \\ &= \left\| \widehat{J(-(\cdot))} \cdot \hat{\nu} \right\|_{L_2(\mathbb{R})} \|v\|_{W_2^1(\mathbb{R})}. \end{aligned}$$

By density of $C_c^\infty(\mathbb{R})$ in $W_2^1(\mathbb{R})$ we obtain the assertion. //

4.2.4 Lemma ([5, Lemma 1])**.** *Let ν, ν_k be finite signed Radon measures $(k \in \mathbb{N})$, $\nu_k \to \nu$ weakly. Then $\sup_{k \in \mathbb{N}} \|\hat{\nu}_k\|_\infty < \infty$.*

Proof. Weak convergence of (ν_k) is exactly pointwise convergence of the corresponding linear functionals. The uniform boundedness principle yields $\sup_{k\in\mathbb{N}} \|\nu_k\|_{C_b(\mathbb{R})'} < \infty$. Furthermore, for $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$|\hat{\nu}_k(t)| = \left|\frac{1}{\sqrt{2\pi}}\nu_k(e^{-it(\cdot)})\right| \leq \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} \left|e^{-its}\right| \, d\,|\nu_k|\,(s) \leq \frac{1}{\sqrt{2\pi}} \sup_{k\in\mathbb{N}} \|\nu_k\|_{C_b(\mathbb{R})'} \,.$$

Hence,

$$\sup_{k\in\mathbb{N}} \|\hat{\nu}_k\|_\infty < \infty.$$ //

We recall a theorem we will use.

4.2.5 Theorem ([59, Theorem A.1])**.** *Let $(\mathcal{H}, (\cdot\,|\,\cdot))$ be a Hilbert space, $\tau \geq 1$ a densely defined closed symmetric form on $\mathcal{H}$. Let $(\tau_k)_{k\in\mathbb{N}\cup\{\infty\}}$ be a sequence of densely defined closed symmetric forms on $\mathcal{H}$ such that*

(a) *there exists $c \geq 1$ such that $\tau \leq \tau_k \leq c\tau$ $(k \in \mathbb{N}\cup\{\infty\})$,*

(b) *there exists $D \subseteq D_\tau$ dense such that for all $u \in D$ we have $\tau_k(u,\cdot) \to \tau_\infty(u,\cdot)$ in D'_τ.*

Let H_k be the self-adjoint operator associated with τ_k $(k \in \mathbb{N}\cup\{\infty\})$. Then $H_k \to H_\infty$ in strong resolvent sense.

Proof. Let D'_τ denote the dual of D_τ, the set of all conjugate linear continuous forms dualized by the inner product of $\mathcal{H}$. Let $J\colon \mathcal{H} \to D'_\tau$, $u \mapsto (u\,|\,\cdot)$ be the embedding, $\tilde{H}_k\colon D_\tau \to D'_\tau$, $\tilde{H}_k u := \tau_k(u,\cdot)$ $(k \in \mathbb{N}\cup\{\infty\})$. Then $\left\|\tilde{H}_k^{-1}\right\| \leq 1$ and $\tilde{H}_k^{-1}J = H_k^{-1}$ for all $k \in \mathbb{N}\cup\{\infty\}$, since

$$\left(\tilde{H}_k u \,\middle|\, u\right) = \tau_k(u,u) \geq 1 \quad (u \in D_\tau)$$

and

$$\tilde{H}_k H_k^{-1} u(v) = \tau_k(H_k^{-1}u, v) = (u\,|\,v) = Ju(v) \quad (u \in \mathcal{H}, v \in D_\tau).$$

Consequently, we have

$$H_k^{-1} - H_\infty^{-1} = \tilde{H}_k^{-1}\left(\tilde{H}_\infty - \tilde{H}_k\right)\tilde{H}_\infty^{-1}J.$$

Now, condition (b) implies $\tilde{H}_k \to \tilde{H}_\infty$ strongly on D. Since D is dense in D_τ and $(\tilde{H}_k)$ is uniformly bounded by c, the assertion follows. //

Now, we can show that vague convergence of the measures implies strong resolvent convergence of the corresponding operators.

4.2.6 Proposition. *Let (μ_n) in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be bounded, $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, $\mu_n \to \mu$ vaguely. Then $H_{\mu_n} \to H_\mu$ in strong resolvent sense.*

Proof. (i) Let $u \in C_c^\infty(\mathbb{R})$ and $v \in W_2^1(\mathbb{R})$. Then

$$|\tau_{\mu_k}(u,v) - \tau_\mu(u,v)| = \left|\int\limits_{\mathbb{R}} u\overline{v}\,d(\mu_k - \mu)\right|.$$

Let $\nu_k := u\mu_k$ ($k \in \mathbb{N}$), $\nu := u\mu$. Then ν_k, ν are finite signed Radon measures on $\mathbb{R}$ ($k \in \mathbb{N}$), $\nu_k \to \nu$ weakly by Lemma 4.2.1 and $\sup_{k\in\mathbb{N}} \|\hat{\nu}_k\|_\infty < \infty$ by Lemma 4.2.4.

(ii) Lemma 4.2.3 yields

$$\left|\int u\overline{v}\, d(\mu_k - \mu)\right| = \left|\int \overline{v}\, d(\nu_k - \nu)\right| \leq \left\|\widehat{J(-(\cdot))} \cdot (\hat{\nu}_k - \hat{\nu})\right\|_{L_2(\mathbb{R})} \|v\|_{W_2^1(\mathbb{R})}.$$

Since $\nu_k \to \nu$ weakly, we have $\hat{\nu}_k(p) \to \hat{\nu}(p)$ for all $p \in \mathbb{R}$. Furthermore,

$$\sup_{k\in\mathbb{N}} \|\hat{\nu}_k - \hat{\nu}\|_\infty < \infty.$$

Thus,

$$\left\|\widehat{J(-(\cdot))} \cdot (\hat{\nu}_k - \hat{\nu})\right\|_{L_2(\mathbb{R})} \to 0$$

by Lebesgue's dominated convergence theorem. This implies that

$$|\tau_{\mu_k}(u,\cdot) - \tau_\mu(u,\cdot)| \to 0$$

in $W_2^1(\mathbb{R})'$. As $k \to \infty$, we conclude by Theorem 4.2.5 that $H_{\mu_k} \to H_\mu$ in strong resolvent sense. Note that $\frac{1}{2}\tau_0 + 1 \geq 1$, $\tau_0 + \mu_k + \gamma + 1 \geq \frac{1}{2}\tau_0 + 1$ and $\tau_0 + \mu_k + \gamma + 1 \leq C(\frac{1}{2}\tau_0 + 1)$ for some C and γ by boundedness of the sequence (μ_k). //

Let us introduce some terminology from dynamical systems.

Definition. Let Ω be a compact metric space and $\alpha\colon \mathbb{R} \times \Omega \to \Omega$ a continuous group action on Ω. Then (Ω, α) is called *dynamical system*. A *dynamical system* (Ω, α) is called *ergodic* with *ergodic measure* $\mathbb{P}$ if every α-invariant measurable subset $A \subseteq \Omega$ satisfies $\mathbb{P}(A) \in \{0, 1\}$. If the ergodic measure $\mathbb{P}$ is unique, then $(\Omega, \alpha, \mathbb{P})$ is said to be *uniquely ergodic*. We call the dynamical system (Ω, α) *minimal*, if every orbit $\mathcal{O}(\omega) := \{\alpha_t(\omega);\ t \in \mathbb{R}\}$ is dense in Ω. If $(\Omega, \alpha, \mathbb{P})$ is uniquely ergodic and minimal, then we call it *strictly ergodic*.

We will need some more notions for operator families.

Definition. Let $(\Omega, \mathbb{P})$ be a probability space. For $\omega \in \Omega$ let H_ω be a self-adjoint operator in $L_2(\mathbb{R})$ and let $\omega \mapsto (H_\omega - z)^{-1}$ be weakly measurable for all $z \in \mathbb{C} \setminus \mathbb{R}$ (i.e., the $\omega \mapsto H_\omega$ is *measurable*). The family $(H_\omega)_{\omega\in\Omega}$ is said to be *ergodic* if there exists an ergodic family $(T_\iota)_{\iota\in I}$ on $(\Omega, \mathbb{P})$ (i.e., T_ι is measurable ($\iota \in I$), and if $A \subseteq \Omega$ is measurable and $T_\iota^{-1}(A) = A$ for all $\iota \in I$, then $\mathbb{P}(A) \in \{0, 1\}$), and a family $(U_\iota)_{\iota\in I}$ of unitaries on $L_2(\mathbb{R})$ such that

$$H_{T_\iota(\omega)} = U_\iota^* H_\omega U_\iota \quad (\omega \in \Omega, \iota \in I).$$

There are several equivalent characterizations for being a measurable operator family, see e.g. [9, Section V.1]. Since our canonical example will be measurable we don't state these properties. The following results hold true in much more generality. Nevertheless, we restrict to our case of measure perturbed Schrödinger operators.

4.2.7 Lemma. *Let $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be $\|\cdot\|_{\mathrm{loc}}$-bounded, vaguely closed and translation invariant, α the group action of $\mathbb{R}$ on Ω. Then $\omega \mapsto H_\omega$ is measurable.*

Proof. By Proposition 4.2.6, $\omega \mapsto (H_\omega - z)^{-1}$ is strongly continuous for all $z \in \mathbb{C} \setminus \mathbb{R}$ and hence weakly measurable. //

The next lemma relates ergodicity of (Ω, α) with ergodicity of $(H_\omega)_{\omega\in\Omega}$.

4.2.8 Lemma. *Let $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be $\|\cdot\|_{\mathrm{loc}}$-bounded, vaguely closed and translation invariant, α the group action of $\mathbb{R}$ on Ω. Let $(\Omega, \alpha, \mathbb{P})$ be ergodic. Then $(H_\omega)_{\omega\in\Omega}$ is ergodic.*

Proof. For $t \in \mathbb{R}$ the mappings $\alpha_t \colon \Omega \to \Omega$, $\alpha_t(\omega) := \alpha(t, \omega) = \omega(\cdot + t)$ are ergodic. Furthermore, $U(t) \colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $U(t)f := f(\cdot - t)$ is unitary $(t \in \mathbb{R})$. We have

$$H_{\alpha_t(\omega)} = U(-t)H_\omega U(t) \quad (t \in \mathbb{R}, \omega \in \Omega).$$

Therefore, (H_ω) is ergodic. //

A well-known fact for ergodic operator families is that the spectrum is almost surely constant, see e.g. [9].

4.2.9 Proposition ([9, Proposition V.2.4 and Remark V.2.5])**.** *Let $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be $\|\cdot\|_{\mathrm{loc}}$-bounded, vaguely closed and translation invariant, α the group action of $\mathbb{R}$ on Ω. Let $(\Omega, \alpha, \mathbb{P})$ be ergodic. Then there exists $\Sigma, \Sigma_\bullet \subseteq \mathbb{R}$ closed such that for $\mathbb{P}$-a.a. $\omega \in \Omega$ we have*

$$\sigma(H_\omega) = \Sigma, \quad \sigma_\bullet(H_\omega) = \Sigma_\bullet \quad (\bullet \in \{s, c, ac, sc, pp\}).$$

However, if the underlying dynamical system is minimal we obtain constancy of the spectrum which is a much stronger result. This is the main theorem of this section.

4.2.10 Theorem. *Let $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be $\|\cdot\|_{\mathrm{loc}}$-bounded, vaguely closed and translation invariant, α the group action of $\mathbb{R}$ on Ω. Let (Ω, α) be minimal. Then there is $\Sigma \subseteq \mathbb{R}$ such that*

$$\sigma(H_\omega) = \Sigma \quad (\omega \in \Omega).$$

Proof. (i) Define $U \colon \mathbb{R} \to L(\mathcal{H})$ by $U(t)f = f(\cdot - t)$. Then U is a group of unitaries and

$$H_{\alpha_t(\omega)} = U(-t)H_\omega U(t) \quad (t \in \mathbb{R}, \omega \in \Omega).$$

(ii) Let $\omega, \omega' \in \Omega$. If ω and ω' are on the same orbit, i.e. $\mathcal{O}(\omega) = \mathcal{O}(\omega')$, we obtain $\sigma(H_\omega) = \sigma(H_{\omega'})$ by (i). Otherwise, by minimality, there exists (ω_k) in $\mathcal{O}(\omega)$ such that $\omega_k \to \omega'$. Then $\sigma(H_{\omega_k}) = \sigma(H_\omega)$ for all $k \in \mathbb{N}$ by (i).

(iii) By Proposition 4.2.6 we have $H_{\omega_k} \to H_{\omega'}$ in strong resolvent sense.

By [46, Theorem VIII.24] for $E \in \sigma(H_{\omega'})$ there is $E_k \in \sigma(H_{\omega_k})$ $(k \in \mathbb{N})$ with $E_k \to E$. But $\sigma(H_{\omega_k}) = \sigma(H_\omega)$ for all $k \in \mathbb{N}$ and $\sigma(H_\omega)$ is closed, so $E \in \sigma(H_\omega)$. Thus, we have shown $\sigma(H_{\omega'}) \subseteq \sigma(H_\omega)$. Interchanging the roles of ω and ω' yields $\sigma(H_\omega) \subseteq \sigma(H_{\omega'})$ and therefore $\sigma(H_\omega) = \sigma(H_{\omega'})$. //

4.2.11 Remark. In [33] it is proven that for almost periodic bounded potentials in the minimally ergodic case the absolutely continuous part of the spectrum is constant. By [23] the singular continuous and the pure point spectra need not be constant.

4.3. Continuity of solutions of the Schrödinger equation

This section is a preparation for the main results in Chapter 6. In fact, the properties in this and the following section give rise to the objects (i.e., cocycles) studied more abstractly in Chapter 5. Let $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be $\|\cdot\|_{\mathrm{loc}}$-bounded, vaguely closed and translation invariant, α the group action of $\mathbb{R}$ on Ω.

4.3.1 Lemma. *Let $z \in \mathbb{C}$, $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, u the solution of $H_\mu u = zu$ for some fixed initial condition at 0, (μ_n) in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ be $\|\cdot\|_{\mathrm{loc}}$-bounded, $\mu_n \to \mu$, u_n the solution of $H_{\mu_n} u_n = zu_n$ subject to the same initial conditions at 0 as u. Then, for all $K \subseteq \mathbb{R}$ compact, $u_n \to u$ in $C(K)$ and $u_n'(t+) \to u'(t+)$ for all $t \in \mathbb{R} \setminus \operatorname{spt} \mu_{pp}$ (i.e., for all $t \in \mathbb{R}$ not in the countable set $\operatorname{spt} \mu_{pp} := \{t \in \mathbb{R}; \mu(\{t\}) \neq 0\}$).*

Proof. There exists $C \geq 0$ such that $\|\mu_n\|_{\mathrm{loc}}, \|\mu\|_{\mathrm{loc}} \leq C$ $(n \in \mathbb{N})$.

We will only prove the case $t > 0$ and $K = [0,t]$. The case $t < 0$ can be treated analogously. For $K \subseteq \mathbb{R}$ compact then choose $t > 0$ such that $K \subseteq [-t,t]$.

(i) For $t \geq 0$ we have

$$u(t) = u(0) + u'(0+)t + \int_{(0,t]} (t-s)u(s)\, d(\mu - z\lambda)(s),$$

and analogously, for $n \in \mathbb{N}$,

$$u_n(t) = u_n(0) + u_n'(0+)t + \int_{(0,t]} (t-s)u_n(s)\, d(\mu_n - z\lambda)(s).$$

(ii) By vague convergence of $\mu_n \to \mu$ we obtain $\mathbb{1}_{(0,t]}\mu_n \to \mathbb{1}_{(0,t]}\mu$ weakly for all $t > 0$ such that $\mu(\{t\}) = 0$ $(n \in \mathbb{N})$; cf. [7, Section 28].

(iii) Let $t \geq 0$. For $n \in \mathbb{N}$ define

$$g_n(s) := \int_{(0,s]} u(r)\, d(\mu - \mu_n)(r) \quad (s \in [0,t]).$$

For λ-a.a. $s \in [0,t]$ we have $g_n(s) \to 0$ by (ii). Furthermore,

$$\begin{aligned} |g_n(s)| &\leq s\,\|u\|_{\infty,[0,s]}\,(|\mu| + |\mu_n|)((0,s]) \leq t\,\|u\|_{\infty,[0,t]}\,(|\mu| + |\mu_n|)([0,t]) \\ &\leq t\,\|u\|_{\infty,[0,t]}\,2C(t+1) < \infty. \end{aligned}$$

By Lebesgue's dominated convergence theorem, $g_n \to 0$ in $L_2(0,t)$.

For $n \in \mathbb{N}$ define

$$f_n(s) := \int_0^s g_n(r)\, dr = \int_{(0,s]} (s-r)u(r)\, d(\mu - \mu_n)(r) \quad (s \in [0,t]).$$

For λ-a.a. $s \in [0,t]$ we have $f_n(s) \to 0$ by (ii). Furthermore,

$$|f_n(s)| \leq t\,\|u\|_{\infty,[0,t]}\,(|\mu| + |\mu_n|)([0,t]) \leq t\,\|u\|_{\infty,[0,t]}\,2C(t+1) < \infty.$$

Again by Lebesgue's dominated convergence theorem, $f_n \to 0$ in $L_2(0,t)$. As $f_n' = g_n$ ($n \in \mathbb{N}$), we obtain $f_n \to 0$ in $W_2^1(0,t)$, and by Sobolev's embedding, $f_n \to 0$ in $C[0,t]$.

(iv) Let $n \in \mathbb{N}$. We have

$$(u_n - u)(s) = f_n(s) + \int\limits_{(0,s]} (s-r)(u_n - u)(r)\, d(\mu_n - z\lambda)(r).$$

Hence,

$$|u_n - u|(s) \le \|f_n\|_{\infty,[0,t]} + \int\limits_{[0,s)} |t|\, |u_n - u|(r)\, d(|\mu_n| + |z|\,\lambda)(r) \quad (s \in [0,t]).$$

By Gronwall's inequality (Lemma A.1), we obtain

$$|u_n - u|(s) \le \|f_n\|_{\infty,[0,t]} + \int\limits_{[0,s]} \|f_n\|_{\infty,[0,t]}\, e^{|t|(|\mu_n|+|z|\lambda)([r,s])}\, |t|\; d(|\mu_n| + |z|\,\lambda)(r)$$

for all $s \in [0,t]$. Therefore,

$$\sup_{s\in[0,t]} |u_n - u|(s) \le \|f_n\|_{\infty,[0,t]} \Big(1 + e^{t(Ct+|z|t)} t(C(t+1) + |z|\, t)\Big).$$

As $n \to \infty$,

$$\sup_{s\in[0,t]} |u_n - u|(s) \to 0$$

by (iii). Hence, $u_n \to u$ in $C[0,t]$.

(v) For $t \ge 0$ we have

$$(u' - u_n')(t+) = \int\limits_{(0,t]} u(s)\, d(\mu - \mu_n)(s) + \int\limits_{(0,t]} (u - u_n)(s)\, d(\mu_n - z\lambda)(s) \quad (t \ge 0).$$

Since $\mathbb{1}_{(0,t]}\mu_n \to \mathbb{1}_{(0,t]}\mu$ in $\sigma(C[0,t]', C[0,t])$ for λ-a.a. $t \ge 0$ by (ii) and $u_n \to u$ in $C[0,t]$ we obtain $u_n'(t+) \to u'(t+)$ for λ-a.a. $t \ge 0$. More precisely, $u_n'(t+) \to u'(t+)$ if $\mu(\{t\}) = 0$. //

4.3.2 Remark. A slight modification of the proof of Lemma 4.3.1 yields the following: Let $K \subseteq \mathbb{C}$ be compact, $\mu_n, \mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ ($n \in \mathbb{N}$), $\mu_n \to \mu$, $u(\cdot, z)$ a solution of $H_\mu u = zu$, $u_n(\cdot, z)$ a solution of $H_{\mu_n} u = zu$, for $z \in K$, all obeying the same initial conditions at 0. Then

$$\sup_{s\in[\min\{t,0\},\max\{t,0\}]} \sup_{z\in K} |u_n(s,z) - u(s,z)| \to 0 \quad (t \in \mathbb{R})$$
$$\sup_{z\in K} \left|u_n'(t+,z) - u'(t+,z)\right| \to 0 \quad (t \in \mathbb{R} \setminus \operatorname{spt} \mu_{pp}).$$

4.3.3 Lemma. *For $z \in \mathbb{C}$ and $\omega \in \Omega$ let $u_z(\cdot,\omega)$ be the solution of $H_\omega u_z = z u_z$ subject to some fixed initial conditions at 0. Let (z_n) in $\mathbb{R}$, $z_n \to z$. Then*

$$\sup_{-1\le t\le 1}\sup_{\omega\in\Omega} |u_z(t,\omega) - u_{z_n}(t,\omega)| \to 0$$

and

$$\sup_{-1\le t\le 1}\sup_{\omega\in\Omega} \left|u_z'(t+,\omega) - u_{z_n}'(t+,\omega)\right| \to 0.$$

Proof. Let $z, w \in \mathbb{C}$, u_z and u_w the solutions of $H_\omega u_z = z u_z$ and $H_\omega u_w = w u_w$ subject to the same initial conditions at 0. We suppress the dependence of ω in the notation. For $-1 \le t \le 1$ we have

$$\begin{aligned}
&\left|u_z'(t+) - u_w'(t+)\right| \\
&= \left| \int_0^t u_z(s)\, d(\omega - z\lambda)(s) - \int_0^t u_w(s)\, d(\omega - w\lambda)(s) \right| \\
&= \left| \int_0^t (u_z(s) - u_w(s))\, d\omega(s) - (z-w)\int_0^t u_z(s)\, ds - w\int_0^t (u_z(s) - u_w(s))\, ds \right| \\
&\le \int_0^t |u_z(s) - u_w(s)|\, d\,|\omega|\,(s) + |z-w| \int_0^t |u_z(s)|\, ds + |w| \int_0^t |u_z(s) - u_w(s)|\, ds \\
&\le \int_{-1}^1 |u_z(s) - u_w(s)|\, d\,|\omega|\,(s) + |z-w| \int_{-1}^1 |u_z(s)|\, ds + |w| \int_{-1}^1 |u_z(s) - u_w(s)|\, ds.
\end{aligned}$$

Thus, it suffices to show the uniform convergence for the functions.

We have

$$u_z(t) = u_z(0) + A_\omega u_z(0) t + \int_0^t (t-s) u_z(s)\, d(\omega - z\lambda)(s) \quad (t \in \mathbb{R}).$$

Hence, for $0 \le t \le 1$ we obtain by Gronwall's inequality (Lemma A.1)

$$\begin{aligned}
&|u_z(t) - u_w(t)| \\
&\le \int_{[0,t)} |u_z(s) - u_w(s)|\, d\,|\omega|\,(s) + |z-w| \int_0^t |u_z(s)|\, ds + |w| \int_{[0,t)} |u_z(s) - u_w(s)|\, ds \\
&\le |z-w| \int_0^t |u_z(s)|\ ds \\
&\quad + \int_0^t \left(|z-w| \int_0^s |u_z(r)|\ dr \right) e^{(|\omega| + |w|\lambda)([0,t])}\, d(|\omega| + |w|\,\lambda)(s) \\
&\le |z-w| \int_0^1 |u_z(s)|\ ds \left(1 + e^{\|\omega\|_{\mathrm{loc}} + |w|} (\|\omega\|_{\mathrm{loc}} + |w|)\right).
\end{aligned}$$

Since Ω is $\|\cdot\|_{\text{loc}}$-bounded, there exists $C \geq 0$ such that

$$\sup_{\omega\in\Omega} |u_z(s)| \leq C \quad (s \in [0,1]).$$

Hence, as $w \to z$,

$$\sup_{0\leq t\leq 1} \sup_{\omega\in\Omega} |u_z(t) - u_w(t)| \to 0.$$

Similar reasoning shows convergence for $-1 \leq t \leq 0$. //

4.4. Transfer matrices

Let $\Omega \subseteq \mathcal{M}_{\text{loc,unif}}(\mathbb{R})$ be $\|\cdot\|_{\text{loc}}$-bounded, vaguely closed and translation invariant, α the group action of $\mathbb{R}$ on Ω as above. We collect some facts concerning the transfer matrices.

Let $z \in \mathbb{C}$ and $\omega \in \Omega$. We consider the eigenvalue equation $H_\omega u = zu$.

The solution of this equation is determined by the values of u and u' at $t = 0$ by Lemma 1.2.6.

As in Chapter 1, for $z \in \mathbb{C}$, $\omega \in \Omega$ and $t \in \mathbb{R}$ let

$$T_z(t,\omega) = \begin{pmatrix} u_N(t) & u_D(t) \\ u_N'(t+) & u_D'(t+) \end{pmatrix}$$

denote the transfer matrix, i.e., the 2-by-2-matrix satisfying

$$\begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} = T_z(t,\omega) \begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix},$$

where u is the solution of $H_\omega u = zu$. Note that for u_N and u_D we dropped the dependence on z and ω to simplify notation.

4.4.1 Remark. As in Chapter 1 we have

$$T_z(0,\omega) = I, \quad T_z(s+t,\omega) = T_z(s,\alpha_t(\omega))T_z(t,\omega) \quad (s,t \in \mathbb{R}, \omega \in \Omega, z \in \mathbb{C}),$$

i.e., the transfer matrices form a *cocycle*. Furthermore, solutions of the equation $H_\omega u = zu$ may not be continuously differentiable due to possible point masses. Hence $T_z(\cdot,\omega)$ may not be continuous anymore. Thus, although the group action α is continuous, the cocycle T_z may not be continuous.

4.4.2 Lemma. *Let $z \in \mathbb{C}$, $\omega \in \Omega$. Then there exists a countable set $N_\omega \in \mathbb{R}$, such that $T_z(t,\cdot)$ is continuous at ω for all $t \in \mathbb{R} \setminus N_\omega$.*

Proof. Without loss of generality, let $t \geq 0$. Let $\omega \in \Omega$ and (ω_n) in Ω with $\omega_n \to \omega$. Let u be the solution of $H_\omega u = zu$ and u_n be the solution of $H_{\omega_n} u_n = zu_n$ $(n \in \mathbb{N})$, all satisfying the same initial conditions at $t = 0$. By Lemma 4.3.1, all entries of $T_z(t,\omega_n)$ converge to the corresponding entries of $T_z(t,\omega)$ for $t \in \mathbb{R} \setminus \operatorname{spt} \omega_{pp}$. Hence $T_z(t,\omega_n) \to T_z(t,\omega)$ for $t \in \mathbb{R} \setminus \operatorname{spt} \omega_{pp}$. //

Note that the countable set N_ω is given by $N_\omega = \operatorname{spt} \omega_{pp} = \{t \in \mathbb{R};\ \omega(\{t\}) \neq 0\}$.

4.4.3 Lemma. *Let $z \in \mathbb{C}$, $t \in \mathbb{R}$. Then $T_z(t,\cdot)$ is measurable.*

Proof. Let $u(\cdot,\omega)$ be a solution of $H_\omega u = zu$. By Lemma 4.3.1, $\omega \mapsto u(t,\omega)$ is continuous and hence measurable. By Lemma 1.2.5, $u'(t+,\omega) = \lim_{h\to 0+} \frac{u(t+h,\omega)-u(t,\omega)}{h}$. Since $\omega \mapsto \frac{u(t+h,\omega)-u(t,\omega)}{h}$ is continuous, $\omega \mapsto u'(t+,\omega)$ is measurable. Therefore, all entries of $T_z(t,\cdot)$ are measurable and so $T_z(t,\cdot)$ is measurable. //

We summarize the properties of T_z obtained above.

Definition. Let $(\Omega, (\alpha_t)_{t\in\mathbb{R}})$ be a dynamical system and $A\colon \mathbb{R}\times\Omega \to SL(2,\mathbb{C})$. Then A is called an *almost continuous cocycle* if

$$A(0,\omega) = I, \quad A(s+t,\omega) = A(s,\alpha_t(\omega))A(t,\omega) \quad (s,t \in \mathbb{R}, \omega \in \Omega),$$

$A(t,\cdot)$ is measurable for all $t \in \mathbb{R}$, $A(\cdot,\omega)$ is right continuous for all $\omega \in \Omega$ and for all $\omega \in \Omega$ there exists $N_\omega \subseteq \mathbb{R}$ countable such that $A(t,\cdot)$ is continuous at ω for $t \in \mathbb{R}\setminus N_\omega$.

If $N_\omega = \varnothing$ ($\omega \in \Omega$), i.e., if $A(t,\cdot)$ is continuous for all $t \in \mathbb{R}$, then we say that A is a *continuous cocycle*.

There are different notions for continuous cocycles in the literature. Sometimes they require that A is continuous. Note that we just assume that $A(t,\cdot)$ is continuous ($t \in \mathbb{R}$) and $A(\cdot,\omega)$ is right continuous ($\omega \in \Omega$).

Definition. We call $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ *atomless* if $\omega(\{t\}) = 0$ for all $t \in \mathbb{R}$, $\omega \in \Omega$.

4.4.4 Theorem. *Let $z \in \mathbb{C}$. Then T_z is an almost continuous cocycle. In case that Ω is atomless T_z is even a continuous cocycle.*

Proof. From Chapter 1 we know

$$\det T_z(t,\omega) = 1 \quad (t \in \mathbb{R}).$$

Clearly, $T_z(0,\omega) = I$ and T_z defines a cocycle. Right continuity of $T_z(\cdot,\omega)$ is a direct consequence of Lemma 1.2.4. Continuity except on a countable set was proven in Lemma 4.4.2. Measurability was shown in Lemma 4.4.3.

Let Ω be atomless. Then $N_\omega = \varnothing$ for all $\omega \in \Omega$. Hence, Lemma 4.4.2 yields continuity of $T_z(t,\cdot)$ for all $t \in \mathbb{R}$. //

The whole next chapter will focus on (almost) continuous cocycles. Since we are mainly interested in the Schrödinger case of cocycles we conclude this chapter with some of its properties.

4.4.5 Remark. If $E \in \mathbb{R}$, then $T_E\colon \mathbb{R}\times\Omega \to SL(2,\mathbb{R})$, i.e., the entries of the transfer matrices at energy E are real.

4.4.6 Proposition. *Let $z \in \mathbb{C}$. Then*

$$\sup_{-1\le t\le 1} \sup_{\omega\in\Omega} \|T_z(t,\omega)\| < \infty.$$

Proof. Again, for simplicity, we do not state the dependence of solutions on ω and z explicitly. Let $t \in [-1,1]$, $\omega \in \Omega$. Then

$$\|T_z(t,\omega)\|^2 \le |u_D(t)|^2 + |u_N(t)|^2 + \left|u_D'(t+)\right|^2 + \left|u_N'(t+)\right|^2 .$$

Let u be a solution of $H_\omega u = zu$ with normalized initial condition at 0. Then

$$u(t) = u(0) + A_\omega u(0)t - z\int_0^t (t-s)u(s)\,ds + \int_0^t (t-s)u(s)\,d\omega(s).$$

Hence, for $0 \le t \le 1$ we have

$$|u(t)| \le |u(0)| + \left|u'(0+) - u(0)\omega(\{0\})\right| + |z| \int_{[0,t)} |u(s)|\;ds + \int_{[0,t)} |u(s)|\;d\,|\omega|\,(s).$$

Gronwall's inequality, i.e. Lemma A.1, yields $|u(t)| \le C$, where C depends on $|z|$ and $\|\omega\|_{\mathrm{loc}}$. As the same argument can be applied for $t < 0$ and Ω is $\|\cdot\|_{\mathrm{loc}}$-bounded, we obtain

$$\sup_{-1\le t\le 1}\sup_{\omega\in\Omega} |u(t)| < \infty.$$

Since

$$u'(t+) = u'(0+) - u(0)\omega(\{0\}) - z\int_0^t u(s)\,ds + \int_0^t u(s)\,d\omega(s),$$

we have (for $0 \le t \le 1$)

$$\left|u'(t+)\right| \le \left|u'(0+) - u(0)\omega(\{0\})\right| + |z|\int_0^1 |u(s)|\;ds + \int_0^1 |u(s)|\;d\,|\omega|\,(t).$$

Since u is bounded on $[-1,1]$, also u' is bounded on $[-1,1]$, where the bound depends on $|z|$ and $\|\omega\|_{\mathrm{loc}}$. Boundedness of Ω yields

$$\sup_{-1\le t\le 1}\sup_{\omega\in\Omega} \left|u'(t+)\right| < \infty.$$

Hence, $\sup_{-1\le t\le 1}\sup_{\omega\in\Omega}\|T_z(t,\omega)\| < \infty$. //

4.4.7 Proposition. *Let (z_n) in $\mathbb{C}$, $z_n \to z$. Then*

$$\sup_{-1\le t\le 1}\sup_{\omega\in\Omega}\|T_{z_n}(t,\omega) - T_z(t,\omega)\| \to 0.$$

Proof. This is a direct consequence of Lemma 4.3.3. //

4.4.8 Corollary. *Let $t \in \mathbb{R}$, (z_n) in $\mathbb{C}$, $z_n \to z$. Then*

$$\sup_{\omega\in\Omega}\|T_{z_n}(t,\omega) - T_z(t,\omega)\| \to 0.$$

Proof. W.l.o.g. let $t > 0$. Let $\omega \in \Omega$. Write $t = k + s$ with $k \in \mathbb{N}_0$ and $s \in (0, 1]$. First, assume $k = 0$. Then Proposition 4.4.7 yields the assertion. For the step from k to $k + 1$ note that by the cocycle property of T_z we have

$$T_z(t, \omega) = T_z(k + 1 + s, \omega) = T_z(1, \alpha_{k+s}(\omega))T_z(k + 1, \omega).$$

Hence,

$$\begin{aligned}
&T_{z_n}(t, \omega) - T_z(t, \omega) \\
&= T_{z_n}(1, \alpha_{k+s}(\omega)) \left(T_{z_n}(k + 1, \omega) - T_z(k + 1, \omega)\right) \\
&\quad + \left(T_{z_n}(1, \alpha_{k+s}(\omega)) - T_z(1, \alpha_{k+s}(\omega))\right) T_z(k + 1, \omega) \\
&\to 0 \quad (n \to \infty),
\end{aligned}$$

where the convergence is uniformly in $\omega \in \Omega$ by assumption (convergence for $t = k+1$) and by Proposition 4.4.7. //

Chapter 5

Cocycles

This chapter provides abstract results for cocycles. One may always think of Schrödinger cocycles (i.e., the transfer matrices). However, the results do not depend on the underlying Schrödinger equation.

We start with some ergodic theory and then show (semi)uniform estimates of continuous (sub)additive processes. After that we define the Lyapunov exponent for a cocycle and introduce the notion of uniform hyperbolicity. With the help of the ergodic theorems provided in the first section we characterize uniform hyperbolicity for continuous cocycles in different ways. Finally we prove that for continuous cocycles uniform hyperbolicity is stable under small perturbations.

For the whole chapter let Ω be a compact metric space and $\alpha\colon \mathbb{R}\times\Omega\to\Omega$ be a continuous group action on Ω.

5.1. Ergodic theorems

We start with Kingman's subadditive ergodic theorem.

5.1.1 Theorem ([9, Theorem IV.1.2]). *Let $(\Omega,\alpha,\mathbb{P})$ be ergodic. Let $(X_n)_{n\in\mathbb{N}_0}$ be a subadditive process on Ω with discrete time, i.e.,*

$$X_0 = 0,\quad X_{m+n}\le X_m + X_n\circ\alpha_m \quad (m,n\in\mathbb{N}_0),$$

such that X_n is integrable ($n\in\mathbb{N}_0$) and $(\frac{1}{n}\mathbb{E}(X_n))_{n\in\mathbb{N}}$ is bounded from below. Then there exists $Z\in\mathbb{R}$ such that $\frac{1}{n}X_n\to Z$ $\mathbb{P}$-a.s. and in expectation. Moreover, $Z=\inf_{n\in\mathbb{N}}\frac{1}{n}\mathbb{E}(X_n)$.

Proof. For the proof, see [56]. //

5.1.2 Proposition ([9, Corollary IV.1.3]). *Let $(\Omega,\alpha,\mathbb{P})$ be ergodic. Let $(X_t)_{t\ge0}$ be a* subadditive process *on Ω, i.e.,*

$$X_0 = 0,\quad X_{t+s}\le X_t + X_s\circ\alpha_t \quad (s,t\ge 0),$$

such that $X_t\in L_1(\mathbb{P})$ for all $t\ge0$, $\left\{\frac{1}{t}\mathbb{E}(X_t);\ t>0\right\}$ is bounded from below, and there exists $M\in L_1(\mathbb{P})$, such that $|X_t|\le M$ for all $0\le t\le 1$. Then there exists $Z\in\mathbb{R}$ such that $\frac{1}{t}X_t\to Z$ $\mathbb{P}$-a.s. and in expectation and $Z=\inf_{t\ge0}\frac{1}{t}\mathbb{E}(X_t)$.

Proof. We apply Theorem 5.1.1 to the process $(X_n)_{n\in\mathbb{N}_0}$. For $t \geq 0$ let $n \leq t < n+1$ and by subadditivity

$$X_{n+1} - X_{n+1-t} \circ \alpha_t \leq X_t \leq X_n + X_{t-n} \circ \alpha_n.$$

Note that $|X_{n+1-t} \circ \alpha_t| \leq M$ and $|X_{t-n} \circ \alpha_n| \leq M$. Since $\frac{1}{n} M \to 0$ $\mathbb{P}$-a.s., the result follows by Lebesgue's dominated convergence theorem. //

In Theorem 5.1.4 below we generalize the result of [20, Theorem 1 and Corollary 2] to continuous time processes.

We need the following well-known proposition.

5.1.3 Proposition. *Let $(\Omega, \alpha, \mathbb{P})$ be uniquely ergodic, $f \in C(\Omega)$. Then*

$$\lim_{S\to\infty} \sup_{\omega\in\Omega} \left| \frac{1}{S} \int_0^S f(\alpha_t(\omega))\, dt - \int f \, d\mathbb{P} \right| = 0.$$

Proof. For the proof see [39]. //

5.1.4 Theorem. *Let $(\Omega, \alpha, \mathbb{P})$ be uniquely ergodic and $(X_t)_{t\geq 0}$ be a continuous subadditive process on Ω, i.e.*

$$X_0 = 0, \quad X_{t+s} \leq X_t + X_s \circ \alpha_t \quad (s, t \geq 0),$$

and $X_t \in C(\Omega)$ for $t \geq 0$. Furthermore, assume that

$$M := \sup_{t\in[0,1]} \sup_{\omega\in\Omega} |X_t(\omega)| < \infty.$$

Then there exists $Z \in \mathbb{R}$ such that $\frac{1}{t} X_t \to Z$ $\mathbb{P}$-a.s., and we have

$$\limsup_{t\to\infty} \sup_{\omega\in\Omega} \frac{1}{t} X_t(\omega) \leq Z.$$

Proof. Let $\varepsilon > 0$. For $t > 0$ define $\overline{X}_t := \frac{1}{t} \int X_t \, d\mathbb{P}$. By Proposition 5.1.2 there exists $Z \in \mathbb{R}$ such that $\overline{X}_t \to Z$. So, there exists $S \in \mathbb{N}$ such that $\overline{X}_t \leq Z + \varepsilon$ for $t \geq S$. Let $K := \sup_{t\in[0,S]} \sup_{\omega\in\Omega} |X_t(\omega)| < \infty$ (which is finite by subadditivity).

Let $\omega \in \Omega$. By subadditivity, for $k \in \mathbb{N}$ and $t \in [0, S]$ we have

$$X_{kS}(\omega) \leq X_t(\omega) + \sum_{j=0}^{k-2} X_S(\alpha_{jS+t}(\omega)) + X_{S-t}(\alpha_{(k-1)S+t}(\omega)).$$

Integrating with respect to t and dividing by S gives

$$X_{kS}(\omega) \leq 2K + \sum_{j=0}^{k-2} \frac{1}{S} \int_0^S X_S(\alpha_{jS+t}(\omega))\, dt = 2K + \frac{1}{S} \int_0^{(k-1)S} X_S(\alpha_t(\omega))\, dt.$$

Since X_S is continuous, by Proposition 5.1.3 there exists $S' > 0$ not depending on ω such that for all $t \geq S'$ we have

$$\frac{1}{t}\int_0^t \frac{1}{S} X_S(\alpha_r(\omega))\,dr \leq \int_\Omega \frac{1}{S} X_S(\omega)\,d\mathbb{P}(\omega) + \varepsilon.$$

Choose $k \in \mathbb{N}$ such that $(k-1)S > S'$. Then

$$\begin{aligned} X_{kS}(\omega) &\leq 2K + (k-1)S\frac{1}{(k-1)S}\int_0^{(k-1)S} \frac{1}{S} X_S(\alpha_t(\omega))\,dt \\ &\leq 2K + (k-1)S\overline{X}_S + (k-1)S\varepsilon. \end{aligned}$$

Now, for $t \geq S' + 2S$ write $t = kS + r$ with $k \in \mathbb{N}$ and $0 \leq r < S$. Then $(k-1)S = t - r - S > S'$ and therefore

$$X_t(\omega) \leq X_{kS}(\omega) + X_r(\alpha_{kS}(\omega)) \leq 3K + (k-1)S\overline{X}_S + (k-1)S\varepsilon.$$

Since $t > (k-1)S$ we obtain

$$\frac{1}{t}X_t(\omega) \leq \overline{X}_S + \varepsilon + \frac{3K}{t} \leq Z + 2\varepsilon + \frac{3K}{t}.$$

For $t \geq T := \max\left\{3K\varepsilon^{-1}, S' + 2S\right\}$ we finally arrive at

$$\frac{1}{t}X_t(\omega) \leq Z + 3\varepsilon.$$

Thus,

$$\sup_{\omega\in\Omega} \frac{1}{t}X_t(\omega) \leq Z + 3\varepsilon \quad (t \geq T).$$

So,

$$\limsup_{t\to\infty} \sup_{\omega\in\Omega} \frac{1}{t}X_t(\omega) \leq Z + 3\varepsilon.$$

For $\varepsilon \to 0$ we obtain the assertion. //

In case we have an additive process we even obtain uniform convergence for $(\frac{1}{t}X_t)$. The main difference in the proof is that we need uniform control of the lower bound.

5.1.5 Theorem. *Let $(\Omega, \alpha, \mathbb{P})$ be uniquely ergodic and $(X_t)_{t\geq 0}$ be a continuous additive process on Ω, i.e.,*

$$X_0 = 0, \quad X_{t+s} = X_t + X_s \circ \alpha_t \quad (s, t \geq 0),$$

and $X_t \in C(\Omega)$ for $t \geq 0$. Furthermore, assume that

$$M := \sup_{t\in[0,1]} \sup_{\omega\in\Omega} |X_t(\omega)| < \infty.$$

Then there exists $Z \in \mathbb{R}$ such that $\frac{1}{t}X_t \to Z$ $\mathbb{P}$-a.s., and

$$\lim_{t\to\infty} \sup_{\omega\in\Omega} \left|\frac{1}{t}X_t(\omega) - Z\right| = 0.$$

Proof. Let $\varepsilon > 0$. For $t > 0$ define $\overline{X}_t := \frac{1}{t}\int X_t \, d\mathbb{P}$. Again by Proposition 5.1.2 there exists $Z \in \mathbb{R}$ such that $\overline{X}_t \to Z$. Hence, there exists $S \in \mathbb{N}$ such that $\left|\overline{X}_t - Z\right| \leq \varepsilon$ for $t \geq S$. Let $K := \sup_{t\in[0,S]} \sup_{\omega\in\Omega} |X_t(\omega)| < \infty$.

Let $\omega \in \Omega$. By additivity, for $k \in \mathbb{N}$ and $t \in [0, S]$ we have

$$X_{kS}(\omega) = X_t(\omega) + \sum_{j=0}^{k-2} X_S(\alpha_{jS+t}(\omega)) + X_{S-t}(\alpha_{(k-1)S+t}(\omega)).$$

Integrating with respect to t and dividing by S gives

$$-2K + \frac{1}{S}\int_0^{(k-1)S} X_S(\alpha_t(\omega))\, dt \leq X_{kS}(\omega) \leq 2K + \frac{1}{S}\int_0^{(k-1)S} X_S(\alpha_t(\omega))\, dt.$$

Since X_S is continuous, by Proposition 5.1.3 there exists $S' > 0$ (not depending on ω) such that for all $t \geq S'$ we have

$$\left|\frac{1}{t}\int_0^t \frac{1}{S} X_S(\alpha_r(\omega))\, dr - \int_\Omega \frac{1}{S} X_S(\omega)\, d\mathbb{P}(\omega)\right| \leq \varepsilon.$$

Choose $k \in \mathbb{N}$ such that $(k-1)S > S'$. Then

$$-2K + (k-1)S\overline{X}_S - (k-1)S\varepsilon \leq X_{kS}(\omega) \leq 2K + (k-1)S\overline{X}_S + (k-1)S\varepsilon.$$

Now, for $t \geq S' + 2S$ write $t = kS + r$ with $k \in \mathbb{N}$ and $0 \leq r < S$. Then we have $(k-1)S = t - r - S > S'$ and therefore

$$-3K + (k-1)S\overline{X}_S - (k-1)S\varepsilon \leq X_t(\omega) \leq 3K + (k-1)S\overline{X}_S + (k-1)S\varepsilon.$$

Since $t > (k-1)S$ we obtain

$$Z - 2\varepsilon - \frac{3K}{t} \leq \frac{1}{t}X_t(\omega) \leq Z + 2\varepsilon + \frac{3K}{t}.$$

For $t \geq T := \max\left\{3K\varepsilon^{-1}, S' + 2S\right\}$ we finally arrive at

$$\left|\frac{1}{t}X_t(\omega) - Z\right| \leq 3\varepsilon.$$

So,

$$\sup_{\omega\in\Omega}\left|\frac{1}{t}X_t(\omega) - Z\right| \leq 3\varepsilon. \qquad //$$

5.1.6 Remark. One would like to prove the previous two theorems for almost continuous (sub)additive processes, i.e., just assuming that for all $\omega \in \Omega$ there exists $N_\omega \subseteq [0,\infty)$ countable such that X_t is continuous at ω for all $t \in [0,\infty) \setminus N_\omega$. In the discrete case there seem to exist proofs of such (semi)uniform ergodic theorems for not necessarily continuous functions, see [11]. However, these proofs do not generalize directly to the continuous case.

5.2. Characterization of uniform cocycles

We introduce the notions of Lyapunov exponents and uniform hyperbolicity for cocycles. Then we characterize uniform hyperbolicity by means of a continuous exponential splitting.

Definition. Let $(\Omega, \alpha, \mathbb{P})$ be ergodic. Let $A\colon \mathbb{R} \times \Omega \to SL(2, \mathbb{C})$ be an almost continuous cocycle satisfying

$$D := \sup_{t\in[-1,1]} \sup_{\omega\in\Omega} \|A(t,\omega)\| < \infty.$$

By Proposition 5.1.2 there exists $\Lambda(A) \in \mathbb{R}$, called the *Lyapunov exponent* of A, such that

$$\Lambda(A) = \lim_{t\to\infty} \frac{1}{t} \ln \|A(t,\omega)\|$$

for $\mathbb{P}$-a.a. $\omega \in \Omega$ (just consider the process defined by $X_t := \ln \|A(t,\cdot)\|$ $(t \in \mathbb{R})$).

We say that $\Lambda(A) \in \mathbb{R}$ exists *uniformly* if the limit $\Lambda(A) = \lim_{t\to\infty} \frac{1}{t} \ln \|A(t,\omega)\|$ exists for all $\omega \in \Omega$ (and is independent of ω) and the convergence is uniform in Ω, i.e.,

$$\sup_{\omega\in\Omega} \left| \frac{1}{t} \ln \|A(t,\omega)\| - \Lambda(A) \right| \to 0.$$

The cocycle is called *uniform* if $\Lambda(A)$ exists uniformly, and *hyperbolic* if $\Lambda(A) > 0$. Finally, a cocycle A is called *uniformly hyperbolic* if A is uniform and hyperbolic.

Our definition of the Lyapunov exponent only takes into account the positive half-axis. One could also define a Lyapunov exponent for the negative half-axis. Since the cocycle in question takes values in the special linear group the determinant of the cocycle matrices equals one, i.e., the transformation is *volume preserving.* Therefore, the Lyapunov exponents for both half-lines are equal (which is the statement of the next lemma).

5.2.1 Lemma. *Let $(\Omega, \alpha, \mathbb{P})$ be ergodic. Let $A\colon \mathbb{R} \times \Omega \to SL(2, \mathbb{C})$ be an almost continuous cocycle satisfying*

$$D := \sup_{t\in[-1,1]} \sup_{\omega\in\Omega} \|A(t,\omega)\| < \infty.$$

Then also

$$\lim_{t\to-\infty} \left| \frac{1}{|t|} \ln \|A(t,\omega)\| - \Lambda(A) \right| = 0.$$

In case A is uniform, then the convergence is also uniform.

Proof. For $\omega \in \Omega$ and $t \in \mathbb{R}$ we have

$$I = A(0,\omega) = A(-t, \alpha_t(\omega))A(t,\omega).$$

Hence,

$$A(t,\omega)^{-1} = A(-t, \alpha_t(\omega)).$$

Let $|A(t,\omega)| = (A(t,\omega)^* A(t,\omega))^{1/2}$. Then

$$\|A(t,\omega)\| = \||A(t,\omega)|\| = \max\{|\lambda|;\ \lambda \text{ eigenvalue of } |A(t,\omega)|\},$$
$$\left\|A(t,\omega)^{-1}\right\|^{-1} = \left\||A(t,\omega)|^{-1}\right\|^{-1} = \min\{|\lambda|;\ \lambda \text{ eigenvalue of } |A(t,\omega)|\}.$$

Since $\det A(t,\omega) = \det|A(t,\omega)| = 1$ we obtain

$$\|A(t,\omega)\| \cdot \left\|A(t,\omega)^{-1}\right\|^{-1} = 1$$

and therefore

$$\|A(t,\omega)\| = \left\|A(t,\omega)^{-1}\right\| = \|A(-t,\alpha_t(\omega))\|.$$

Now, we conclude for $\mathbb{P}$-a.a. $\omega \in \Omega$ (or, if A is uniform, then uniformly on Ω)

$$\begin{aligned}\lim_{t\to-\infty}\left|\frac{1}{|t|}\ln\|A(t,\omega)\| - \Lambda(A)\right| &= \lim_{t\to-\infty}\left|\frac{1}{|t|}\ln\|A(-t,\alpha_t(\omega))\| - \Lambda(A)\right| \\ &= \lim_{s\to\infty}\left|\frac{1}{s}\ln\|A(s,\alpha_{-s}(\omega))\| - \Lambda(A)\right| \\ &= 0.\end{aligned}$$
//

5.2.2 Remark. The Lyapunov exponent describes the typical exponential growth rate, i.e., $\|A(t,\omega)\| \sim e^{\Lambda(A)t}$ as $t \to \infty$. We will exploit this fact later in more detail when we introduce an exponential splitting.

We now apply the results of the previous section to processes generated by cocycles. Note that since $\omega \mapsto A(t,\omega)$ is measurable for all $t \in \mathbb{R}$ we can associate with A a stochastic process $X_t := \ln\|A(t,\cdot)\|$ $(t \in \mathbb{R})$.

5.2.3 Corollary. *Let* $(\Omega,\alpha,\mathbb{P})$ *be uniquely ergodic,* $A\colon \mathbb{R}\times\Omega \to SL(2,\mathbb{C})$ *a continuous cocycle such that*

$$D := \sup_{t\in[0,1]}\sup_{\omega\in\Omega}\|A(t,w)\| < \infty.$$

Then

$$\limsup_{t\to\infty}\sup_{\omega\in\Omega}\frac{1}{t}\ln\|A(t,\omega)\| \le \Lambda(A).$$

Proof. Take $X_t := \ln\|A(t,\cdot)\|$ $(t \ge 0)$ in Theorem 5.1.4. //

5.2.4 Lemma. *Let* $(\Omega,\alpha,\mathbb{P})$ *be uniquely ergodic and* $A\colon \mathbb{R}\times\Omega \to SL(2,\mathbb{C})$ *a continuous cocycle satisfying* $D := \sup_{t\in[0,1]}\sup_{\omega\in\Omega}\|A(t,\omega)\| < \infty$. *Assume that* $\Lambda(A) = 0$. *Then* A *is uniform.*

Proof. For all $\omega \in \Omega$ and uniformly on Ω we have

$$0 \le \liminf_{t\to\infty}\frac{1}{t}\ln\|A(t,\omega)\| \le \limsup_{t\to\infty}\frac{1}{t}\ln\|A(t,\omega)\| \le \Lambda(A) = 0.$$

Hence, A is uniform. //

Next, we aim for a characterization of uniform hyperbolicity. This will require some preparation. Since we want to apply the results only to Schrödinger cocycles corresponding to real energies, we restrict to $SL(2,\mathbb{R})$-valued cocycles. However, we believe that all the results to follow remain true for $SL(2,\mathbb{C})$-valued cocycles.

5.2.5 Remark. The projective line $\mathcal{P}(\mathbb{K}^2)$ is the set of one-dimensional subspaces of $\mathbb{K}^2$ and can be considered as the set of equivalence classes of directions in $\mathbb{K}^2$. It can be equipped with a metric. If $\mathbb{K} = \mathbb{R}$ one may identify $\mathcal{P}(\mathbb{K}^2)$ with $[0,\pi)$ by computing the angle between the the subspace and $\mathbb{R} \times \{0\}$. If $\mathbb{K} = \mathbb{C}$ one can think of $\mathcal{P}(\mathbb{K}^2)$ as the Riemann sphere via the stereographic projection.

5.2.6 Proposition ([35, Proposition 4.1])**.** *Let $(A_t)_{t\geq 0}$ be a family in $SL(2,\mathbb{R})$. Then there exists at most one $v \in \mathcal{P}(\mathbb{K}^2)$ with $\|A_t V\| \to 0$ as $t \to \infty$ for every $V \in v$.*

Proof. Assume the contrary. Then there exist linearly independent vectors $V_1, V_2 \in \mathbb{K}^2$ with $\|A_t V_j\| \to 0$ for $j \in \{1,2\}$. Thus, $\|A_t\| \to 0$ contradicting $\|A_t\| \geq 1$ for all $t \geq 0$ (since $\det A_t = 1$ for all $t \geq 0$). //

5.2.7 Proposition (compare [35, Proposition 4.3])**.** *Let $A\colon \mathbb{R} \times \Omega \to SL(2,\mathbb{R})$ be an almost continuous cocycle (which may be either left continuous or right continuous) and assume that*

$$D := \sup_{-1\leq t\leq 1} \sup_{\omega\in\Omega} \|A(t,\omega)\| < \infty.$$

For $t \in \mathbb{R}$ and $\omega \in \Omega$ let $u(t,\omega)$ be the eigenspace of $|A(t,\omega)| := (A(t,\omega)^* A(t,\omega))^{1/2}$ to the corresponding eigenvalue $a(t,\omega) := \||A(t,\omega)|\|^{-1} = \|A(t,\omega)\|^{-1}$.

(a) Assume there exists $\delta > 0$ and $t_0 \geq 0$ with $\delta \leq \frac{1}{t} \ln \|A(t,\omega)\|$ ($\omega \in \Omega$, $t \geq t_0$). Then $u(t,\omega)$ is one-dimensional for $t \geq t_0$ and $(u(t,\cdot))_{t\geq t_0}$ converges uniformly to a continuous function $u \in C(\Omega, \mathcal{P}(\mathbb{K}^2))$.

(b) Assume there exists $\delta > 0$ and $t_0 \geq 0$ with $\delta \leq \frac{1}{t} \ln \|A(t,\omega)\| \leq \frac{3}{2}\delta$ ($\omega \in \Omega$, $t \geq t_0$). Then there exist $\kappa, C > 0$ with $\|A(t,\omega)U\| \leq Ce^{-\kappa t} \|U\|$ ($t \geq 0$, $\omega \in \Omega$, $U \in u(\omega)$).

Proof. For $\vartheta \in [0,2\pi)$ define $u_\vartheta := (\cos\vartheta, \sin\vartheta)$. Let $\vartheta_{t,\omega} \in [0,2\pi)$ such that $u_{\vartheta_{t,\omega}}$ is an eigenvector of $|A(t,\omega)|$ to the eigenvalue $a(t,\omega)$. Then $u_{\vartheta_{t,\omega}+\frac{\pi}{2}}$ is an eigenvector of $|A(t,\omega)|$ to the eigenvalue $a(t,\omega)^{-1}$. Writing u_ϑ as a linear combination of these two eigenvectors we conclude

$$\|A(t,\omega)u_\vartheta\|^2 = a(t,\omega)^{-2} \sin^2(\vartheta - \vartheta_{t,\omega}) + a(t,\omega)^2 \cos^2(\vartheta - \vartheta_{t,\omega}).$$

Hence, for $s \in [0,1]$ we have

$$\begin{aligned} a(t+s,\omega)^{-2} \sin^2(\vartheta_{t,\omega} - \vartheta_{t+s,\omega}) &\leq \left\|A(t+s,\omega)u_{\vartheta_{t,\omega}}\right\|^2 \\ &\leq D^2 \left\|A(t,\omega)u_{\vartheta_{t,\omega}}\right\|^2 = D^2 a(t,\omega)^2. \end{aligned}$$

Since

$$\begin{aligned} a(t,\omega)^{-1} &= \|A(t,\omega)\| = \|A(-s,\alpha_{t+s}(\omega))A(t+s,\omega)\| \\ &\leq D\,\|A(t+s,\omega)\| = Da(t+s,\omega)^{-1}, \end{aligned}$$

we obtain

$$a(t,\omega)^{-2}\sin^2(\vartheta_{t,\omega}-\vartheta_{t+s,\omega})\le D^4 a(t,\omega)^2.$$

Note that the angle between two one-dimensional subspaces of $\mathbb{K}^2$ is at most $\frac{\pi}{2}$. Since $\sin^2(x)\ge\left(\frac{2x}{\pi}\right)^2$ for $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$ we have

$$|\vartheta_{t,\omega}-\vartheta_{t+s,\omega}|\le\frac{\pi}{2}D^2a(t,\omega)^2.$$

(In fact, one has to choose the "right" $\vartheta_{t+s,\omega}$ corresponding to the eigenspace.)

(a) By assumption we have

$$1<e^{\delta t}\le\|A(t,\omega)\|=a(t,\omega)^{-1}\quad(\omega\in\Omega,\,t\ge t_0).$$

Hence, the eigenspace $u(t,\omega)$ is one-dimensional for $t\ge t_0$ and $\omega\in\Omega$.

Furthermore, for $s\in[0,1]$,

$$|\vartheta_{t,\omega}-\vartheta_{t+s,\omega}|\le\frac{\pi}{2}D^2e^{-2\delta t}\quad(\omega\in\Omega,\,t\ge t_0).$$

Now, for $t'>t\ge t_0$ and $\omega\in\Omega$ we conclude

$$\begin{aligned}\left|\vartheta_{t,\omega}-\vartheta_{t',\omega}\right|&\le\sum_{j=0}^{\lceil t'-t\rceil-2}|\vartheta_{t+j,\omega}-\vartheta_{t+j+1,\omega}|+\left|\vartheta_{t+\lceil t'-t\rceil-1,\omega}-\vartheta_{t',\omega}\right|\\&\le\sum_{j=0}^{\lceil t'-t\rceil-1}\frac{\pi}{2}D^2e^{-2\delta t}e^{-2\delta j}\\&\le\frac{\pi}{2}D^2e^{-2\delta t}\frac{1}{1-e^{-2\delta}}.\end{aligned}$$

Therefore, $(\vartheta_{t,\omega})_{t\ge0}$ is convergent to some ϑ_ω, uniformly in ω. Let $u(\omega):=u_{\vartheta_\omega}$.

In terms of the projective space we conclude

$$\sup_{\omega\in\Omega}d_{\mathcal{P}(\mathbb{K}^2)}([u_{\vartheta_{t,\omega}}]_{\mathcal{P}(\mathbb{K}^2)},[u(\omega)]_{\mathcal{P}(\mathbb{K}^2)})\le C\frac{\pi}{2}D^2e^{-2\delta t}\frac{1}{1-e^{-2\delta}}.\tag{5.1}$$

Now, we show continuity of $\omega\mapsto u(\omega)$. Let $\omega\in\Omega$. Let $\varepsilon>0$. There exists $T\ge t_0$, such that

$$|\vartheta_{t,\omega}-\vartheta_\omega|\le\varepsilon\quad(t\ge T).$$

There exists $t\in[T,\infty)\setminus N_\omega$ such that $A(t,\cdot)$ is continuous at ω. Therefore, also $u(t,\cdot)$ and $u_{\vartheta_{t,(\cdot)}}$ are continuous at ω. Hence, there exists $\delta>0$ such that for $\omega'\in B(\omega,\delta)$ we have

$$\left|\vartheta_{t,\omega}-\vartheta_{t,\omega'}\right|\le\varepsilon.$$

Now,

$$|\vartheta_\omega-\vartheta_{\omega'}|\le|\vartheta_\omega-\vartheta_{t,\omega}|+\left|\vartheta_{t,\omega}-\vartheta_{t,\omega'}\right|+\left|\vartheta_{t,\omega'}-\vartheta_{\omega'}\right|\le3\varepsilon.$$

(b) Let $U \in u(\omega)$ with $\|U\| = 1$ and $t \geq t_0$. Since $u_{\vartheta_{t,\omega}} \to u(\omega)$ uniformly in ω by means of equation (5.1) we find $U_t \in u(t,\omega)$ with $\|U_t\| = 1$ and a C independent of t and ω such that

$$\|U - U_t\| \leq Ce^{-2\delta t}.$$

Since $a(t,\omega) \leq e^{-\delta t}$ uniformly in ω we have

$$\|A(t,\omega)U_t\| = \| |A(t,\omega)| \, U_t\| = \|a(t,\omega)U_t\| \leq e^{-\delta t}.$$

By assumption, $\|A(t,\omega)\| \leq e^{\frac{3}{2}\delta t}$, so we obtain

$$\|A(t,\omega)U\| \leq \|A(t,\omega)(U - U_t)\| + \|A(t,\omega)U_t\| \leq Ce^{-\frac{1}{2}\delta t} + e^{-\delta t} \leq (C+1)e^{-\frac{1}{2}\delta t}.//$$

We now state and prove the main theorem of this section: the characterization of uniform hyperbolicity. A similar theorem was formulated in [35] for the case of discrete time cocycles.

5.2.8 Theorem (compare [35, Theorem 3]). *Let $(\Omega, \alpha, \mathbb{P})$ be uniquely ergodic and $A\colon \mathbb{R} \times \Omega \to SL(2,\mathbb{R})$ a continuous cocycle and assume that*

$$D := \sup_{-1 \leq t \leq 1} \sup_{\omega \in \Omega} \|A(t,\omega)\| < \infty.$$

Then the following are equivalent:
(a) A is uniformly hyperbolic.
(b) There exist constants $\kappa, C > 0$ and $u, v \in C(\Omega, \mathcal{P}(\mathbb{K}^2))$ with

$$\|A(t,\omega)U\| \leq Ce^{-\kappa t} \|U\| \quad \textit{and} \quad \|A(-t,\omega)V\| \leq Ce^{-\kappa t} \|V\|$$

for all $\omega \in \Omega$, $t \geq 0$, $U \in u(\omega)$ and $V \in v(\omega)$.
(c) There exist $\delta > 0$ and $t_0 \geq 0$ such that

$$0 < \delta < \frac{1}{t} \ln \|A(t,\omega)\| \leq \frac{3}{2}\delta$$

for all $\omega \in \Omega$ and $t \geq t_0$.
In case (b) holds true we have $u(\omega) \neq v(\omega)$ $(\omega \in \Omega)$ and $A(t,\omega)u(\omega) \subseteq u(\alpha_t(\omega))$, $A(t,\omega)v(\omega) \subseteq v(\alpha_t(\omega))$ for all $t \in \mathbb{R}$, $\omega \in \Omega$.

The statement (b) in the theorem is called *continuous exponential splitting* and will be exploited in more detail in the next section.

Proof. (a) $\Rightarrow$ (c): This is clear.
(c) $\Rightarrow$ (b): By Proposition 5.2.7 there exist $\kappa, C > 0$ and $u \in C(\Omega, \mathcal{P}(\mathbb{K}^2))$ such that $\|A(t,\omega)U\| \leq Ce^{-\kappa t} \|U\|$ for all $t \geq 0$, $\omega \in \Omega$ and $U \in u(\omega)$). The construction of v is similar (backward time).
(b) $\Rightarrow$ (a): By (b),

$$\|A(s, \alpha_t(\omega))A(t,\omega)U\| = \|A(s+t,\omega)U\| \to 0 \quad (s \to \infty).$$

Proposition 5.2.6 implies

$$[A(t,\omega)U]_{\mathcal{P}(\mathbb{K}^2)} = u(\alpha_t(\omega)) \quad \text{and} \quad [A(t,\omega)V]_{\mathcal{P}(\mathbb{K}^2)} = v(\alpha_t(\omega))$$

for all $\omega \in \Omega$, $t \in \mathbb{R}$, $U \in u(\omega)$ and $V \in v(\omega)$ with $U, V \neq 0$.

Let $\omega \in \Omega$, $U \in u(\omega)$ and $t \geq 0$. We have

$$\|U\| = \|A(t, \alpha_{-t}(\omega))A(-t, \omega)U\| \leq Ce^{-\kappa t} \|A(-t, \omega)U\| .$$

Hence,

$$\|A(-t, \omega)U\| \geq C^{-1}e^{\kappa t} \|U\| . \tag{5.2}$$

We conclude that $u(\omega) \neq v(\omega)$.

For $\omega \in \Omega$ choose $U(\omega) \in u(\omega)$, $V(\omega) \in v(\omega)$ with $\|U(\omega)\| = \|V(\omega)\| = 1$.

There exist $a, d \colon \mathbb{R} \times \Omega \to \mathbb{K} \setminus \{0\}$ with

$$\begin{aligned} A(t, \omega)U(\omega) &= a(t, \omega)U(\alpha_t(\omega)), \\ A(t, \omega)V(\omega) &= d(t, \omega)V(\alpha_t(\omega)). \end{aligned}$$

Since $u(\omega) \neq v(\omega)$, the matrix $C(\omega) = (U(\omega), V(\omega))$ is invertible and we have

$$C(\alpha_t(\omega))^{-1}A(t, \omega)C(\omega) = \begin{pmatrix} a(t, \omega) & 0 \\ 0 & d(t, \omega) \end{pmatrix} . \tag{5.3}$$

As $\|U(\omega)\| = \|V(\omega)\| = 1$, $U(\omega)$ and $V(\omega)$ are unique up to a multiplication by a complex number r of modulus 1.

By continuity of u and v, for fixed $\omega \in \Omega$ we can choose a neighborhood of ω on which U and V can be chosen continuously. As the functions

$$\omega \mapsto \|C(\omega)\| , \quad \omega \mapsto \left\|C(\omega)^{-1}\right\| , \quad \omega \mapsto |a(t, \omega)| , \quad \omega \mapsto |b(t, \omega)| \quad (t \geq 0)$$

are invariant under the replacement of $U(\omega)$ by $rU(\omega)$ or $V(\omega)$ by $rV(\omega)$, they are continuous. Thus, uniformity of

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

is sufficient for uniformity of A, as $\omega \mapsto \|C(\omega)\|$ and $\omega \mapsto \left\|C(\omega)^{-1}\right\|$ are uniformly bounded. Positivity of $\Lambda(A)$ is immediate, since $\|A(\cdot, \omega)\|$ grows exponentially as $\Lambda(A) \geq \kappa$ by (5.2).

The cocycle property of A implies the cocycle property for a and d. Since they are scalar valued, the processes $(\ln |a(t, \cdot)|)_{t \geq 0}$ and $(\ln |d(t, \cdot)|)_{t \geq 0}$ are additive. Furthermore, $\ln |a(t, \cdot)| , \ln |d(t, \cdot)| \in C(\Omega)$ for all $t \geq 0$. Since

$$D = \sup_{-1 \leq t \leq 1} \sup_{\omega \in \Omega} \|A(t, \omega)\| < \infty,$$

by formula (5.3) and the uniform bound on $\|C(\cdot)\|$ and $\left\|C(\cdot)^{-1}\right\|$, we have

$$\sup_{-1 \leq t \leq 1} \sup_{\omega \in \Omega} (|a| (t, \omega) + |d| (t, \omega)) < \infty.$$

By Theorem 5.1.5, $(\frac{1}{t} \ln |a(t, \cdot)|)_{t \geq 0}$ and $(\frac{1}{t} \ln |d(t, \cdot)|)_{t \geq 0}$ converge uniformly.

Hence,

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

is uniform and, therefore, A is uniform as well. //

5.2.9 Remark. As soon as one can extend the (semi)uniform estimates given in Theorems 5.1.4 and 5.1.5 one obtains the same characterization for almost continuous cocycles.

5.3. A stability result for uniform cocycles

For the whole section let $A\colon \mathbb{R}\times\Omega \to SL(2,\mathbb{R})$ be an almost continuous cocycle satisfying

$$D_A := \sup_{-1\le t\le 1}\sup_{\omega\in\Omega}\|A(t,\omega)\| < \infty.$$

Assume A admits an *exponential splitting*, i.e., there exist constants $\kappa, C > 0$ and $u,v\colon \Omega\to\mathcal{P}(\mathbb{K}^2)$ with

$$\|A(t,\omega)U\| \le Ce^{-\kappa t}\|U\| \quad\text{and}\quad \|A(-t,\omega)V\| \le Ce^{-\kappa t}\|V\|$$

for all $\omega\in\Omega$, $t\ge 0$, $U\in u(\omega)$ and $V\in v(\omega)$ and $u(\omega)\ne v(\omega)$ $(\omega\in\Omega)$.

The aim is first to show that u and v are in fact continuous and then to prove a stability result: continuous exponential splittings (and hence uniform hyperbolicity) is preserved under small perturbations.

Note that for $\omega\in\Omega$ there exists $N_\omega\subseteq\mathbb{R}$ countable such that $A(t,\cdot)$ is continuous at ω for $t\in\mathbb{R}\setminus N_\omega$.

We will need a variety of lemmas (which are well-known for the case of continuous cocycles).

5.3.1 Lemma. *Let $t\in\mathbb{R}\setminus N_\omega$. Then $\Omega\times\mathbb{K}^2\ni(\omega,x)\mapsto A(t,\omega)x$ is continuous.*

Proof. Let $((\omega_k,x_k))$ in $\Omega\times\mathbb{K}^2$, $(\omega_k,x_k)\to(\omega,x)$. Then

$$\|A(t,\omega_k)x_k - A(t,\omega)x\| \le \|A(t,\omega_k)-A(t,\omega)\|\,\|x_k\| + \|A(t,\omega)\|\,\|x_k-x\| \to 0. //$$

5.3.2 Lemma. *Let $K\subseteq\mathbb{R}\times\Omega$ be compact. Then $\{\|A(t,\omega)\|;\ (t,\omega)\in K\}$ is bounded.*

Proof. (i) By induction on $n\in\mathbb{N}$ we prove

$$\sup_{-n\le t\le n}\sup_{\omega\in\Omega}\|A(t,\omega)\| \le D_A^n.$$

For $n=1$ this is just the assumption. Now, assume

$$\sup_{-n\le t\le n}\sup_{\omega\in\Omega}\|A(t,\omega)\| \le D_A^n.$$

For $t=n+s$ with $s\in(0,1]$ we obtain

$$\|A(t,\omega)\| = \|A(n,\alpha_s(\omega))A(s,\omega)\| \le \|A(n,\alpha_s(\omega))\|\,\|A(s,\omega)\| \le D_A^n\cdot D_A = D_A^{n+1}.$$

similarly, for $t=-n+s$ with $s\in[-1,0)$ we obtain

$$\|A(t,\omega)\| \le D_A^{n+1}.$$

Since $D_A\ge 1$ (as $A(0,\omega)=I$), we arrive at

$$\sup_{-n-1\le t\le n+1}\sup_{\omega\in\Omega}\|A(t,\omega)\| \le D_A^{n+1}.$$

(ii) There exists $n\in\mathbb{N}$ such that $K\subseteq[-n,n]\times\Omega$. Now (i) proves the assertion. //

Define

$$\mathcal{S} := \left\{(\omega, x) \in \Omega \times \mathbb{K}^2;\ \lim_{t\to\infty} \|A(t,\omega)x\| = 0\right\},$$
$$\mathcal{U} := \left\{(\omega, x) \in \Omega \times \mathbb{K}^2;\ \lim_{t\to-\infty} \|A(t,\omega)x\| = 0\right\}.$$

These sets may be called extended stable and unstable subsets.

5.3.3 Lemma. *We have $\mathcal{S} \cap \mathcal{U} \subseteq \Omega \times \{0\}$.*

Proof. Let $(\omega, x) \in \mathcal{S} \cap \mathcal{U}$. Then there exist $U \in u(\omega)$, $V \in v(\omega)$ with $x = U + V$. Then, for $t \leq 0$,

$$C^{-1}e^{\kappa t}\|U\| \leq \|A(t,\omega)U\| \leq \|A(t,\omega)(U+V)\| + \|A(t,\omega)V\| \to 0 \quad (t \to -\infty),$$

i.e., $U = 0$. Similarly, $V = 0$ and therefore $x = 0$. //

5.3.4 Remark. The same proof shows that

$$\mathcal{S} = \left\{(\omega, x) \in \Omega \times \mathbb{K}^2;\ x \in u(\omega)\right\},$$
$$\mathcal{U} = \left\{(\omega, x) \in \Omega \times \mathbb{K}^2;\ x \in v(\omega)\right\}.$$

This characterization also justifies the notion introduced above: $\mathcal{S}$ encodes the stable directions and $\mathcal{U}$ the unstable directions.

5.3.5 Lemma (compare [49, Lemma 1])**.** *Let $K \subseteq \mathbb{K}^2$ be compact, $((\omega_k, x_k))$ in $\Omega \times K$, $(\omega_k, x_k) \to (\omega, x)$, (t_k) in $(0, \infty)$, $t_k \to \infty$.*

(a) Assume $A(t, \omega_k)x_k \in K$ for all $t \in [0, t_k]$, $k \in \mathbb{N}$. Then $(\omega, x) \in \mathcal{S}$.

(b) Assume $A(-t, \omega_k)x_k \in K$ for all $t \in [0, t_k]$, $k \in \mathbb{N}$. Then $(\omega, x) \in \mathcal{U}$.

Proof. (a) Let $t \in [0, \infty) \setminus N_\omega$. By Lemma 5.3.1, $A(t, \omega)x \in K$. Since $A(\cdot, \omega)x$ is right continuous and N_ω is countable we conclude $A(t, \omega)x \in K$ for all $t \in [0, \infty)$. In particular, $\{\|A(t,\omega)x\|;\ t \geq 0\}$ is bounded. Since A admits an exponential splitting, $x \in u(\omega)$ and hence $(\omega, x) \in \mathcal{S}$.

(b) The proof of (b) is similar. Just note that $t \mapsto A(-t, \omega)x$ is left continuous. //

Define

$$\mathcal{A}^+ := \{(\omega, x) \in \mathcal{S};\ \|A(t,\omega)x\| \leq 1 \quad (t \geq 0)\},$$
$$\mathcal{A}^- := \{(\omega, x) \in \mathcal{U};\ \|A(-t,\omega)x\| \leq 1 \quad (t \geq 0)\}.$$

The aim of the definition of these two subsets is to shrink the possible x to some compact subset of $\mathbb{K}^2$. Since $A(0, \omega) = I$ for all $\omega \in \Omega$ we necessarily have $\|x\| \leq 1$ for $(\omega, x) \in \mathcal{A}^\pm$.

5.3.6 Lemma (compare [49, Lemma 2])**.** *$\mathcal{A}^\pm$ is compact.*

Proof. Since $\mathcal{A}^+ \subseteq \Omega \times B_{\mathbb{K}^2}[0, 1]$, it suffices to show that $\mathcal{A}^+$ is closed. Let $((\omega_k, x_k))$ in $\mathcal{A}^+$, $(\omega_k, x_k) \to (\omega, x)$. Let $t \in [0, \infty) \setminus N_\omega$. Then $(\omega, x) \mapsto A(t, \omega)x$ is continuous by Lemma 5.3.1. Hence, $\|A(t,\omega)x\| \leq 1$. Since $A(\cdot, \omega)x$ is right continuous, $\|A(t,\omega)x\| \leq 1$ $(t \geq 0)$. For $k \in \mathbb{N}$ we have $A(t, \omega_k)x_k \in B_{\mathbb{K}^2}[0, 1]$ for all $t \in [0, k]$. By Lemma 5.3.5, $(\omega, x) \in \mathcal{S}$. We conclude that $(\omega, x) \in \mathcal{A}^+$ and hence that $\mathcal{A}^+$ is closed.

Analogously, $\mathcal{A}^-$ is compact. //

5.3.7 Lemma (compare [49, Lemma 3])**.** *Let* $0 < \lambda \leq 1$*,* (t_k) *in* $(0, \infty)$ *with* $t_k \to \infty$*.*

(a) Let $((\omega_k, x_k))$ *in* $\mathcal{A}^+$*. Then there exists* $k \in \mathbb{N}$ *such that* $\|A(t_k, \omega_k)x_k\| < \lambda$*.*

(b) Let $((\omega_k, x_k))$ *in* $\mathcal{A}^-$*. Then there exists* $k \in \mathbb{N}$ *such that* $\|A(-t_k, \omega_k)x_k\| < \lambda$*.*

Proof. (a) Assume the contrary. Define $\xi_k := A(t_k, \omega_k)x_k$ $(k \in \mathbb{N})$. Then $\|\xi_k\| \geq \lambda$ for all $k \in \mathbb{N}$. Furthermore, $(\alpha_{t_k}(\omega_k), \xi_k) \in \mathcal{A}^+$ by the cocycle property for all $k \in \mathbb{N}$. Since $\mathcal{A}^+$ is compact, there exists a subsequence $((\alpha_{t_{k_l}}(\omega_{k_l}), \xi_{k_l}))$ such that $(\alpha_{t_{k_l}}(\omega_{k_l}), \xi_{k_l}) \to (\omega, \xi) \in \mathcal{A}^+ \subseteq \mathcal{S}$. Then $\|\xi\| \geq \lambda$. Furthermore,

$$\|A(t, \alpha_{k_l}(\omega_{k_l}))\xi_{k_l}\| = \|A(t, \alpha_{k_l}(\omega_{k_l}))A(t_{k_l}, \omega_{k_l})x_{k_l}\| = \|A(t + t_{k_l}, \omega_{k_l})x_{k_l}\| \leq 1$$

for all $t \in [-t_{k_l}, 0]$ and $l \in \mathbb{N}$. By Lemma 5.3.5, $(\omega, \xi) \in \mathcal{U}$. Hence, $(\omega, \xi) \in \mathcal{S} \cap \mathcal{U}$. By Lemma 5.3.3, $\xi = 0$. This is a contradiction.

(b) The proof of part (b) is analogous. //

5.3.8 Lemma (compare [49, Lemma 5])**.** *(a) There exists* $0 < \nu \leq 1$ *such that for all* $(\omega, x) \in \mathcal{S}$ *with* $\|x\| \leq \nu$ *we have* $(\omega, x) \in \mathcal{A}^+$*.*

(b) There exists $0 < \nu \leq 1$ *such that for all* $(\omega, x) \in \mathcal{U}$ *with* $\|x\| \leq \nu$ *we have* $(\omega, x) \in \mathcal{A}^-$*.*

Proof. (a) Assume the contrary. Then there exists a sequence $((\omega_k, x_k))$ in $\mathcal{S}$ with $\|x_k\| \to 0$ such that $(\omega_k, x_k) \notin \mathcal{A}^+$ for all $k \in \mathbb{N}$. Hence, $x_k \neq 0$ for all $k \in \mathbb{N}$ and

$$\lambda_k := \left(\sup_{t \geq 0} \|A(t, \omega_k)x_k\| \right)^{-1} \in (0, 1) \quad (k \in \mathbb{N}).$$

Then $(\omega_k, \lambda_k x_k) \in \mathcal{A}^+$, but $(\omega_k, \theta\lambda_k x_k) \notin \mathcal{A}^+$ whenever $\theta > 1$. Let $0 < \varepsilon < 1$. For $k \in \mathbb{N}$, there exists $t_k \geq 0$ such that

$$\|A(t_k, \omega_k)(\lambda_k x_k)\| \geq 1 - \varepsilon.$$

(Just choose $\theta = \frac{1}{1-\varepsilon}$). The sequence (t_k) is unbounded by Lemma 5.3.2, since

$$1 - \varepsilon \leq \|A(t_k, \omega_k)(\lambda_k x_k)\| \leq \|A(t_k, \omega_k)\| \, \|\lambda_k x_k\| \quad (k \in \mathbb{N})$$

and $\|\lambda_k x_k\| \leq \|x_k\| \to 0$. Lemma 5.3.7 with $\lambda = 1 - \varepsilon$ yields a contradiction.

(b) The proof of part (b) is similar. //

5.3.9 Proposition (compare [49, Theorem 1])**.** $\mathcal{S}$ *and* $\mathcal{U}$ *are closed.*

Proof. Let $((\omega_k, x_k))$ in $\mathcal{S}$, $(\omega_k, x_k) \to (\omega, x)$. If $x = 0$ then $(\omega, x) \in \mathcal{S}$. Otherwise, let ν be the constant from Lemma 5.3.8 and set $\theta := \frac{\nu}{2\|x\|}$. Then for large $k \in \mathbb{N}$ we have $(\omega_k, \theta x_k) \in \mathcal{A}^+$ by Lemma 5.3.8. Since $\mathcal{A}^+$ is closed, also $(\omega, \theta x) \in \mathcal{A}^+ \subseteq \mathcal{S}$. Hence, also $(\omega, x) \in \mathcal{S}$.

Closedness of $\mathcal{U}$ is proven analogously. //

The next observation will be crucial. Having an exponential splitting implies that the splitting is actually continuous (and by Theorem 5.2.8 cocycle is uniformly hyperbolic).

5.3.10 Proposition (compare [49, Lemma 7])**.** *The splitting is continuous, i.e., the mappings* $u, v \colon \Omega \to \mathcal{P}(\mathbb{K}^2)$ *are continuous.*

Proof. Assume u is not continuous at $\omega \in \Omega$. Then there exists $\varepsilon > 0$ and (ω_n) in Ω with $\omega_n \to \omega$ such that

$$d_H(B_{\mathbb{K}^2}[0,1] \cap u(\omega), B_{\mathbb{K}^2}[0,1] \cap u(\omega_n)) \geq \varepsilon \quad (n \in \mathbb{N}),$$

where

$$d_H(A, B) := \max\{\max\{\operatorname{dist}(a, B);\ a \in A\}, \max\{\operatorname{dist}(A, b);\ b \in B\}\}$$

denotes the Hausdorff distance of two non-empty compact subsets $A, B \subseteq \mathbb{K}^2$. Hence, there are two possibilities.

(i) There exists a sequence (x_n) in $B_{\mathbb{K}^2}[0,1]$ with $x_n \in u(\omega_n)$ $(n \in \mathbb{N})$ such that

$$\operatorname{dist}(x_n, B_{\mathbb{K}^2}[0,1] \cap u(\omega)) \geq \varepsilon \quad (n \in \mathbb{N}).$$

Since (x_n) is bounded by 1, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x$ for some $x \in B_{\mathbb{K}^2}[0,1]$. Hence, $\operatorname{dist}(x, B_{\mathbb{K}^2}[0,1] \cap u(\omega)) \geq \varepsilon$ and therefore also $x \neq 0$ (and x cannot be in $u(\omega)$).

As $x_{n_k} \in u(\omega_{n_k})$ for all $k \in \mathbb{N}$ we have $((\omega_{n_k}, x_{n_k}))_k$ in $\mathcal{S}$. Since S is closed by Proposition 5.3.9 and $(\omega_{n_k}, x_{n_k}) \to (\omega, x) \in \mathcal{S}$, i.e., $x \in u(\omega)$. This is a contradiction.

(ii) There exists a subsequence (n_k) and $x \in B_{\mathbb{K}^2}[0,1] \cap u(\omega)$, $x \neq 0$ such that

$$\operatorname{dist}(x, B_{\mathbb{K}^2}[0,1] \cap u(\omega_{n_k})) \geq \frac{\varepsilon}{2} \quad (k \in \mathbb{N}).$$

Let $e_{n_k} \in B_{\mathbb{K}^2}[0,1] \cap u(\omega_{n_k})$ be a unit vector $(k \in \mathbb{N})$. Then for a subsequence (k_j), $e_{n_{k_j}} \to e$ with $\|e\| = 1$. By the argument in (i) we have $e \in u(\omega)$. Hence, there exists $\theta \in \mathbb{K}$ with $|\theta| \leq 1$ such that $x = \theta e$. Then $x_{n_{k_j}} := \theta e_{n_{k_j}} \to x$, contradicting $\operatorname{dist}(x, B_{\mathbb{K}^2}[0,1] \cap u(\omega_{n_{k_j}})) \geq \frac{\varepsilon}{2}$ $(j \in \mathbb{N})$.

Therefore, u is continuous.

The argument for v is exactly the same. //

Having shown continuity of u and v we now aim for the perturbation result: uniform hyperbolicity will be preserved under small perturbations of the cocycle. The idea to prove the result is to "lift" the action of the cocycle to some Banach space of functions and then to split the Banach space into two subspaces according to some Riesz projection. We begin with two lemmas. Note that for a Banach space X we write $L(X)$ for the set of bounded linear operators. If A is a linear operator in X then $R(A) := \{Ax;\ x \in D(A)\}$ and $N(A) := \{x \in D(A);\ Ax = 0\}$ denotes the range and the null space of A. For a closed linear operator A in X (hence, especially for bounded operators) we write

$$\varrho(A) := \{z \in \mathbb{C};\ (z - A) \text{ is one-to-one}, R(z - A) = X\}$$

for the resolvent set and for $z \in \varrho(A)$ the resolvent is given by $R(z, A) = (z - A)^{-1}$.

5.3.11 Lemma. *Let X be a Banach space, $A, B \in L(X)$, $z \in \varrho(A) \cap \varrho(B)$. Assume that $\|R(z, A)(B - A)\| < 1$. Then*

$$R(z, B) - R(z, A) = \sum_{n=1}^{\infty} R(z, A)^n (B - A)^n R(z, A).$$

Proof. By Neumann's series,

$$\sum_{n=0}^{\infty} R(z,A)^n (B-A)^n = (I - R(z,A)(B-A))^{-1} = R(z,B)(z-A).$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} R(z,A)^n (B-A)^n R(z,A) &= \sum_{n=0}^{\infty} R(z,A)^n (B-A)^n R(z,A) - R(z,A)\\ &= R(z,B) - R(z,A).\end{aligned}$$ //

5.3.12 Lemma. *Let X be a Banach space, P, Q continuous projections in X satisfying $\|P-Q\| < 1$. Then $\dim R(P) = \dim R(Q)$, and there exists a bounded linear map $h\colon R(P) \to N(P)$ such that $R(Q) = \{f + h(f);\ f \in R(P)\}$.*

Proof. (i) We show $Q\colon R(P) \to R(Q)$ is injective. Let $f \in R(P)$, $Qf = 0$. Then

$$\|f\| = \|Pf - Qf\| \leq \|P - Q\| \, \|f\| .$$

Hence, $\|f\| = 0$. Therefore, $\dim R(P) \leq \dim R(Q)$. Interchanging the roles of P and Q yields $\dim R(P) = \dim R(Q)$.

(ii) First we show that for $f \in R(P)$ there exists a unique element $g \in N(P)$ such that $f + g \in R(Q)$.

Let $S := I - P$. Then S is a projection with $R(S) = N(P)$ and $Q + S = I - (P - Q)$ is invertible. Let $f \in X$. Then

$$f = (Q+S)(Q+S)^{-1}f = Q(Q+S)^{-1}f + S(Q+S)^{-1}f,$$

where the first term is in $R(Q)$ and the second one in $N(P)$.

We now show that this decomposition of f is unique. Note that

$$\|f\| = \|Qf - Pf\| \leq \|Q - P\| \, \|f\|$$

for $f \in R(Q) \cap N(P)$. As $\|Q - P\| < 1$, necessarily we have $f = 0$ and hence $X = R(Q) \oplus N(P)$. Therefore, each $f \in R(P)$ can be uniquely expressed as $f = f' + (-g)$ with $f' \in R(Q)$ and $g \in N(P)$.

Hence, we can define $h(f)$ to be the unique element $g \in N(P)$ such that $f + g \in R(Q)$. This shows $R(Q) \supseteq \{f + h(f);\ f \in R(P)\}$. For the converse inclusion note that for $g \in R(Q)$ we have $Pg \in R(P)$ and $g - Pg \in N(P)$, since P is a projection. Therefore, $R(Q) = \{f + h(f);\ f \in R(P)\}$.

Let $f_1, f_2 \in R(P)$, $z \in \mathbb{K}$. Then

$$zf_1 + f_2 + zh(f_1) + h(f_2) = zf_1 + zh(f_1) + f_2 + h(f_2) \in R(Q).$$

By uniqueness, $h(zf_1 + f_2) = zh(f_1) + h(f_2)$, i.e., h is linear.

It remains to show that h is continuous. To this end, we show that h is closed. First, note that $R(P)$, $N(P)$ and $R(Q)$ are closed, since P and Q are continuous projections (see [2, 7.14]). Let (f_n) in $R(P)$, $f_n \to f$ in $R(P)$, $h(f_n) \to g$ in $N(P)$. Then $(f_n + h(f_n))$ is in $R(Q)$ and $f_n + h(f_n) \to f + g$ in X and therefore also in $R(Q)$, since $R(Q)$ is a closed subspace. By uniqueness of $h(f)$, we have $h(f) = g$, i.e., h is closed. The closed graph theorem yields continuity of h. //

The following stability theorem is the main result in this section. This type of result is also called roughness of uniform hyperbolicity (roughness of exponential dichotomy) or Coppel's Theorem. In the language of exponential splittings it was formulated in [25, Theorem 3.1].

5.3.13 Theorem (see [25, Theorem 3.1])**.** *Let (Ω, α) be uniquely ergodic and $A, B\colon \mathbb{R}\times \Omega \to SL(2,\mathbb{R})$ continuous cocycles satisfying $D_A := \sup_{-1\le t\le 1}\sup_{\omega\in\Omega}\|A(t,\omega)\| < \infty$. Let A be uniformly hyperbolic. Then there exists $\delta > 0$ such that if*

$$D := \sup_{-1\le t\le 1}\sup_{\omega\in\Omega}\|A(t,\omega) - B(t,\omega)\| < \delta,$$

then also B is uniformly hyperbolic.

Proof. (i) Let $X := \big\{f\colon \Omega \to \mathbb{K}^2;\ f \text{ bounded}\big\}$ be a Banach space of bounded functions, equipped with supremum norm. For $t \in \mathbb{R}$, define $T_A(t)\colon X \to X$ by

$$T_A(t)f(\omega) := A(t, \alpha_{-t}(\omega))f(\alpha_{-t}(\omega)).$$

Then $T_A(t) \in L(X)$ $(t \in \mathbb{R})$ and $T_A(t+s) = T_A(t)T_A(s)$ $(s,t \in \mathbb{R})$.

Since A is uniform with $\Lambda(A) > 0$, Theorem 5.2.8 yields continuous and linearly independent mappings $u, v \in C(\Omega, \mathcal{P}(\mathbb{K}^2))$. Let $x \in \mathbb{K}^2$. Then there exist unique $x_u, x_v \in C(\Omega;\mathbb{K}^2)$ such that

$$x = x_u(\omega) + x_v(\omega)$$

and $x_u(\omega) \in u(\omega)$, $x_v(\omega) \in v(\omega)$ for all $\omega \in \Omega$. For $\omega \in \Omega$ and $x \in \mathbb{K}^2$ define $P_\omega x := x_u(\omega)$. Then P_ω is a projection and $\omega \mapsto P_\omega$ is continuous. Define $\tilde{P}\colon X \to X$ by $\tilde{P}f(\omega) := P_\omega(f(\omega))$. Then $\tilde{P}$ is a continuous projection on X, and the additional statement in Theorem 5.2.8 yields that $\tilde{P}$ commutes with $T_A(t)$ for all $t \in \mathbb{R}$.

Note that $X = R(\tilde{P}) \oplus N(\tilde{P})$. Hence, we can consider the restrictions of $T_A(t)$ to the (closed) subspaces $R(\tilde{P})$ and $N(\tilde{P})$ $(t \in \mathbb{R})$. By Theorem 5.2.8 we have

$$\lim_{n\to\infty}\left\|\Big(T_A(t)|_{R(\tilde{P})}\Big)^n\right\|^{1/n} = \lim_{n\to\infty}\left\|\Big(T_A(nt)|_{R(\tilde{P})}\Big)\right\|^{1/n} \le \lim_{n\to\infty}(Ce^{-\kappa nt})^{1/n} = e^{-\kappa t}$$

and

$$\lim_{n\to\infty}\left\|\Big(T_A(-t)|_{N(\tilde{P})}\Big)^n\right\|^{1/n} = \lim_{n\to\infty}\left\|\Big(T_A(-nt)|_{N(\tilde{P})}\Big)\right\|^{1/n} \le \lim_{n\to\infty}(Ce^{-\kappa nt})^{1/n} = e^{-\kappa t}.$$

The spectral radius formula ([46, Theorem VI.6]) yields $\sigma(T_A(t)|_{R(\tilde{P})}) \subseteq B_{\mathbb{C}}(0, e^{-\kappa\frac{t}{2}})$ and $\sigma(T_A(t)^{-1}|_{N(\tilde{P})}) \subseteq B_{\mathbb{C}}(0, e^{-\kappa\frac{t}{2}})$, i.e., $\sigma(T_A(t)|_{N(\tilde{P})}) \subseteq \mathbb{C} \setminus B_{\mathbb{C}}(0, e^{\kappa\frac{t}{2}})$.

Hence, the unit circle $S := \{z \in \mathbb{C};\ |z| = 1\}$ (positively orientated) separates the spectrum of $T_A(1)$ into two disjoint closed subsets, one inside and one outside S. Define the projection $P_*\colon X \to X$ by

$$P_* = \frac{1}{2\pi i}\int_S R(z, T_A(1))\,dz.$$

By [48, page 406] we have

$$R(P_*) = \left\{ f \in X;\ \lim_{t\to\infty} \|T_A(t)f\| = 0 \right\},$$
$$N(P_*) = \left\{ f \in X;\ \lim_{t\to\infty} \|T_A(-t)f\| = 0 \right\}.$$

Furthermore, $X = R(P_*) \oplus N(P_*)$. Let $f \in R(P_*)$ and $\varepsilon > 0$. Then there exists $t_0 \geq 0$ such that for all $t \geq t_0$ and $\omega \in \Omega$ we have

$$\|A(t, \alpha_{-t}(\omega))f(\alpha_{-t}(\omega)\| = \|T_A(t)f(\omega)\| \leq \varepsilon.$$

For $\omega = \alpha_t(\omega')$ we conclude

$$\left\|A(t, \omega')f(\omega')\right\| \leq \varepsilon \quad (\omega' \in \Omega).$$

Hence, $f(\omega') \in u(\omega')$ for all $\omega' \in \Omega$, i.e., $\tilde{P}f(\omega') = f(\omega')$ for all $\omega' \in \Omega$. Hence, $f \in R(\tilde{P})$. On the other hand, $f \in R(\tilde{P})$ implies $T_A(t)f = T_A(t)\tilde{P}f \to 0$. Hence, $R(P_*) = R(\tilde{P})$. Analogously, $N(\tilde{P}) = N(P_*)$. Let $f \in X$. Then $f = \tilde{P}f + (1 - \tilde{P})f$ and

$$\tilde{P}f = P_*\tilde{P}f = P_*(\tilde{P}f + (1 - \tilde{P})f) = P_*f.$$

Thus, $\tilde{P} = P_*$.

(ii) Similarly, for $t \in \mathbb{R}$ let $T_B(t) \in L(X)$ be defined by

$$T_B(t)f(\omega) = B(t, \alpha_{-t}(\omega))f(\alpha_{-t}(\omega)) \quad (\omega \in \Omega, f \in X).$$

Then also $T_B(s + t) = T_B(t)T_B(s)$ for all $s, t \in \mathbb{R}$. There exists $\delta \in (0, 1)$ such that if $\|T_B(1) - T_A(1)\| < \delta$, then the spectrum of $T_B(1)$ is also separated by S into two closed disjoint subsets. Let

$$Q_* = \frac{1}{2\pi i} \int_S R(z, T_B(1))\, dz.$$

Choosing δ sufficiently small, by Lemma 5.3.11,

$$\|Q_* - P_*\| \leq \sup_{z\in S} \sum_{n=1}^{\infty} \|R(z, T_A(1))\|^n \|T_B(1) - T_A(1)\|^n \|R(z, T_A(1))\|.$$

So, as $S \ni z \mapsto \|R(z, T_A(1))\|$ is continuous and hence bounded, there exists a constant K such that

$$\|Q_* - P_*\| \leq K \|T_B(1) - T_A(1)\| \leq K\delta.$$

We shrink δ such that $\|Q_* - P_*\| < 1$.

(iii) Since S separates the spectrum of $T_B(1)$ into two parts there exists $\kappa' > 0$ such that $\sigma(T_B(1)|_{R(Q_*)}) \subseteq B_{\mathbb{C}}(0, e^{-2\kappa'})$ and $\sigma(T_B(-1)|_{N(Q_*)}) \subseteq B_{\mathbb{C}}(0, e^{-2\kappa'})$. Note that $T_B(t)$ commutes with Q_* for all $t \in \mathbb{R}$. By the spectral radius formula there exists $C > 0$ such that for all $n \in \mathbb{N}_0$ we have

$$\left\|T_B(n)|_{R(Q_*)}\right\| = \left\|(T_B(1)|_{R(Q_*)})^n\right\| \leq Ce^{-\kappa' n},$$
$$\left\|T_B(-n)|_{N(Q_*)}\right\| = \left\|(T_B(-n)|_{N(Q_*)})^n\right\| \leq Ce^{-\kappa' n}.$$

For $t \geq 0$ choose $n \in \mathbb{N}_0$ and $s \in [0,1)$ such that $t = n + s$. Then

$$\begin{aligned}\left\|T_B(t)|_{R(Q_*)}\right\| &= \|Q_* T_B(n) T_B(s) Q_*\| \leq \|Q_* T_B(n) Q_*\| \, \|T_B(s)\| \\ &\leq C e^{-\kappa' n}(D_A + D) \leq C(D_A + D) e^{\kappa'} e^{-\kappa' t}.\end{aligned}$$

Analogously,

$$\left\|T_B(-t)|_{N(Q_*)}\right\| \leq C(D_A + D) e^{\kappa'} e^{-\kappa' t}.$$

(iv) By Lemma 5.3.12 there exists $h \colon R(P_*) \to N(P_*)$ linear and continuous, such that $h(f)$ is the unique element with $R(Q_*) = \{f + h(f);\ f \in R(P_*)\}$, i.e., $T_B(t)(f + h(f)) \to 0$ as $t \to \infty$.

(v) The next aim is to show that h "fibers" over Ω. Note that for $\omega \in \Omega$ and $x \in R(P_\omega)$ we have $\mathbb{1}_{\{\omega\}} x \in R(P_*)$. For $\omega \in \Omega$ define $h_\omega \colon R(P_\omega) \to N(P_\omega)$ by $h_\omega(x) := h(\mathbb{1}_{\{\omega\}} x)(\omega)$. Now, let $f \in R(P_*)$. Then $f(\omega) \in R(P_\omega)$ $(\omega \in \Omega)$.

For $t \in \mathbb{R}$ and $\omega \in \Omega$ define $\tilde{f}_{t,\omega} \in R(P_*)$ by $\tilde{f}_{t,\omega} := \mathbb{1}_{\{\alpha_{-t}(\omega)\}}(\cdot) f(\alpha_{-t}(\omega))$. Define $\tilde{h} \in X$ by $\tilde{h}(\omega) := h(\mathbb{1}_{\{\omega\}}(\cdot) f(\omega))(\omega) = h_\omega(f(\omega))$. Then $\tilde{h} \in N(P_*)$ and

$$\begin{aligned}&T_B(t)(f + \tilde{h})(\omega) \\ &= B(t, \alpha_{-t}(\omega)) \Big(f(\alpha_{-t}(\omega)) + \tilde{h}(\alpha_{-t}(\omega)) \Big) \\ &= B(t, \alpha_{-t}(\omega)) \big(f(\alpha_{-t}(\omega)) + h(\mathbb{1}_{\{\alpha_{-t}(\omega)\}}(\cdot) f(\alpha_{-t}(\omega)))(\alpha_{-t}(\omega)) \big) \\ &= B(t, \alpha_{-t}(\omega)) \Big(\tilde{f}_{t,\omega}(\alpha_{-t}(\omega)) + h(\tilde{f}_{t,\omega})(\alpha_{-t}(\omega)) \Big) \\ &= T_B(t)(\tilde{f}_{t,\omega} + h(\tilde{f}_{t,\omega}))(\omega).\end{aligned}$$

Let $\varepsilon > 0$. Then by (iii) there exists $t_0 > 0$ such that

$$\left\|T_B(t)|_{R(Q_*)}\right\| \leq \varepsilon \quad (t \geq t_0).$$

Furthermore, $\left\|\tilde{f}_{t,\omega}\right\| \leq \|f\|$. Hence,

$$\sup_{\omega \in \Omega} \left| T_B(t)(\tilde{f}_{t,\omega} + h(\tilde{f}_{t,\omega}))(\omega) \right| \leq \varepsilon(\|f\| + \|h\| \, \|f\|) \quad (t \geq t_0).$$

By uniqueness of $h(f)$ and (iv) we obtain $h(f) = \tilde{h}$, i.e.,

$$h(f)(\omega) = h_\omega(f(\omega)) \quad (\omega \in \Omega).$$

(vi) Now, let us show that Q_* "fibers" over Ω. Let $f \in X$ and set $g := Q_* f$. Then

$$Q_* g = g = P_* g + (1 - P_*) g$$

and hence $(1 - P_*) g = h(P_* g)$. Therefore, for $\omega \in \Omega$,

$$Q_* g(\omega) = P_* g(\omega) + h(P_* g)(\omega) = P_\omega(g(\omega)) + h_\omega(P_\omega(g(\omega))) = (P_\omega + h_\omega P_\omega)(g(\omega)).$$

Define $Q_\omega := P_\omega + h_\omega P_\omega$. Then

$$Q_* g(\omega) = Q_\omega(g(\omega)) \quad (\omega \in \Omega).$$

Therefore, $Q_* f(\omega) = Q_\omega(Q_* f(\omega))$.

Let $f \in N(Q_*)$. Shrinking δ such that $\delta + e^{-2\kappa'} < 1$ we obtain

$$\|T_A(-1)f\| \leq \delta \|f\| + \|T_B(-1)\| \|f\| \leq (\delta + e^{-2\kappa'}) \|f\|,$$

and hence $T_A(-t)f \to 0$, i.e., $f \in N(P_*)$. Therefore, $N(Q_*) \subseteq N(P_*)$ and hence $P_*(1 - Q_*) = 0$. We conclude that $Q_* f(\omega) = Q_\omega(Q_* f(\omega)) = Q_\omega(f(\omega))$ for all $f \in X$, $\omega \in \Omega$. It is easy to see that Q_ω is a projection for all $\omega \in \Omega$.

(vii) Since $\sup_{\omega\in\Omega} \|Q_\omega - P_\omega\| \leq \|Q_* - P_*\| < 1$, we have $\dim Q_\omega = \dim P_\omega = 1$ $(\omega \in \Omega)$. For $\omega \in \Omega$ let $u_B(\omega), v_B(\omega) \in \mathcal{P}(\mathbb{K}^2)$ be defined by $u_B(\omega) = R(Q_\omega)$ and $v_B(\omega) = N(Q_\omega)$.

By (iii), for all $\omega \in \Omega$, $U \in u_B(\omega)$ and $V \in v_B(\omega)$ we have

$$\|B(t,\omega)U\| \leq C' e^{-\kappa' t} \|U\| \quad \text{and} \quad \|B(-t,\omega)V\| \leq C' e^{-\kappa' t} \|V\| \quad (t \geq 0),$$

Proposition 5.3.10 shows that u_B, v_B are continuous.

By Theorem 5.2.8, B is uniformly hyperbolic. //

5.3.14 Remark. We would like to prove the theorem also for the case of almost continuous cocycles. In fact, the only reason for the restriction to continuous cocycles was the application of Theorem 5.2.8. As soon as one can generalize this theorem one directly obtains the generalized perturbation result.

5.4. The set of cocycles as a metric space

In this final section of the present chapter we investigate the set of (almost) continuous cocycles. For the whole section let $(\Omega, \alpha, \mathbb{P})$ be ergodic.

Let $\mathcal{C}$ be the set of all almost continuous cocycles $A\colon \mathbb{R} \times \Omega \to SL(2,\mathbb{C})$ satisfying

$$D_A := \sup_{-1\leq t\leq 1} \sup_{\omega\in\Omega} \|A(t,\omega)\| < \infty.$$

Define $d_{\mathcal{C}}\colon \mathcal{C} \times \mathcal{C} \to [0,\infty)$,

$$d_{\mathcal{C}}(A,B) := \sup_{-1\leq t\leq 1} \sup_{\omega\in\Omega} \|A(t,\omega) - B(t,\omega)\|.$$

The next Lemma is obvious.

5.4.1 Lemma. *$d_{\mathcal{C}}$ is a metric on $\mathcal{C}$.*

5.4.2 Lemma. *Let $A, B \in \mathcal{C}$, $\omega \in \Omega$, $t \geq 0$. Then*

$$\|A(t,\omega) - B(t,\omega)\| \leq (D_A + D_B)^{\lceil t-1 \rceil} d_{\mathcal{C}}(A,B).$$

Proof. We prove this by induction. Write $t = n + s$ with $n \in \mathbb{N}_0$ and $s \in (0,1]$. For $n = 0$ there is nothing to prove. For $n+1$, we have

$$\begin{aligned}
&\|A(n+1+s,\omega) - B(n+1+s,\omega)\| \\
&\leq \|A(s,\alpha_{n+1}(\omega)) - B(s,\alpha_{n+1}(\omega))\| \|A(n+1,\omega)\| \\
&\quad + \|B(s,\alpha_{n+1}(\omega))\| \|A(n+1,\omega) - B(n+1,\omega)\|.
\end{aligned}$$

Hence,

$$\begin{aligned}&\|A(n+1+s,\omega)-B(n+1+s,\omega)\|\\&\le d_{\mathcal{C}}(A,B)D_A^{n+1}+D_B(D_A+D_B)^n d_{\mathcal{C}}(A,B)\\&\le (D_A+D_B)^{n+1}d_{\mathcal{C}}(A,B).\end{aligned}$$

//

5.4.3 Lemma. *For $t>0$ define $\Lambda_t\colon \mathcal{C}\to[0,\infty)$,*

$$\Lambda_t(A):=\frac{1}{t}\int\limits_{\Omega}\ln\|A(t,\omega)\|\,d\mathbb{P}(\omega)=\frac{1}{t}\mathbb{E}(\ln\|A(t,\cdot)\|).$$

Then Λ_t is continuous.

Proof. For $x,y\ge 1$ we have $|\ln x-\ln y|\le|x-y|$. By Lemma 5.4.2 we conclude

$$|\ln\|A(t,\omega)\|-\ln\|B(t,\omega)\||\le (D_A+D_B)^{\lceil t-1\rceil}d_{\mathcal{C}}(A,B).$$

Thus,

$$|\Lambda_t(A)-\Lambda_t(B)|\le\frac{1}{t}(D_A+D_B)^{\lceil t-1\rceil}d_{\mathcal{C}}(A,B).$$

//

5.4.4 Lemma. *The mapping $\Lambda\colon\mathcal{C}\to[0,\infty)$, $\Lambda(A)=\lim_{t\to\infty}\frac{1}{t}\ln\|A(t,\omega)\|$ is upper semicontinuous.*

Proof. For $A\in\mathcal{C}$ we have $\Lambda_t(A)\to\Lambda(A)=\inf_{t>0}\Lambda_t(A)$ by Kingman's ergodic theorem (Proposition 5.1.2). Since the infimum of upper semicontinuous functions is upper semicontinuous, $\Lambda=\inf_{t>0}\Lambda_t$ is upper semicontinuous. //

5.4.5 Lemma. *Let $(\Omega,\alpha,\mathbb{P})$ be uniquely ergodic, $A\in\mathcal{C}$ be a continuous cocycle, $\Lambda(A)=0$. Then Λ is continuous at A.*

Proof. Let (A_k) in $\mathcal{C}$, $A_k\to A$. Since Λ is upper semicontinuous,

$$0\le\liminf_{k\to\infty}\Lambda(A_k)\le\limsup_{k\to\infty}\Lambda(A_k)\le\Lambda(A)=0.$$

//

In the discrete case, Furman proved in [20] continuity of Λ at all uniform continuous cocycles.

Definition. Let $f\colon\mathbb{C}\to[-\infty,\infty)$. Then f is *subharmonic*, if f is upper semicontinuous and for all $z\in\mathbb{C}$ and $r>0$:

$$f(z)\le\frac{1}{2\pi}\int\limits_0^{2\pi}f(z+re^{i\varphi})\,d\varphi.$$

5.4.6 Lemma. *Let $T\colon\mathbb{C}\to\mathcal{C}$ such that $T(\cdot)(t,\omega)$ is holomorphic for all $t\in\mathbb{R}$, $\omega\in\Omega$. Then $z\mapsto\Lambda(T(z))$ is subharmonic.*

Proof. By [9, Lemma V.4.4 iii)], $\ln\|T(\cdot)(t,\omega)\|$ is subharmonic for all $t\in\mathbb{R}$, $\omega\in\Omega$. By Fatou's lemma, also $z\mapsto\mathbb{E}\left(\ln\|T(z)(t,\cdot)\|\right)$ is subharmonic for all $t\in\mathbb{R}$. By [9, Lemma V.4.4 ii)], $z\mapsto\Lambda(T(z))=\inf_{t>0}\mathbb{E}\left(\ln\|T(z)(t,\cdot)\|\right)$ is subharmonic. //

A nice feature of subharmonic functions is that if two subharmonic functions are equal λ^2-a.e., then they are equal, see [9, Lemma V.4.4 i)]. We will apply this fact in the next chapter to relate spectral properties of random Schrödinger operators with the Lyapunov exponent.

Chapter 6

Random Schrödinger Operators 2

This final chapter focuses on Schrödinger operators again. It contains results connecting dynamical properties of (Ω, α) to spectral properties of $(H_\omega)_{\omega\in\Omega}$.

We will characterize the spectrum (basically) in terms of the Lyapunov exponent. This characterization is the same as in [34], where it was proven for the discrete case. Similar results can be found in [24] for some special potentials.

We introduce the Titchmarsh-Weyl m-functions for the half-line problems and prove various statements concerning these functions. Then we extend Kotani theory to the case of atomless measures as potentials, thus characterizing an essential support of the absolutely continuous part of the spectrum. Last but not least we focus on Delone dynamical systems inducing operator families modeling quasicrystalline materials. We will use results of all the previous chapters to conclude (almost surely) purely singular continuous spectrum and also Cantor sets as spectrum (in case of atomless potentials) for such types of operator families.

We end this chapter by some remarks on open problems and further directions.

For the rest of this chapter let (Ω, α) be as in Chapter 4, i.e., $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ is $\|\cdot\|_{\mathrm{loc}}$-bounded, closed w.r.t. the vague topology and translation invariant, and $\alpha\colon \mathbb{R}\times\Omega \to \Omega$, $\alpha_t(\omega) = \omega(\cdot + t)$ is the continuous group action on Ω.

6.1. The spectrum as a set

In this section we characterize the spectrum of $(H_\omega)_{\omega\in\Omega}$ as a set in terms of the Lyapunov-Exponent (and non-uniformity of the transfer matrices). We follow the ideas developed in [34] for the case of discrete Schrödinger operators. For $z \in \mathbb{C}$ and $\omega \in \Omega$ let $T_z(\cdot, \omega)$ be the transfer matrix for H_ω. Note that by Proposition 4.4.6 we have

$$\sup_{-1\leq t\leq 1}\sup_{\omega\in\Omega} \|T_z(t, \omega)\| < \infty.$$

For $(\Omega, \alpha, \mathbb{P})$ ergodic define

$$\gamma(z) := \Lambda(T_z).$$

Recall that by minimality there exists $\Sigma \subseteq \mathbb{R}$ closed such that $\sigma(H_\omega) = \Sigma$ for all $\omega \in \Omega$, see Theorem 4.2.10.

6.1.1 Lemma. *Let $(\Omega, \alpha, \mathbb{P})$ be strictly ergodic and atomless, T_E uniform for every E in $\mathbb{R}$. Then for the (ω-independent) spectrum we have $\Sigma = \{E \in \mathbb{R};\ \gamma(E) = 0\}$ and γ is continuous on Σ.*

Proof. Set $\Gamma := \{E \in \mathbb{R};\ \gamma(E) = 0\}$. By Proposition 4.4.7 and Lemma 5.4.5 we obtain continuity of γ on Γ.

"$\Gamma \subseteq \Sigma$": Let $\omega \in \Omega$. Write

$$A := \big\{E \in \mathbb{R};\ \text{for all solutions } u \text{ of } H_\omega u = Eu \text{ and all } \kappa > 0 \text{ there is } C > 0\text{:}\\ |u(t)| \leq Ce^{\kappa|t|}\ (t \in \mathbb{R})\big\},$$

for the set of energies such that there exists a subexponentially bounded solution.

First of all we show that $\Gamma \subseteq A$. Let $E \in \Gamma$. Then

$$\lim_{t \to \pm\infty} \frac{1}{|t|} \ln \|T_E(t, \omega)\| = 0.$$

Hence, for all $\kappa > 0$ there is $t_0 > 0$ such that

$$\frac{1}{|t|} \ln \|T_E(t, \omega)\| \leq \kappa \quad (|t| > t_0),$$

i.e. $\|T_E(t, \omega)\| \leq e^{\kappa|t|}$ for $|t| > t_0$. There exists $C > 1$ such that $\|T_E(t, \omega)\| \leq C$ for $|t| \leq t_0$, since solutions remain bounded on compact intervals (see also Proposition 4.4.6 and note that T_E is a cocycle). This implies

$$\|T_E(t, \omega)\| \leq Ce^{\kappa|t|} \quad (t \in \mathbb{R}),$$

i.e., $E \in A$.

Let $E \in \Gamma \subseteq A$ and $u \neq 0$ be a solution of $H_\omega u = Eu$. Then u is subexponentially bounded and by Proposition A.3.4 we conclude that $E \in \sigma(H_\omega)$. By minimality, the spectrum does not depend on ω and hence $\Gamma \subseteq \Sigma$.

"$\Sigma \subseteq \Gamma$": Let $\omega \in \Omega$. We have to show that $\sigma(H_\omega) \subseteq \Gamma$. We prove this by contradiction. Assume there is spectrum in $\complement\Gamma$. By Theorem 5.3.13 and Proposition 4.4.7 we can deduce that $\complement\Gamma$ is open and hence the spectral measures of H_ω give weight to $\complement\Gamma$. Therefore, there is $E \in \complement\Gamma \cap \sigma(H_\omega)$ admitting a subexponentially bounded solution $u \neq 0$ of $H_\omega u = Eu$ (see Proposition A.3.5). We have

$$\begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} = T_E(t, \omega) \begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix} \quad (t \in \mathbb{R}).$$

By Theorem 5.2.8, there exist $\kappa, C > 0$ and $u(\omega), v(\omega) \in \mathcal{P}(\mathbb{K}^2)$ such that

$$\|T_E(t, \omega)U\| \leq Ce^{-\kappa t} \|U\|, \quad \|T_E(-t, \omega)V\| \leq Ce^{-\kappa t} \|V\|$$

for all $t \geq 0$, $U \in u(\omega)$, $V \in v(\omega)$, and $u(\omega) \neq v(\omega)$. Hence, there exist $U \in u(\omega)$ and $V \in v(\omega)$ such that

$$\begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix} = U + V.$$

Furthermore,

$$\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} \right\| = \left\| T_E(t,\omega) \begin{pmatrix} u(0) \\ u'(0+) \end{pmatrix} \right\| \geq \left| \|T_E(t,\omega)U\| - \|T_E(t,\omega)V\| \right| \quad (t \in \mathbb{R}).$$

For $t \geq 0$ large, $\|T_E(t,\omega)U\|$ becomes small, so

$$\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} \right\| \geq \|T_E(t,\omega)V\| - \|T_E(t,\omega)U\| \geq \tilde{C} e^{\frac{1}{2}\kappa t}.$$

For $-t \geq 0$ large, $\|T_E(t,\omega)V\|$ becomes small, so

$$\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} \right\| \geq \|T_E(t,\omega)U\| - \|T_E(t,\omega)V\| \geq \tilde{C} e^{\frac{1}{2}\kappa t}.$$

Hence, u is exponentially growing in at least one direction. This contradicts the fact that u is subexponentially bounded. //

6.1.2 Lemma. *Let $(\Omega, \alpha, \mathbb{P})$ be uniquely ergodic and atomless, $E \in \mathbb{R}$, $\gamma(E) = 0$. Then T_E is uniform.*

Proof. Since $\gamma(E) = 0$, by Lemma 5.2.4, T_E is uniform. //

6.1.3 Lemma. *Let (Ω, α) be strictly ergodic and atomless, $E \in \mathbb{R} \setminus \Sigma$. Then T_E is uniformly hyperbolic.*

Proof. By minimality, $E \in \rho(H_\omega)$ for all $\omega \in \Omega$.

Let $\omega \in \Omega$. We show: there exist vectors $U(\omega), V(\omega) \in \mathbb{K}^2$ such that $\|T_E(t,\omega)U(\omega)\|$ decays exponentially for $t \to \infty$ and $\|T_E(t,\omega)V(\omega)\|$ decays exponentially for $t \to -\infty$.

Let $t_0 < 0$. Define the restriction $H_\omega|_{[t_0,0]}$ of H_ω to $[t_0, 0]$ by

$$\begin{aligned} D(H_\omega|_{[t_0,0]}) &:= \left\{ u \in L_2(t_0,0);\ u, A_\omega u \in W^1_{1,\mathrm{loc}}[t_0,0],\ -(A_\omega u)' \in L_2(t_0,0) \right\}, \\ H_\omega|_{[t_0,0]} u &:= -(A_\omega u)'. \end{aligned}$$

Since we have limit point case at $-\infty$ (see Proposition 1.3.5) there exists $(a,b) \in \mathbb{K}^2 \setminus \{(0,0)\}$ such that for solutions u of $H_\omega u = Eu$ with $(u(t_0), u'(t_0+)) \in \operatorname{lin}\{(a,b)\}$, the linear span of $\{(a,b)\}$, we have $u \notin L_2(-\infty, t_0)$. Let $v \in D(H_\omega|_{[t_0,0]}) \subseteq L_2(t_0,0)$ such that

$$\begin{pmatrix} v(t_0) \\ v'(t_0+) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} v(0) \\ v'(0-) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Set $\tilde{v} := (H_\omega|_{[t_0,0]} - E)v \in L_2(t_0,0) \subseteq L_2(\mathbb{R})$, where we extended $\tilde{v}$ by zero. Define $u := (H_\omega - E)^{-1}\tilde{v} \in L_2(\mathbb{R})$. Note that u is a solution of $H_\omega u = Eu + \tilde{v}$ and hence a solution of $H_\omega u = Eu$ on $[0,\infty)$. Then $(u(0), u'(0+)) \neq (0,0)$, for if $(u(0), u'(0+)) = (0,0)$, then $u|_{(t_0,0)} = v|_{(t_0,0)}$ and hence

$$\begin{pmatrix} u(t_0) \\ u'(t_0+) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

But this would imply $u \notin L_2(-\infty, t_0)$ and therefore $u \notin L_2(\mathbb{R})$. Therefore, u cannot vanish on $[0,\infty)$.

By Combes-Thomas arguments, see Proposition A.3.1, there exist $C \geq 0$ and $\kappa > 0$ (not depending on ω) such that

$$\left\| \mathbb{1}_{(t-\frac{1}{2},t+\frac{1}{2})} u \right\|_{L_2(\mathbb{R})} \leq C e^{-\kappa t} \quad (t \geq 0).$$

Note that in the following the constant C may increase from line to line.

Since

$$A_\omega u(t) = u'(t+) - \int_0^t u(s)\, d\omega(s) \quad (t \in \mathbb{R}),$$

for $t \geq \frac{1}{2}$ and $s \in [-\frac{1}{2}, \frac{1}{2}]$ we have

$$\begin{aligned} \left| u'(t+) - u'((t+s)+) \right| &\leq |A_\omega u(t) - A_\omega u(t+s)| + \left| \int_t^{t+s} u(r)\, d\omega(r) \right| \\ &= |E| \left| \int_t^{t+s} u(r)\, dr \right| + \left| \int_t^{t+s} u(r)\, d\omega(r) \right| \\ &\leq |E| \left\| \mathbb{1}_{(t-\frac{1}{2},t+\frac{1}{2})} u \right\|_{L_2(\mathbb{R})} + \left\| \mathbb{1}_{(t-\frac{1}{2},t+\frac{1}{2})} u \right\|_{L_\infty(\mathbb{R})} \|\omega\|_{\mathrm{loc}} . \end{aligned}$$

By Caccioppoli's inequality for local solutions (see Proposition A.3.3), we have

$$\left\| \mathbb{1}_{(t-\frac{1}{4},t+\frac{1}{4})} u' \right\|_{L_2(\mathbb{R})} \leq C \left\| \mathbb{1}_{(t-\frac{1}{2},t+\frac{1}{2})} u' \right\|_{L_2(\mathbb{R})} \leq C e^{-\kappa t} \quad (t \geq \frac{1}{2}).$$

Thus, by Sobolev's inequality,

$$|u(t)| \leq C e^{-\kappa t} \quad (t \geq \frac{1}{2}),$$

and hence

$$\left| u'(t+)) - u'((t+s)+) \right| \leq C e^{-\kappa t} \quad (t \geq \frac{1}{2}).$$

Therefore,

$$\left| u'(t+) \right| \leq C e^{-\kappa t} + \left| u'((t+s)+) \right|$$

and integration with respect to $s \in [-\frac{1}{4}, \frac{1}{4}]$ and an application of Hölder's inequality yields

$$\left| u'(t+) \right| \leq C e^{-\kappa t} + \left\| \mathbb{1}_{(t-\frac{1}{4},t+\frac{1}{4})} u' \right\|_{L_1(\mathbb{R})} \leq C e^{-\kappa t} \quad (t \geq \frac{1}{2}).$$

We end up with

$$\left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} \right\| \leq C e^{-\kappa t} \quad (t \geq \frac{1}{2}).$$

Hence, the initial condition $U(\omega) = (u(0), u'(0+))$ gives rise to a solution of the Schrödinger equation $H_\omega u = Eu$ which decays exponentially for $t \to \infty$ and does not vanish on $[0, \infty)$. This yields an element $u(\omega) = [U(\omega)]_{\mathcal{P}(\mathbb{K}^2)} \in \mathcal{P}(\mathbb{K}^2)$.

Analogously, we find $v(\omega) \in \mathcal{P}(\mathbb{K}^2)$ such that the corresponding solutions decay exponentially for $t \to -\infty$.

We have $u(\omega) \neq v(\omega)$. Indeed, in case $u(\omega) = v(\omega)$, such an initial condition yields an $L_2(\mathbb{R})$-solution of $H_\omega u = Eu$, i.e., E is an eigenvalue of H_ω. But $E \notin \sigma(H_\omega)$, so $u(\omega) \neq v(\omega)$.

Therefore, T_E admits an exponential splitting (note that the constants κ and C can be chosen uniformly on Ω).

By Proposition 5.3.10, $\omega \mapsto u(\omega)$ and $\omega \mapsto v(\omega)$ are continuous.

By Theorem 5.2.8, T_E is uniformly hyperbolic. //

As a consequence of the previous lemmas we obtain the following characterization.

6.1.4 Theorem. *Let $(\Omega, \alpha, \mathbb{P})$ be strictly ergodic and atomless. Then the following are equivalent:*

(a) T_E is uniform for all $E \in \mathbb{R}$.

(b) $\Sigma = \{E \in \mathbb{R};\ \gamma(E) = 0\}$.

In this case the Lyapunov exponent $\gamma\colon \mathbb{R} \to [0, \infty)$ is continuous on Σ.

Proof. "(a) ⇒ (b)": This follows from Lemma 6.1.1, which also shows continuity of γ.

"(b) ⇒ (a)": This is a direct consequence of Lemma 6.1.2 and Lemma 6.1.3. //

As a sharpening of Theorem 6.1.4 we obtain the following.

6.1.5 Theorem. *Let $(\Omega, \alpha, \mathbb{P})$ be strictly ergodic and atomless. Then*

$$\Sigma = \{E \in \mathbb{R};\ \gamma(E) = 0\} \cup \{E \in \mathbb{R};\ T_E \textit{ is not uniform}\},$$

where the union is disjoint.

Proof. By Lemma 6.1.2 the union is disjoint.

"$\supseteq$": This is a direct consequence of Lemma 6.1.3.

"$\subseteq$": Let $E \in \mathbb{R}$ with $\gamma(E) > 0$ and T_E uniform.

Let $\delta > 0$. By Proposition 4.4.7, as soon as $|E - F|$ is small enough, we have

$$D := \sup_{-1 \le t \le 1} \sup_{\omega \in \Omega} \|T_E(t, \omega) - T_F(t, \omega)\| < \delta.$$

By Theorem 5.3.13, T_F is uniformly hyperbolic for all F in a small open interval I containing E. Now, we can repeat the proof of Theorem 6.1.1 replacing $\complement\Gamma$ with I. Assume there is spectrum in I. Fix $\omega \in \Omega$. Then the spectral measures of H_ω give weight to I. By Proposition A.3.5 there exists $F \in I \cap \sigma(H_\omega)$ admitting a subexponentially bounded solution. But $\gamma(F) > 0$, a contradiction. So, in particular, $E \notin \Sigma$. //

As soon as one can prove the (semi)uniform estimates given in Theorems 5.1.4 and 5.1.5 for almost continuous processes one can omit the assumption that Ω has to be atomless.

6.2. Hyperbolicity

In this section we focus on one particular Schrödinger operator. It can be seen as the first preliminary section for Kotani theory. In fact, later we will prove the Ishii-Pastur-Kotani theorem, which states that an essential support of the absolutely continuous part of the spectrum is given by the set of zeros of the Lyapunov exponent. This section provides some tools to prove the Ishii-Pastur "half" of the theorem.

Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Define

$$\mathrm{hyp}(H_\mu) := \left\{ E \in \mathbb{R};\ \exists \gamma(E) > 0 : \lim_{t\to\pm\infty} \frac{1}{|t|} \ln \|T_E(t,\mu)\| = \gamma(E) \right\},$$

the set of hyperbolic values of H_μ.

6.2.1 Lemma ([9, Proposition III.4.10]). *Let $E \in \mathrm{hyp}(H_\mu)$. Then there exist two one-dimensional subspaces $V^+(E)$ and $V^-(E)$ of $\mathbb{K}^2$, such that for $0 \neq v \in \mathbb{K}^2$ we have*

$$v \in V^\pm(E) \Longleftrightarrow \lim_{t\to\pm\infty} \frac{1}{|t|} \ln \|T_E(t,\mu)v\| = -\gamma(E),$$
$$v \notin V^\pm(E) \Longleftrightarrow \lim_{t\to\pm\infty} \frac{1}{|t|} \ln \|T_E(t,\mu)v\| = \gamma(E).$$

Proof. This is a direct consequence of Osedelec's Theorem; cf. [9, Theorem IV.2.4 and Proposition III.4.10]. //

6.2.2 Lemma ([9, Lemma III.4.11]). *Let $E \in \mathrm{hyp}(H_\mu)$. The following are equivalent:*
(a) E is an eigenvalue of H_μ.
(b) There exists $v \in \mathbb{K}^2 \setminus \{0\}$ such that

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \ln \|T_E(t,\mu)v\| = -\gamma(E).$$

Proof. Let u be a non-zero (generalized) solution of $H_\mu u = Eu$. Set $v := (u(0), u'(0+))$. Then $v \neq 0$. By Lemma 6.2.1, the alternative (i) $v \in V^+(E) \cap V^-(E)$ or (ii) $v \notin V^+(E) \cap V^-(E)$ yields the existence of $\alpha, t_0 > 0$ such that in case of (i) we have

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \ln \|T_E(t,\mu)v\| = -\gamma(E) \Longrightarrow \left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} \right\| \leq e^{-\alpha|t|} \quad (|t| \geq t_0),$$

while in case of (ii) we conclude

$$\lim_{t\to+\infty} \frac{1}{|t|} \ln \|T_E(t,\mu)v\| = \gamma(E) \quad \text{or} \quad \lim_{t\to-\infty} \frac{1}{|t|} \ln \|T_E(t,\mu)v\| = \gamma(E)$$
$$\Longrightarrow \left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} \right\| \geq e^{\alpha t} \quad (t \geq t_0) \quad \text{or} \quad \left\| \begin{pmatrix} u(t) \\ u'(t+) \end{pmatrix} \right\| \geq e^{-\alpha t} \quad (t \leq -t_0).$$

Since u is a solution of $H_\mu u = Eu$, in the first case we have $u \in L_2(\mathbb{R})$ and therefore $u \in D(T_\mu) = D(H_\mu)$, i.e., E is an eigenvalue. In the second case, $u \notin W_2^1(\mathbb{R})$ and hence $u \notin D(H_\mu)$. //

6.2.3 Theorem ([9, Theorem III.4.12]). *Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. Let m be a non-negative continuous Borel measure on $\mathbb{R}$ which is supported by* $\mathrm{hyp}(H_\mu)$*. Then m is orthogonal to the spectral measure ϱ_μ of H_μ.*

Proof. Since m is continuous and the set of eigenvalues of H_μ is at most countable, we can conclude that $m(\{E \in \mathbb{R};\ E \text{ is an eigenvalue of } H_\mu\}) = 0$. By Lemma 6.2.2 it follows that for m-a.a. $E \in \mathbb{R}$ and for any non-zero (generalized) solution u of $H_\mu u = Eu$, $\left\|\begin{pmatrix} u(\cdot) \\ u'(\cdot+) \end{pmatrix}\right\|$ is growing exponentially fast in at least one direction of $\mathbb{R}$. Without loss of generality, let $\left\|\begin{pmatrix} u(\cdot) \\ u'(\cdot+) \end{pmatrix}\right\|$ grow exponentially fast for $t \to \infty$.

By Proposition A.3.5, for ϱ_μ-a.a. $E \in \mathbb{R}$ there exists a non-zero subexponentially bounded solution of $H_\mu u = Eu$. Since

$$u'(t+) = u'(0+) + \int_0^t u(s)\, d(\mu - E\lambda)(s) \quad (t \in \mathbb{R})$$

and $|\mu - E\lambda|\,([0,t]) \le (|t| + 1)(\|\mu\|_{\mathrm{loc}} + |E|)$ $(t \in \mathbb{R})$, $|u'(\cdot+)|$ is subexponentially bounded as well. Hence, we have

$$\varrho_\mu(\mathrm{hyp}(H_\mu) \setminus \{E \in \mathbb{R};\ E \text{ is an eigenvalue of } H_\mu\}) = 0$$

and, therefore, ϱ_μ is orthogonal to m. //

6.3. Titchmarsh-Weyl m-functions

This section provides the tools for the Kotani "half" of the Ishii-Pastur-Kotani theorem. We investigate the Titchmarsh-Weyl m-functions and the kernel of the resolvent. Then we prove various auxiliary results concerning these functions. The so-called w-function describing the exponential behavior of the (unique) L_2-solutions at $\pm\infty$ will be introduced and the connection with the Lyapunov exponent will be established. In this section we follow [31] and [9, Chapter 7]. Since we deal with measures as potentials (in contrast to the stated sources), we give full proofs of the results.

Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$. As proven in Chapter 1 the operator H_μ is in the limit point case at $\pm\infty$.

Let $z \in \mathbb{C}$. Denote by $u_D(\cdot, z)$, $u_N(\cdot, z)$ the solutions of the Schrödinger equation $H_\mu u = zu$ subject to

$$\begin{aligned} u_D(0,z) &= 0 & \qquad u_N(0,z) &= 1, \\ u_D'(0+,z) &= 1, & \qquad u_N'(0+,z) &= 0. \end{aligned}$$

Also consider $H_\mu^+ := H_\mu|_{[0,\infty)}$ and $H_\mu^- := H_\mu|_{(-\infty,0]}$ with Dirichlet boundary conditions at 0. These two operators are self-adjoint on $L_2([0,\infty))$ and $L_2((-\infty,0])$, respectively. Furthermore, $H_\mu^\pm$ is in the limit point case at $\pm\infty$.

For $z \in \mathbb{C} \setminus \mathbb{R}$, there is a unique solution $u_\pm(\cdot, z)$ of $H_\mu^\pm u = zu$ which is L_2 at $\pm\infty$ and satisfies $u_\pm(0, z) = 1$. Thus, there exist unique $m_\pm(z)$ such that

$$u_\pm(\cdot, z) = u_N(\cdot, z) \pm m_\pm(z) u_D(\cdot, z).$$

The functions $z \mapsto m_\pm(z)$ are called *Titchmarsh-Weyl m-functions.*

The Wronskian between $u_+(\cdot, z)$ and $u_-(\cdot, z)$ may be computed and satisfies

$$W(u_+(\cdot, z), u_-(\cdot, z)) = -(m_+(z) + m_-(z)).$$

6.3.1 Lemma (see also [16, Theorem 8.3])**.** *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then the resolvent $(H_\mu - z)^{-1}$ has an integral kernel $G_\mu(\cdot, \cdot, z)$ satisfying*

$$G_\mu(s, t, z) = \begin{cases} \frac{u_+(t,z)u_-(s,z)}{W(u_+(\cdot,z),u_-(\cdot,z))} & s \leq t, \\ \frac{u_+(s,z)u_-(t,z)}{W(u_+(\cdot,z),u_-(\cdot,z))} & s > t. \end{cases}$$

In particular,

$$G_\mu(0, 0, z) = -\frac{1}{m_+(z) + m_-(z)}.$$

Proof. Since $(H_\mu - z)^{-1} \colon L_2(\mathbb{R}) \to D(H_\mu) \subseteq W_2^1(\mathbb{R}) \subseteq L_\infty(\mathbb{R})$, $(H_\mu - z)^{-1}$ maps $L_2(\mathbb{R})$ to $L_\infty(\mathbb{R})$. Now, since $|\mu|$ is form small with respect to the classical Dirichlet form,

$$\int |g|^2 \, d\,|\mu| \leq \frac{1}{2} \|g\|_{W_2^1(\mathbb{R})}^2 + C \|g\|_2^2 \quad (g \in W_2^1(\mathbb{R})).$$

Hence, for $f \in L_2(\mathbb{R})$,

$$\begin{aligned}
&\left\|(H_\mu - z)^{-1} f\right\|_{W_2^1(\mathbb{R})}^2 \\
&= \left\|(H_\mu - z)^{-1} f\right\|_{L_2(\mathbb{R})}^2 + \tau_\mu((H_\mu - z)^{-1} f, (H_\mu - z)^{-1} f) - \int \left|(H_\mu - z)^{-1} f\right|^2 d\mu \\
&\leq \left\|(H_\mu - z)^{-1}\right\|^2 \|f\|_{L_2(\mathbb{R})}^2 + \left|\left(f \mid (H_\mu - z)^{-1} f\right)\right| + |z| \left\|(H_\mu - z)^{-1} f\right\|_{L_2(\mathbb{R})}^2 \\
&\quad + \frac{1}{2} \left\|(H_\mu - z)^{-1} f\right\|_{W_2^1(\mathbb{R})}^2 + C \left\|(H_\mu - z)^{-1} f\right\|_{L_2(\mathbb{R})}^2 \\
&\leq \frac{1}{2} \left\|(H_\mu - z)^{-1} f\right\|_{W_2^1(\mathbb{R})}^2 \\
&\quad + \Big((1 + C + |z|) \left\|(H_\mu - z)^{-1}\right\|^2 + \left\|(H_\mu - z)^{-1}\right\|\Big) \|f\|_{L_2(\mathbb{R})}^2 .
\end{aligned}$$

Therefore,

$$\left\|(H_\mu - z)^{-1} f\right\|_{W_2^1(\mathbb{R})}^2 \leq 2 \Big((1 + C + |z|) \left\|(H_\mu - z)^{-1}\right\|^2 + \left\|(H_\mu - z)^{-1}\right\|\Big) \|f\|_{L_2(\mathbb{R})}^2 .$$

We conclude

$$\begin{aligned}
&\left\|(H_\mu - z)^{-1} f\right\|_{L_\infty(\mathbb{R})}^2 \leq \left\|(H_\mu - z)^{-1} f\right\|_{W_2^1(\mathbb{R})}^2 \\
&\leq 2 \Big((1 + C + |z|) \left\|(H_\mu - z)^{-1}\right\|^2 + \left\|(H_\mu - z)^{-1}\right\|\Big) \|f\|_{L_2(\mathbb{R})}^2 ,
\end{aligned}$$

i.e., $(H_\mu - z)^{-1} \in L(L_2(\mathbb{R}), L_\infty(\mathbb{R}))$ and therefore has an integral kernel $G_\mu(\cdot, \cdot, z)$.

Let $f \in L_2(\mathbb{R})$ and define $g \colon \mathbb{R} \to \mathbb{K}$ by

$$g(s) := u_+(s, z) \int_{-\infty}^{s} u_-(t, z) f(t) \, dt + u_-(s, z) \int_{s}^{\infty} u_+(t, z) f(t) \, dt.$$

For $f \in L_{2,c}(\mathbb{R})$ we have $g \in W^1_{1,\mathrm{loc}}(\mathbb{R})$ and $(A_\mu g)' = -zg - W(u_+(\cdot,z),u_-(\cdot,z))f$, i.e., $g \in L_2(\mathbb{R})$ and $(H_\mu - z)^{-1}f = W(u_+(\cdot,z),u_-(\cdot,z))^{-1}g$. For general $f \in L_2(\mathbb{R})$ we approximate f by a sequence (f_n) in $L_{2,c}(\mathbb{R})$ such that $f_n \to f$ in $L_2(\mathbb{R})$ and pointwise a.e. Since $(H_\mu - z)^{-1}$ is continuous,

$$(H_\mu - z)^{-1}f = \lim_{n\to\infty}(H_\mu - z)^{-1}f_n = \lim_{n\to\infty} W(u_+(\cdot,z),u_-(\cdot,z))^{-1}g_n$$

and $g_n \to g$ pointwise a.e. (at least for a subsequence). Therefore, $(H_\mu - z)^{-1}f = W(u_+(\cdot,z),u_-(\cdot,z))^{-1}g$.

The Wronskian of two solutions is constant. Therefore,

$$\begin{aligned} W(u_+(\cdot,z),u_-(\cdot,z)) &= u_+(0,z)u_-'(0+,z) - u_+'(0+,z)u_-(0,z) \\ &= -m_-(z) - m_+(z) = -(m_+(z) + m_-(z)). \end{aligned}$$

Hence,

$$G_\mu(0,0,z) = -\frac{1}{m_+(z) + m_-(z)}. \qquad //$$

6.3.2 Remark. Let $z \in \mathbb{C}\setminus\mathbb{R}$. Note that

$$W(u_+(\cdot,z),u_D(\cdot,z)) = W(u_-(\cdot,z),u_D(\cdot,z)) = 1.$$

Again by [16, Theorem 8.3], the kernels $G^\pm_\mu(\cdot,\cdot,z)$ of the resolvent of $H^\pm_\mu$ satisfy

$$G^+_\mu(s,t,z) = \begin{cases} u_+(t,z)u_D(s,z) & s \le t, \\ u_+(s,z)u_D(t,z) & s > t, \end{cases} \qquad G^-_\mu(s,t,z) = \begin{cases} u_-(s,z)u_D(t,z) & s \le t, \\ u_-(t,z)u_D(s,z) & s > t. \end{cases}$$

6.3.3 Proposition (see also [16, Theorem 9.1 and Corollary 9.5])**.** *The m-functions $m_\pm \colon \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}$ are holomorphic, $m_+(z) = \overline{m_+(\overline{z})}$ for all $z \in \mathbb{C}\setminus\mathbb{R}$ and we have*

$$\frac{\operatorname{Im} m_\pm(z)}{\operatorname{Im} z} = \|u_\pm\|^2_{L_2} > 0 \quad (z \in \mathbb{C}\setminus\mathbb{R}).$$

Proof. (i) We only prove the assertions for m_+. The proofs for m_- are the same. Let $a,b \in (0,\infty)$, $a < b$. For $t \in (a,b)$ we have

$$\begin{aligned} (H^+_\mu - z)^{-1}\mathbb{1}_{(a,b)}(t) &= u_+(t,z)\int_a^t u_D(s,z)\,ds + u_D(t,z)\int_t^b u_+(s,z)\,ds \\ &= \Big(u_N(t,z) + m_+(z)u_D(t,z)\Big)\int_a^t u_D(s,z)\,ds \\ &\quad + u_D(t,z)\int_t^b \big(u_N(s,z) + m_+(z)u_D(s,z)\big)\,ds \\ &= m_+(z)u_D(t,z)\int_a^b u_D(s,z)\,ds + \int_a^b \tilde{G}(t,s,z)\,ds, \end{aligned}$$

where

$$\tilde{G}(t,s,z) = \begin{cases} u_D(s,z)u_N(t,z) & s < t, \\ u_D(t,z)u_N(s,z) & s \geq t. \end{cases}$$

Therefore,

$$\big((H_\mu^+ - z)^{-1}\mathbb{1}_{(a,b)} \,\big|\, \mathbb{1}_{(a,b)}\big) = m_+(z)\left(\int_a^b u_D(s,z)\,ds\right)^2 + \int_a^b\int_a^b \tilde{G}(t,s,z)\,ds\,dt.$$

Note that u_D and u_N are holomorphic in z by Lemma 1.2.6 and the resolvent of H_μ^+ is holomorphic on the resolvent set. Furthermore, since $u_N(\cdot,z)$ and $u_D(\cdot,z)$ are locally bounded, the integrals on the right hand side are analytic in z.

Let $z \in \mathbb{C}\setminus\mathbb{R}$. We show that there exist $a,b \in (0,\infty)$ such that $\int_a^b u_D(s,z)\,ds \neq 0$. Assume the contrary. Then $u_D(\cdot,z) = 0$ almost everywhere, contradicting $u_D'(0+,z) = 1$.

Hence, m_+ is holomorphic.

(ii) Let $z \in \mathbb{C}\setminus\mathbb{R}$. Since $u_D(\cdot,\overline{z}) = \overline{u_D(\cdot,z)}$ and $u_N(\cdot,\overline{z}) = \overline{u_N(\cdot,z)}$ we obtain

$$L_2(0,\infty) \ni \overline{u_+(\cdot,z)} = \overline{u_N(\cdot,z) + m_+(z)u_D(\cdot,z)} = u_N(\cdot,\overline{z}) + \overline{m_+(z)}u_D(\cdot,\overline{z})$$

Hence, $m_+(\overline{z}) = \overline{m_+(z)}$.

(iii) We now show $\|u_+(\cdot,z)\|_{L_2(0,\infty)}^2 = \frac{\operatorname{Im} m_+(z)}{\operatorname{Im} z}$. Since u_+ is nontrivial, the last assertion will follow. Let $z_1,z_2 \in \mathbb{C}\setminus\mathbb{R}$. Then

$$W(u_+(\cdot,z_1),u_+(\cdot,z_2))(0) = m_+(z_2) - m_+(z_1).$$

Therefore, with the help of Lemma 1.3.3 for $N \geq 0$ we can compute

$$\begin{aligned}(z_1 - z_2)&\int_0^N u_+(s,z_1)u_+(s,z_2)\,ds \\ &= W(u_+(\cdot,z_1),u_+(\cdot,z_2))(N) - W(u_+(\cdot,z_1),u_+(\cdot,z_2))(0) \\ &= W(u_+(\cdot,z_1),u_+(\cdot,z_2))(N) + m_+(z_1) - m_+(z_2).\end{aligned}$$

As $N \to \infty$ we obtain $W(u_+(\cdot,z_1),u_+(\cdot,z_2))(N) \to 0$ (for example by Lemma 2.3.2, which also holds true for complex energies). Hence,

$$(z_1 - z_2)\int_0^\infty u_+(s,z_1)u_+(s,z_2)\,ds = m_+(z_1) - m_+(z_2).$$

Let $z \in \mathbb{C}\setminus\mathbb{R}$. Then $m_+(z) = \overline{m_+(\overline{z})}$ and therefore

$$u_+(\cdot,z) = \overline{u_+(\cdot,\overline{z})}.$$

We conclude that

$$\|u_+(\cdot,z)\|_{L_2(0,\infty)}^2 = \int_0^\infty u_+(s,z)u_+(s,\overline{z})\,ds = \frac{\operatorname{Im} m_+(z)}{\operatorname{Im} z}. \qquad //$$

Now, consider again the family $(H_\omega)_{\omega\in\Omega}$. Then $m_\pm(z)$ (and also $u_\pm(t,z)$, $u'_\pm(t+,z)$) are random variables for all $z \in \mathbb{C}^+$. We denote by $m_\pm(z)(\omega)$ the m-functions at z for $\omega \in \Omega$, and similarly for the solutions. For $z \in \mathbb{C}^+$, $\omega \in \Omega$ and $t \in \mathbb{R}$ define

$$f_\pm(t,z,\omega) := m_\pm(z)(\alpha_t(\omega)).$$

According to Remark 1.3.8 we have

$$m_+(z)(\alpha_t(\omega)) = -\lim_{s\to\infty} \frac{u_N(s,z)(\alpha_t(\omega))}{u_D(s,z)(\alpha_t(\omega))} \quad (t \in \mathbb{R}, \omega \in \Omega).$$

Since by the cocycle property of T_z we have $T_z(s+t,\omega) = T_z(s,\alpha_t(\omega))T_z(t,\omega)$ we can solve this matrix equation for the elements of $T_z(s,\alpha_t(\omega))$ and obtain

$$\begin{aligned} u_N(s,z)(\alpha_t(\omega)) &= u_N(s+t,z)(\omega)u'_D(t+,z)(\omega) - u_D(s+t,z)(\omega)u'_N(t+,z)(\omega) \\ u_D(s,z)(\alpha_t(\omega)) &= u_D(s+t,z)(\omega)u_N(t,z)(\omega) - u_N(s+t,z)(\omega)u_D(t,z)(\omega) \end{aligned}$$

for all $s,t \in \mathbb{R}$, $\omega \in \Omega$. Thus,

$$\begin{aligned} f_+(t,z,\omega) &= m_+(z)(\alpha_t(\omega)) \\ &= -\lim_{s\to\infty} \frac{u_N(s+t,z)(\omega)u'_D(t+,z)(\omega) - u_D(s+t,z)(\omega)u'_N(t+,z)(\omega)}{u_D(s+t,z)(\omega)u_N(t,z)(\omega) - u_N(s+t,z)(\omega)u_D(t,z)(\omega)} \\ &= -\lim_{s\to\infty} \frac{\frac{u_N(s+t,z)(\omega)}{u_D(s+t,z)(\omega)}u'_D(t+,z)(\omega) - u'_N(t+,z)(\omega)}{u_N(t,z)(\omega) - \frac{u_N(s+t,z)(\omega)}{u_D(s+t,z)(\omega)}u_D(t,z)(\omega)} \\ &= \frac{u'_N(t+,z)(\omega) + m_+(z)(\omega)u'_D(t+,z)(\omega)}{u_N(t,z)(\omega) + m_+(z)(\omega)u_D(t,z)(\omega)} \\ &= \frac{u'_+(t+,z)(\omega)}{u_+(t,z)(\omega)}. \end{aligned}$$

Similarly,

$$f_-(t,z,\omega) = m_-(z)(\alpha_t(\omega)) = -\frac{u'_-(t+,z)(\omega)}{u_-(t,z)(\omega)}.$$

Therefore, $t \mapsto f_\pm(t,z,\omega)$ satisfies (in a distributional sense) the Ricatti equation

$$f'_\pm(\cdot,z,\omega) = \pm(\omega - z - f_\pm(\cdot,z,\omega)^2).$$

6.3.4 Lemma (compare [31, Lemma 1.1]). *Let $z \in \mathbb{C}^+$, $t \in \mathbb{R}$, $\omega \in \Omega$. Then*

(a) $G_\omega(t,t,z) = G_{\alpha_t(\omega)}(0,0,z)$.

(b) $f_+(t,z,\omega) - f_-(t,z,\omega) = \frac{d}{dt}\log G_\omega(t,t,z)$, *where* log *denotes the principal value of the complex logarithm function.*

(c) $G_\omega(t,t,z) + \frac{d}{dz}\frac{1}{2G_\omega(t,t,z)} = -\frac{d}{dt}h(\alpha_t(\omega))$, *where*

$$h(\omega) = \frac{1}{2}G_\omega(0,0,z)\left(\int_0^\infty u_+(t,z)(\omega)^2\,dt - \int_{-\infty}^0 u_-(t,z)(\omega)^2\,dt\right).$$

Proof. (a) By Lemma 6.3.1 we have

$$G_\omega(0,0,z) = -\frac{1}{m_+(z)(\omega) + m_-(z)(\omega)}.$$

Hence,

$$G_{\alpha_t(\omega)}(0,0,z) = -\frac{1}{f_+(t,z,\omega) + f_-(t,z,\omega)}.$$

Since we can write

$$u_\pm(t,z)(\omega) = \exp(\pm \int_0^t f_\pm(s,z,\omega)\,ds) = u_N(t,z)(\omega) \pm m_\pm(z)(\omega)u_D(t,z)(\omega),$$

we obtain

$$f_+(t,z,\omega) + f_-(t,z,\omega) = \frac{m_+(z)(\omega) + m_-(z)(\omega)}{u_+(t,z)(\omega)u_-(t,z)(\omega)} = -G_\omega(t,t,z).$$

(b) Let $\omega \in \Omega$ and $z \in \mathbb{C}^+$. Define $g(t) := \log G_\omega(t,t,z)$. Then we compute

$$\begin{aligned} g'(t) &= \frac{1}{G_\omega(t,t,z)}\left(\partial_1 G_\omega(t,t,z) + \partial_2 G_\omega(t,t,z)\right) \\ &= \frac{u_+(t,z)(\omega)u'_-(t+,z)(\omega) + u'_+(t+,z)(\omega)u_-(t,z)(\omega)}{G_\omega(t,t,z)W(u_+(\cdot,z)(\omega), u_-(\cdot,z)(\omega))} \\ &= \frac{u_+(t,z)(\omega)u'_-(t+,z)(\omega) + u'_+(t+,z)(\omega)u_-(t,z)(\omega)}{u_+(t,z)(\omega)u_-(t,z)(\omega)} \\ &= \frac{u'_+(t+,z)(\omega)}{u_+(t,z)(\omega)} - \left(-\frac{u'_-(t+,z)(\omega)}{u_-(t,z)(\omega)}\right) = f_+(t,z,\omega) - f_-(t,z,\omega). \end{aligned}$$

(c) By part (a) it follows that

$$G_{\alpha_t(\omega)}(0,0,z) = G_\omega(t,t,z) = G_\omega(0,0,z)u_+(t,z)(\omega)u_-(t,z)(\omega).$$

Furthermore, as

$$m_+(z)(\alpha_t(\omega)) = f_+(t,z,\omega) = \frac{u_+(t+,z)(\omega)}{u_+(t,z)(\omega)},$$

we compute

$$\begin{aligned} &u_+(t,z)(\omega)u_+(s,z)(\alpha_t(\omega)) \\ &= u_+(t,z)(\omega)\left(u_N(s,z)(\alpha_t(\omega)) + \frac{u_+(t+,z)(\omega)}{u_+(t,z)(\omega)}u_D(s,z)(\alpha_t(\omega))\right) \\ &= u_N(s,z)(\alpha_t(\omega))u_+(t,z)(\omega) + u_D(s,z)(\alpha_t(\omega))u'_+(t+,z)(\omega). \end{aligned}$$

Since $T_z(s,\alpha_t(\omega))T_z(t,\omega) = T_z(s+t,\omega)$ and

$$T_z(t,\omega)\begin{pmatrix} 1 \\ m_+(z)(\omega) \end{pmatrix} = \begin{pmatrix} u_+(t,\omega) \\ u'_+(t+,\omega) \end{pmatrix},$$

we conclude that $u_+(t,z)(\omega)u_+(s,z)(\alpha_t(\omega)) = u_+(s+t,z)(\omega)$. By the same reasoning we have $u_-(t,z)(\omega)u_-(s,z)(\alpha_t(\omega)) = u_-(s+t,z)(\omega)$. Considering the function $t \mapsto h(\alpha_t(\omega))$, we can write

$$\begin{aligned}
&h(\alpha_t(\omega))\\
&= \frac{1}{2}G_{\alpha_t(\omega)}(0,0,z)\left(\int_0^\infty u_+(s,z)(\alpha_t(\omega))^2\,ds - \int_{-\infty}^0 u_-(s,z)(\alpha_t(\omega))^2\,ds\right)\\
&= \frac{1}{2}G_\omega(0,0,z)\Bigg(\frac{u_-(t,z)(\omega)}{u_+(t,z)(\omega)}\int_0^\infty u_+(t,z)(\omega)^2 u_+(s,z)(\alpha_t(\omega))^2\,ds\\
&\qquad\qquad - \frac{u_+(t,z)(\omega)}{u_-(t,z)(\omega)}\int_{-\infty}^0 u_-(t,z)(\omega)^2 u_-(s,z)(\alpha_t(\omega))^2\,ds\Bigg)\\
&= \frac{1}{2}G_\omega(0,0,z)\left(\frac{u_-(t,z)(\omega)}{u_+(t,z)(\omega)}\int_t^\infty u_+(r,z)(\omega)^2\,dr - \frac{u_+(t,z)(\omega)}{u_-(t,z)(\omega)}\int_{-\infty}^t u_-(r,z)(\omega)^2\,dr\right).
\end{aligned}$$

Note that

$$\left(\frac{u_-(\cdot,z)(\omega)}{u_+(\cdot,z)(\omega)}\right)' = \frac{W(u_+(\cdot,z)(\omega),u_-(\cdot,z)(\omega))}{u_+(\cdot,z)(\omega)^2},$$

and

$$\left(\frac{u_+(\cdot,z)(\omega)}{u_-(\cdot,z)(\omega)}\right)' = -\frac{W(u_+(\cdot,z)(\omega),u_-(\cdot,z)(\omega))}{u_-(\cdot,z)(\omega)^2}.$$

Differentiating $t \mapsto h(\alpha_t(\omega))$ therefore yields

$$\begin{aligned}
&\frac{d}{dt}h(\alpha_t(\omega))\\
&= \frac{1}{2}G_\omega(0,0,z)\\
&\quad\Bigg(\frac{W(u_+(\cdot,z)(\omega),u_-(\cdot,z)(\omega))}{u_+(t,z)(\omega)^2}\int_t^\infty u_+(r,z)(\omega)^2\,dr - \frac{u_-(t,z)(\omega)}{u_+(t,z)(\omega)}u_+(t,z)(\omega)^2\\
&\quad + \frac{W(u_+(\cdot,z)(\omega),u_-(\cdot,z)(\omega))}{u_-(t,z)(\omega)^2}\int_{-\infty}^t u_-(r,z)(\omega)^2\,dr - \frac{u_+(t,z)(\omega)}{u_-(t,z)(\omega)}u_-(t,z)(\omega)^2\Bigg)\\
&= -G_\omega(0,0,z)u_+(t,z)(\omega)u_-(t,z)(\omega)\\
&\qquad + \frac{1}{2}\frac{1}{u_+(t,z)(\omega)^2}\int_t^\infty u_+(r,z)(\omega)^2\,dr + \frac{1}{2}\frac{1}{u_-(t,z)(\omega)^2}\int_{-\infty}^t u_-(r,z)(\omega)^2\,dr\\
&= -G_\omega(t,t,z)\\
&\quad + \frac{1}{2}\frac{1}{u_+(t,z)(\omega)^2}\int_t^\infty u_+(r,z)(\omega)^2\,dr + \frac{1}{2}\frac{1}{u_-(t,z)(\omega)^2}\int_{-\infty}^t u_-(r,z)(\omega)^2\,dr.
\end{aligned}$$

Let $f \in L_2(\mathbb{R})$. Then by the first resolvent identity, for $s \in \mathbb{R}$ and $z, z_0 \in \mathbb{C}^+$,

$$\begin{aligned}
&\int_{\mathbb{R}} G_\omega(s,t,z) f(t)\, dt - \int_{\mathbb{R}} G_\omega(s,t,z_0) f(t)\, dt \\
&= (z - z_0) \int_{\mathbb{R}} G_\omega(s,r,z) \int_{\mathbb{R}} G_\omega(r,t,z_0) f(t)\, dt\, dr \\
&= (z - z_0) \int_{\mathbb{R}} \int_{\mathbb{R}} G_\omega(s,r,z) G_\omega(r,t,z_0)\, dr f(t)\, dt.
\end{aligned}$$

Hence, by continuity of $G_\omega(\cdot,\cdot,z)$

$$G_\omega(s,t,z) - G_\omega(s,t,z_0) = (z - z_0) \int_{\mathbb{R}} G_\omega(s,r,z) G_\omega(r,t,z_0)\, dr \quad (s,t \in \mathbb{R}).$$

We set $s = t$ and differentiate with respect to z. This yields

$$\frac{d}{dz} G_\omega(t,t,z) = \int_{\mathbb{R}} G_\omega(t,r,z) G_\omega(r,t,z_0)\, dr + (z - z_0) \frac{d}{dz} \int_{\mathbb{R}} G_\omega(t,r,z) G_\omega(r,t,z_0)\, dr.$$

Setting $z_0 = z$ and using $G_\omega(r,t,z) = G_\omega(t,r,z)$, we arrive at

$$\frac{d}{dz} G_\omega(t,t,z) = \int_{\mathbb{R}} G_\omega(t,s,z)^2\, ds.$$

Therefore, we obtain

$$\begin{aligned}
\frac{d}{dz} \frac{1}{2G_\omega(t,t,z)} &= -\frac{1}{2G_\omega(t,t,z)^2} \int_{\mathbb{R}} G_\omega(t,s,z)^2\, ds \\
&= -\frac{1}{2} \frac{1}{u_+(t,z)(\omega)^2} \int_t^\infty u_+(r,z)(\omega)^2\, dr - \frac{1}{2} \frac{1}{u_-(t,z)(\omega)^2} \int_{-\infty}^t u_-(r,z)(\omega)^2\, dr.
\end{aligned}$$

Hence, the assertion follows. //

6.3.5 Lemma (compare [31, Lemma 1.2]). *Let Ω be atomless and $K \subseteq \mathbb{C}^+$ be compact. Then there exist $C_1, C_2 > 0$ such that for all $z \in K$, $\omega \in \Omega$ we have*

$$C_1 \leq |m_\pm(z)(\omega)|, \ |\operatorname{Im} m_\pm(z)(\omega)|, \ |G_\omega(0,0,z)| \leq C_2.$$

Proof. By [47, Lemma 1] we observe that $m_\pm \colon K \times \Omega \to \mathbb{C}$ is continuous. Indeed, let $(z_k, \omega_k) \in K \times \Omega$, $(z_k, \omega_k) \to (z, \omega)$. Then

$$\begin{aligned}
&|m_\pm(z_k)(\omega_k) - m_\pm(z)(\omega)| \\
&\leq |m_\pm(z_k)(\omega_k) - m_\pm(z_k)(\omega)| + |m_\pm(z_k)(\omega) - m_\pm(z)(\omega)| \\
&\leq \left(\sup_{z \in K} |m_\pm(z)(\omega_k) - m_\pm(z)(\omega)| \right) + |m_\pm(z_k)(\omega) - m_\pm(z)(\omega)|.
\end{aligned}$$

The first term converges to 0 by [47, Lemma 1], the second one tends to 0 since $m_\pm(\cdot)(\omega)$ is continuous.

Since $K \times \Omega$ is compact there exist $C_2 \geq 0$ such that

$$|m_\pm(z)(\omega)| \leq C_2 \quad (z \in K, \omega \in \Omega).$$

Since also $\operatorname{Im} m_\pm \colon K \times \Omega \to \mathbb{C}$ is continuous there exists $C_1 \geq 0$ such that

$$C_1 \leq |\operatorname{Im} m_\pm(z)(\omega)| \quad (z \in K, \omega \in \Omega).$$

We show that $C_1 > 0$. Assume the contrary, then there exists (z_k, ω_k) in $K \times \Omega$ such that $\operatorname{Im} m_+(z_k, \omega_k) \to 0$. By compactness of $K \times \Omega$ there exists a convergent subsequence with limit $(z, \omega) \in K \times \Omega$. Continuity implies $\operatorname{Im} m_+(z)(\omega) = 0$. This yieds a contradiction to Proposition 6.3.3 as $\operatorname{Im} m_+(z)(\omega) > 0$. Similar reasoning holds true for $\operatorname{Im} m_-$.

Thus,

$$0 < C_1 \leq |\operatorname{Im} m_\pm(z)(\omega)| \leq |m_\pm(z)(\omega)| \leq C_2 \quad (z \in K, \omega \in \Omega).$$

Now,

$$|G_\omega(0,0,z)| = \frac{1}{|m_+(z)(\omega) + m_-(z)(\omega)|} \geq \frac{1}{2C_2}$$

and

$$|G_\omega(0,0,z)| = \frac{1}{|m_+(z)(\omega) + m_-(z)(\omega)|} \leq \frac{1}{\operatorname{Im} m_+(z)(\omega)} \leq \frac{1}{C_1}$$

imply the assertion. //

Let $(\Omega, \alpha, \mathbb{P})$ be ergodic. Define a function w on $\mathbb{C} \setminus \mathbb{R}$ by

$$w(z) := \frac{1}{2}\mathbb{E}(m_+(z) + m_-(z)) = -\frac{1}{2}\mathbb{E}\left(\frac{1}{G_{(\cdot)}(0,0,z)}\right) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$

To prove that w' is a Herglotz function we need to interchange differentiation and integration. The following remark is a consequence of Lebesgue's dominated convergence theorem and the mean value inequality.

6.3.6 Remark. Let $U \subseteq \mathbb{C}$ be open, $(\Omega, \mathbb{P})$ be a measure space, $f \colon U \times \Omega \to \mathbb{C}$ such that $f(z, \cdot) \in L_1(\mathbb{P})$ for all $z \in U$. Define

$$F(z) := \int_\Omega f(z, \omega)\, d\mathbb{P}(\omega) \quad (z \in U).$$

Assume that $f(\cdot, \omega)$ is holomorphic for all $\omega \in \Omega$ and

$$\left|\frac{\partial}{\partial z} f(z, \cdot)\right| \leq g \in L_1(\mathbb{P}).$$

Then F is holomorphic and

$$F'(z) = \int_\Omega \frac{\partial}{\partial z} f(z, \omega)\, d\mathbb{P}(\omega).$$

For the rest of this section let Ω be atomless.

6.3.7 Proposition (compare [9, Lemma VII.1.9]). *Let $(\Omega, \alpha, \mathbb{P})$ be ergodic and atomless. Then w is a Herglotz function, $w'(z) = \mathbb{E}(G_{(\cdot)}(0,0,z)) = \int G_\omega(0,0,z)\,d\mathbb{P}(\omega)$ $(z \in \mathbb{C} \setminus \mathbb{R})$ and w' is again a Herglotz function.*

Proof. Let $z \in \mathbb{C}^+$.

(i) By Lemma 6.3.5, $m_\pm(z) \in L_1(\mathbb{P})$ and also

$$\log G_{(\cdot)}(0,0,z) = \ln\left|G_{(\cdot)}(0,0,z)\right| + i \arg G_{(\cdot)}(0,0,z) \in L_1(\mathbb{P}).$$

(i) We have $w(z) = \frac{1}{2}\mathbb{E}(m_+(z) + m_-(z))$ $(z \in \mathbb{C}^+)$. Lemma 6.3.4 yields that $f_+(t,z,\omega) - f_-(t,z,\omega) = \frac{d}{dt}\log G_{\alpha_t(\omega)}(0,0,z)$. Hence, for $a < b$, we have

$$\int_a^b (f_+(t,z,\omega) - f_-(t,z,\omega))\,dt = \log G_{\alpha_b(\omega)}(0,0,z) - \log G_{\alpha_a(\omega)}(0,0,z).$$

Integration with respect to $\mathbb{P}$ and using the definition of $f_\pm(\cdot,z,\omega)$ yields

$$\begin{aligned}&\int_\Omega \int_a^b m_+(z)(\alpha_t(\omega)) - m_-(z)(\alpha_t(\omega))\,dt\,d\mathbb{P}(\omega)\\ &= \int_\Omega \log G_{\alpha_b(\omega)}(0,0,z) - \log G_{\alpha_a(\omega)}(0,0,z)\,d\mathbb{P}(\omega) = 0.\end{aligned}$$

Fubini's Theorem allows to interchange the integrations. Invariance of $\mathbb{P}$ yields

$$(b-a)\,(\mathbb{E}(m_+(z)) - \mathbb{E}(m_-(z))) = \int_a^b \mathbb{E}(m_+(z)) - \mathbb{E}(m_-(z))\,dt = 0.$$

Thus, $\mathbb{E}(m_+(z)) = \mathbb{E}(m_-(z))$ and $w(z) = \mathbb{E}(m_+(z)) = \mathbb{E}(m_-(z))$.

Since m_+, m_- have positive imaginary parts, also $\operatorname{Im} w(z) \geq 0$ for $z \in \mathbb{C}^+$.

(ii) By Lemma 6.3.4 we have

$$-\frac{d}{dz}\frac{1}{2G_\omega(t,t,z)} = G_\omega(t,t,z) + \frac{d}{dt}h(\alpha_t(\omega)).$$

Hence, for $a < b$, we have

$$\int_a^b -\frac{d}{dz}\frac{1}{2G_\omega(t,t,z)}\,dt = \int_a^b G_\omega(t,t,z)\,dt + h(\alpha_b(\omega)) - h(\alpha_a(\omega)).$$

The right-hand side is integrable with respect to $\mathbb{P}$. Also, by Lemma 6.3.4, the integrand on the left-hand side is $\mathbb{P}$-integrable. Integration with respect to $\mathbb{P}$ yields

$$\int_\Omega \int_a^b -\frac{d}{dz}\frac{1}{2G_\omega(t,t,z)}\,dt\,d\mathbb{P}(\omega) = \int_\Omega \int_a^b G_\omega(t,t,z)\,dt\,d\mathbb{P}(\omega).$$

Fubini's Theorem and again invariance of $\mathbb{P}$ imply

$$\begin{aligned}(b-a)\mathbb{E}\left(-\frac{d}{dz}\frac{1}{2G_{(\cdot)}(0,0,z)}\right) &= \int_a^b \mathbb{E}\left(-\frac{d}{dz}\frac{1}{2G_{(\cdot)}(t,t,z)}\right) dt \\ &= \int_a^b \mathbb{E}\left(G_{(\cdot)}(t,t,z)\right) dt \\ &= (b-a)\mathbb{E}\left(G_{(\cdot)}(0,0,z)\right).\end{aligned}$$

Now, we would like to interchange differentiation and integration.

By the very last line of the proof of Lemma 6.3.4 we have

$$\frac{d}{dz}\frac{1}{2G_\omega(0,0,z)} = -\frac{1}{2}\int_0^\infty u_+(r,z)(\omega)^2\,dr - \frac{1}{2}\int_{-\infty}^0 u_-(r,z)(\omega)^2\,dr.$$

By Lemma 6.3.5 and Proposition 6.3.3 we have

$$\sup_{z\in K}\sup_{\omega\in\Omega}\|u_\pm(\cdot,z)(\omega)\|_{L_2}^2 < \infty.$$

Hence,

$$\sup_{z\in K}\sup_{\omega\in\Omega}\left|\frac{d}{dz}\frac{1}{2G_\omega(0,0,z)}\right| < \infty.$$

Therefore, Remark 6.3.6 yields

$$w'(z) = \frac{d}{dz}\mathbb{E}\left(-\frac{1}{2G_{(\cdot)}(0,0,z)}\right) = \mathbb{E}\left(-\frac{d}{dz}\frac{1}{2G_{(\cdot)}(0,0,z)}\right) = \mathbb{E}(G_{(\cdot)}(0,0,z)).$$

Since $G_\omega(0,0,\cdot)$ is analytic (since the resolvent is analytic) for all $\omega\in\Omega$ and, furthermore,

$$\frac{d}{dz}G_\omega(0,0,z) = \int_{\mathbb{R}} G_\omega(0,s,z)^2\,ds,$$

we estimate

$$\begin{aligned}\left|\frac{d}{dz}G_\omega(0,0,z)\right| &\le \int_{\mathbb{R}} |G_\omega(0,s,z)|^2\,ds \\ &\le \frac{1}{|W(u_+(\cdot,z)(\omega),u_-(\cdot,z)(\omega))|^2}\left(\|u_-(\cdot,z)(\omega)\|_{L_2}^2 + \|u_+(\cdot,z)(\omega)\|_{L_2}^2\right).\end{aligned}$$

Since the Wronskian and the norms of u_+ and u_- are uniformly bounded in z and ω by Lemma 6.3.5 and Proposition 6.3.3, we arrive at

$$\sup_{z\in K}\sup_{\omega\in\Omega}\left|\frac{d}{dz}G_\omega(0,0,z)\right| < \infty.$$

Hence, w' is holomorphic. //

6.3.8 Remark. The function w—in a sense—encodes the asymptotic behaviour of $u_\pm$ at $\pm\infty$, i.e.,

$$u_\pm(t,z)(\omega) \sim e^{w(z)t}$$

for $t \to \pm\infty$. Thus, the real part $\operatorname{Re} w$ of w should describe the exponential decay rate. Our next aim is to prove a similar relation between the w-function and the Lyapunov exponent γ.

6.3.9 Lemma (compare [9, Lemma VII.1.10])**.** *Let* $(\Omega, \alpha, \mathbb{P})$ *be ergodic and atomless. Then*

$$\mathbb{E}\left(\frac{1}{\operatorname{Im} m_\pm(z)}\right) = -\frac{2}{\operatorname{Im} z} \operatorname{Re} w(z) \quad (z \in \mathbb{C}^+).$$

In particular, $\operatorname{Re} w(z) < 0$ *(*$z \in \mathbb{C}^+$*).*

Proof. Taking imaginary parts in the Ricatti equation for $f_+(\cdot, z, w)$ yields

$$\operatorname{Im} f_+'(\cdot,z,w) = -\operatorname{Im} z - 2\operatorname{Re} f_+(\cdot,z,w)\operatorname{Im} f_+(\cdot,z,w)$$

in the sense of distributions. Since the right-hand side is a continuous function, also the left-hand side is continuous and, therefore, both functions are equal. Hence, for $t \in \mathbb{R}$,

$$\frac{d}{dt}\ln\operatorname{Im} f_+(t,z,\omega) = \frac{\operatorname{Im} f_+'(t,z,\omega)}{\operatorname{Im} f_+(t,z,\omega)} = -\frac{\operatorname{Im} z}{\operatorname{Im} f_+(t,z,\omega)} - 2\operatorname{Re} f_+(t,z,\omega),$$

i.e.,

$$\frac{d}{dt}\ln\operatorname{Im} m_+(z)(\alpha_t(\omega)) + \frac{\operatorname{Im} z}{\operatorname{Im} m_+(z)(\alpha_t(\omega))} = -2\operatorname{Re} m_+(z)(\alpha_t(\omega)).$$

Integration over an interval (a,b), then with respect to $\mathbb{P}$ and Fubini's Theorem yield

$$(b-a)\mathbb{E}\left(\frac{\operatorname{Im} z}{\operatorname{Im} m_+(z)}\right) = (b-a)\mathbb{E}\left(-2\operatorname{Re} m_+(z)\right).$$

Since the left-hand side is positive, $\operatorname{Re} w(z) < 0$. The equation for m_- is proven analogously. //

We are now in the position to connect the real part of w with the Lyapunov exponent γ. There is also a nice connection between the imaginary part of w and the integrated density of states of (H_ω).

6.3.10 Remark. Let $(\Omega, \alpha, \mathbb{P})$ be ergodic. For $\omega \in \Omega$ and $l > 0$ let $H_\omega|_{[-l,l]}$ denote the restriction of H_ω to $[-l,l]$ (i.e., to $L_2([-l,l])$) with Dirichlet boundary conditions at $\pm l$. Then $H_\omega|_{[-l,l]}$ is self-adjoint and has purely discrete spectrum (see [6]).

Let $(E_j(l,\omega))_{j\in\mathbb{N}}$ be the nondecreasing sequence of eigenvalues of $H_\omega|_{[-l,l]}$. For $E \in \mathbb{R}$ define

$$N_\omega(E,l) := \frac{1}{2l}\left|\{E_j(l,\omega);\ E_j(l,\omega) \le E\}\right| = \frac{1}{2l}\operatorname{Tr} E_\omega|_{[-l,l]}(E),$$

where $E_\omega|_{[-l,l]}$ is the resolution of identity of $H_\omega|_{[-l,l]}$. It is well-known that

$$N(E) := \mathbb{E}\left(\lim_{l\to\infty} N_{(\cdot)}(E,l)\right) = \lim_{l\to\infty} \mathbb{E}\left(N_{(\cdot)}(E,l)\right)$$

exists for every $E \in \mathbb{R}$. The function N is called the *integrated density of states* for $(H_\omega)_{\omega\in\Omega}$.

6.3.11 Proposition (compare [9, Proposition VII.1.11]). *Let $(\Omega, \alpha, \mathbb{P})$ be ergodic and atomless. Then*

$$N(E) = \frac{1}{\pi} \operatorname{Im} w(E + i0+), \quad \gamma(E) = -\operatorname{Re} w(E + i0+) \quad (E \in \mathbb{R}),$$

where N is the integrated density of states and γ is the Lyapunov exponent. Moreover, there is $a \in \mathbb{R}$ such that

$$\gamma(E) = a + \int_{\mathbb{R}} \ln\left|\frac{E-t}{t-i}\right| dN(t).$$

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Note that we have the Herglotz representation

$$G_\omega(0,0,z) = \int_{\mathbb{R}} \frac{1}{t-z}\, d\varrho_\omega(t).$$

Proposition 6.3.7 yields

$$w'(z) = \mathbb{E}(G_{(\cdot)}(0,0,z)) = \mathbb{E}\left(\int_{\mathbb{R}} \frac{1}{t-z}\, d\varrho_{(\cdot)}(t)\right) = \int_{\mathbb{R}} \frac{1}{t-z}\, d\varrho(t),$$

where $\varrho(A) = \mathbb{E}\left(\varrho_{(\cdot)}(A)\right)$ for $A \subseteq \mathbb{R}$ measurable.

Note that we also have $\varrho(A) = \mathbb{E}(E_{(\cdot)}(0,0,A))$, where $E_\omega(0,0,\cdot)$ is the kernel element of the resolution of the identity of H_ω. Thus, the distribution function of ϱ is the integrated density of states N of (H_ω), see also [43, 42].

Integration by parts yields

$$w'(z) = \int_{\mathbb{R}} \frac{N(t)}{(t-z)^2}\, dt.$$

Integrating both sides, we obtain

$$w(z) = a + \int_{\mathbb{R}} \frac{(1+tz)N(t)}{(t-z)(1+t^2)}\, dt,$$

for some $a \in \mathbb{C}$. Since $w(z) = -\frac{1}{2}\mathbb{E}(G_{(\cdot)}(0,0,z)^{-1})$, we have $w(\overline{z}) = \overline{w(z)}$ and therefore $\operatorname{Im} a = 0$. For $E \in \mathbb{R}$, $\varepsilon > 0$ and $z = E + i\varepsilon$ we obtain

$$\operatorname{Im} w(E + i\varepsilon) = \varepsilon \int_{\mathbb{R}} \frac{N(t)}{(E-t)^2 + \varepsilon^2}\, dt = \int_{\mathbb{R}} \frac{N(E + \varepsilon u)}{1+u^2}\, du.$$

A combination of [16, Corollary 8.4] and [43, Theorem 8] yields that any given $E \in \mathbb{R}$ is $\mathbb{P}$-a.s. not an eigenvalue of H_ω. Hence, N is continuous. We conclude that

$$\pi N(E) = N(E) \int_{\mathbb{R}} \frac{1}{1+u^2}\, du = \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} \frac{N(E+\varepsilon u)}{1+u^2}\, du = \lim_{\varepsilon \to 0+} \operatorname{Im} w(E+i\varepsilon).$$

Furthermore,

$$-\operatorname{Re} w(z) = -\operatorname{Re} a - \int_{\mathbb{R}} \operatorname{Re}\left(\frac{1+tz}{(t-z)(t^2+1)}\right) N(t)\, dt = -\operatorname{Re} a + \int_{\mathbb{R}} \ln\left|\frac{t-z}{t-i}\right| d\varrho(t),$$

by integration by parts. Lemma 6.3.9 yields

$$-\int_{\mathbb{R}} \ln\left|\frac{t-z}{t-i}\right| d\varrho(t) < -\operatorname{Re} a.$$

Let $\varepsilon \in (0,1)$ and put $z = E + i\varepsilon$. Then a similar reasoning as in [31] shows that we can write

$$-\operatorname{Re} w(z) = -\operatorname{Re} a + \int_{\mathbb{R}} \ln\left|\frac{t-E-i\varepsilon}{t-E}\right| d\varrho(t) + \int_{\mathbb{R}} \ln\left|\frac{t-E}{t-i}\right| d\varrho(t).$$

Now, as $\varepsilon \to 0+$, the second term converges monotonically to 0. Hence,

$$-\operatorname{Re} w(E+i0+) = -\operatorname{Re} a + \int_{\mathbb{R}} \ln\left|\frac{t-E}{t-i}\right| d\varrho(t).$$

Note that $z \mapsto \gamma(z)$ and $z \mapsto -\operatorname{Re} w(z)$ are subharmonic on $\mathbb{C}$ (for γ this follows from Lemma 1.4.1 and Lemma 5.4.6, for $-\operatorname{Re} w$ this follows from the monotone convergence above). We compute

$$\frac{1}{t} \ln|u_\pm(t,z)(\omega)| = \pm \operatorname{Re} \frac{1}{t} \int_0^t m_\pm(z)(\alpha_s(\omega))\, ds.$$

Taking expectations, we obtain

$$\frac{1}{t}\mathbb{E}(\ln|u_\pm(t,z)|) = \pm \operatorname{Re} \frac{1}{t} \int_0^t w(z)\, ds = \pm \operatorname{Re} w(z).$$

Thus, as $u_\pm'(t+,z)$ is a multiple (independent of t) of $u_\pm(t,z)$ and u_+ and u_- are linearly independent,

$$\gamma(z) = \inf_{t>0} \frac{1}{t}\mathbb{E}(\ln\|T_z(t,\cdot)\|) = -\operatorname{Re} w(z).$$

Since this equality holds true for all $z \in \mathbb{C} \setminus \mathbb{R}$ we obtain $\gamma = -\operatorname{Re} w$ on $\mathbb{C}$ by [9, Lemma V.4.4] which finishes the proof. //

6.3.12 Remark. The measure ϱ is the *spectral measure* for $(H_\omega)_{\omega\in\Omega}$, whereas the measures ϱ_ω are the spectral measures for H_ω ($\omega \in \Omega$). Also, $\operatorname{Im} w(\cdot + i0+)$ is sometimes called *rotation number* of (H_ω).

6.3.13 Lemma (compare [31, Lemma 4.1])**.** *Let* $(\Omega, \alpha, \mathbb{P})$ *be ergodic and atomless,* $K \subseteq \mathbb{R}$ *be compact with* $\lambda(K) > 0$. *Suppose that* $\gamma(E) = -\operatorname{Re} w(E + i0+) = 0$ *for* λ*-a.a.* $E \in K$. *Then*

$$-\lim_{\varepsilon\to 0+} \int_K \frac{\operatorname{Re} w(E + i\varepsilon)}{\varepsilon}\, dE = \int_K \pi N_{ac}(E)\, dE.$$

Here, N_{ac} *is the density of* ϱ_{ac}.

Proof. Let $v(x, y) := \operatorname{Re} w(x + iy)$ for $x + iy \in \mathbb{C}^+$. The Cauchy-Riemann equations yield

$$w'(x + iy) = -\frac{\partial v}{\partial x}(x, y) + i\frac{\partial v}{\partial y}(x, y).$$

Since w' is a Herglotz function, for λ-a.e. $x \in \mathbb{R}$ the limit $\operatorname{Im} w'(x + i0+)$ exists and we have

$$\operatorname{Im} w'(x + i0+) = \frac{d\varrho_{ac}}{d\lambda}(x).$$

Let $E \in K$ such that $N_{ac}(E) = \frac{1}{\pi}\frac{\partial v}{\partial y}(E, 0+)$ exists and $-v(E, 0+) = \gamma(E) = 0$. Then

$$-\frac{\operatorname{Re} w(E + i\varepsilon)}{\varepsilon} = \frac{v(x, \varepsilon) - v(x, 0+)}{\varepsilon} \to \frac{\partial v}{\partial y}(E, 0+) = \pi N_{ac}(E).$$

Furthermore,

$$-\operatorname{Re} w(E + i\varepsilon) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\varepsilon}{(E - t)^2 + \varepsilon^2}\gamma(t)\, dt.$$

Hence, $\left(-\frac{\operatorname{Re} w(E+i\varepsilon)}{\varepsilon}\right)$ converges monotonically to $\pi N_{ac}(E)$ and by the monotone convergence theorem,

$$\int_K -\frac{\operatorname{Re} w(E + i\varepsilon)}{\varepsilon}\, dE \to \int_K \pi N_{ac}(E)\, dE. \qquad //$$

6.4. Kotani theory

In this section we generalize the Ishii-Pastur-Kotani theorem to the case of measure-perturbed Schrödinger operators. We start with the Ishii-Pastur theorem in its general form for measures.

6.4.1 Theorem (compare [9, Proposition VII.3.1])**.** *Let* $(\Omega, \alpha, \mathbb{P})$ *be ergodic and* m *be a positive Borel measure on* $\mathbb{R}$ *such that the Lyapunov exponent* γ *is strictly positive* m*-a.e. Then* ϱ_ω *is orthogonal to* m *for* $\mathbb{P}$*-a.a.* $\omega \in \Omega$, *where* ϱ_ω *is the spectral measure of* H_ω.

Proof. Define

$$W := \{(\omega, E) \in \Omega \times \mathbb{R};\ E \in \operatorname{hyp}(H_\omega)\}\,.$$

The mapping $E \mapsto T_E(t,\omega)$ ($t \in \mathbb{R}$, $\omega \in \Omega$) is continuous by Lemma 4.3.3 and the mapping $\omega \mapsto T_E(t,\omega)$ is measurable ($E \in \mathbb{R}$, $t \in \mathbb{R}$) by Lemma 4.4.3. Hence, W is measurable for the product-σ-algebra $\mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{R})$ by [22, Theorem 2].

Let $A := \{E \in \mathbb{R};\ \gamma(E) > 0\}$. By the assumption we have $m(A) = m(\mathbb{R})$, i.e., A has full m-measure. For $E \in A$ consider the process $(T_E(t,\cdot))_{t\in\mathbb{R}}$. By Oseledec's Theorem there exists $\Omega_{0,E}$ of full $\mathbb{P}$-measure, such that $\gamma(E)$ exists for all $\omega \in \Omega_{0,E}$. Since $E \in A$, $\gamma(E)$ must be positive. Hence, $(\omega, E) \in W$ for all $\omega \in \Omega_{0,E}$, i.e., $W_E \supseteq \Omega_{0,E}$, where W_E is the section of W for fixed E. Hence, $\mathbb{P}(W_E) = 1$ for all $E \in A$.

Now, we show that the measure m is supported by $\operatorname{hyp}(H_\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$. For $E \in A$ we have $\mathbb{1}_{\complement \operatorname{hyp}(H_\omega)}(E) = 0$ for all $\omega \in W_E$. Hence, $\mathbb{1}_{\complement \operatorname{hyp}(H_\omega)}(E) = 0$ $\mathbb{P}$-a.s. and therefore

$$\int\limits_\Omega \mathbb{1}_{\complement \operatorname{hyp}(H_\omega)}(E)\, d\mathbb{P}(\omega) = 0,$$

for all $E \in A$, i.e., m-almost everywhere. Hence,

$$\int\limits_{\mathbb{R}} \int\limits_\Omega \mathbb{1}_{\complement \operatorname{hyp}(H_\omega)}(E)\, d\mathbb{P}(\omega)\, dm(E) = 0.$$

By Fubini's Theorem,

$$0 = \int\limits_{\mathbb{R}} \int\limits_\Omega \mathbb{1}_{\complement \operatorname{hyp}(H_\omega)}(E)\, d\mathbb{P}(\omega)\, dm(E) = \int\limits_\Omega m(\complement \operatorname{hyp}(H_\omega))\, d\mathbb{P}(\omega).$$

Since m is a positive measure, the integrand $m(\complement \operatorname{hyp}(H_\omega))$ must be equal to 0 for $\mathbb{P}$-almost all $\omega \in \Omega$. This means that m is supported by $\operatorname{hyp}(H_\omega)$ for $\mathbb{P}$-a.a. $\omega \in \Omega$.

We first consider the case that m is continuous, i.e., $m(\{E\}) = 0$ for all $E \in \mathbb{R}$. Theorem 6.2.3 asserts that m is orthogonal to the spectral measure ϱ_ω for all $\omega \in \Omega$ such that m is carried by $\operatorname{hyp}(H_\omega)$, i.e., $\mathbb{P}$-a.s.

In the general case, let $E \in \mathbb{R}$ such that $m(\{E\}) > 0$. Then $E \in \operatorname{hyp}(H_\omega)$ $\mathbb{P}$-a.s. and, therefore, [9, Proposition IV.2.8] yields

$$\max\left\{ \lim_{t\to-\infty} \frac{1}{|t|} \ln \|T_E(t,\omega)v\|\,, \lim_{t\to\infty} \frac{1}{t} \ln \|T_E(t,\omega)v\| \right\} \geq \gamma(E) \geq 0$$

for all $v \neq 0$. Lemma 6.2.2 implies that E is not an eigenvalue of H_ω for $\mathbb{P}$-a.a. $\omega \in \Omega$ and hence for $\mathbb{P}$-almost all ω we have $\varrho_\omega(\{E\}) = 0$.

Putting these two parts together we obtain: The point measure part m_{pp} is orthogonal to ϱ_ω $\mathbb{P}$-a.s. and the continuous part m_c is orthogonal to ϱ_ω $\mathbb{P}$-a.s. Since $\operatorname{spt} m_{pp}$ is countable, $m = m_c + m_{pp}$ is orthogonal to ϱ_ω $\mathbb{P}$-a.s. //

Now, it is easy to prove the first half of the Ishii-Pastur-Kotani theorem.

Definition. Let $A \subseteq \mathbb{R}$ be measurable. The *essential closure* $\overline{A}^{\mathrm{ess}}$ of A is defined as

$$\overline{A}^{\mathrm{ess}} := \{E \in \mathbb{R};\ \forall\, \varepsilon > 0 : \lambda(A \cap (E - \varepsilon, E + \varepsilon)) > 0\}\,.$$

Recall that if $(\Omega, \alpha, \mathbb{P})$ is ergodic then there exists $\Sigma_{ac} \subseteq \mathbb{R}$ such that $\sigma_{ac}(H_\omega) = \Sigma_{ac}$ for $\mathbb{P}$-a.a. $\omega \in \Omega$.

6.4.2 Theorem. *Let $(\Omega, \alpha, \mathbb{P})$ be ergodic. Then*

$$\Sigma_{ac} \subseteq \overline{\{E \in \mathbb{R};\ \gamma(E) = 0\}}^{ess}.$$

Proof. Let $E \notin \overline{\{E \in \mathbb{R};\ \gamma(E) = 0\}}^{\text{ess}}$. Then there exists $\varepsilon > 0$, such that

$$\lambda\left((E-\varepsilon, E+\varepsilon) \cap \{E \in \mathbb{R};\ \gamma(E) = 0\}\right) = 0.$$

Let $m := \mathbb{1}_{\complement\{E\in\mathbb{R};\ \gamma(E)=0\}}\lambda$. Then $m((E-\varepsilon, E+\varepsilon)) > 0$ and by Theorem 6.4.1 we obtain $\varrho_\omega((E-\varepsilon, E+\varepsilon)) = 0$ for $\mathbb{P}$-a.a. $\omega \in \Omega$. Hence, $E \notin \operatorname{spt} \varrho_{\omega,ac}$ for $\mathbb{P}$-a.a. $\omega \in \Omega$, i.e., $E \notin \Sigma_{ac}$. //

Now, we state Kotani's result.

6.4.3 Theorem (compare [9, Proposition VII.3.3])**.** *Let $(\Omega, \alpha, \mathbb{P})$ be ergodic and atomless, $I \subseteq \mathbb{R}$ measurable such that $\gamma(E) = 0$ for λ-almost all $E \in I$. Then there exists Ω_0 of full $\mathbb{P}$-measure such that for $\omega \in \Omega_0$ and λ-a.e. $E \in I$ we have*

$$\frac{d\varrho_{\omega,ac}}{d\lambda}(E) > 0,$$

and

$$m_+(E + i0+)(\omega) = -\overline{m_-(E + i0+)(\omega)}.$$

Proof. Let $K \subseteq I$ be compact with $\lambda(K) > 0$, $\varepsilon > 0$. Then by Lemma 6.3.9 and Tonelli's Theorem,

$$\mathbb{E}\left(\int_K \frac{1}{\operatorname{Im} m_\pm(E + i\varepsilon)}\, dE\right) = -\int_K \frac{2 \operatorname{Re} w(E + i\varepsilon)}{\varepsilon}\, dE.$$

By Lemma 6.3.13 the right-hand side converges to $2\int_K \pi N_{ac}(E)\, dE$. Fatou's Lemma yields

$$\mathbb{E}\left(\int_K \frac{1}{\operatorname{Im} m_\pm(E + i0+)}\, dE\right) \le 2\int_K \pi N_{ac}(E)\, dE.$$

Since the left-hand side is finite, $\operatorname{Im} m_\pm(E + i0+)(\omega) > 0$ for λ-a.a. $E \in K$ with probability 1. By definition of the kernel of the resolvent we have

$$\operatorname{Im} G_\omega(0, 0, E + i\varepsilon) = \frac{\operatorname{Im} m_+(E + i\varepsilon)(\omega) + \operatorname{Im} m_-(E + i\varepsilon)(\omega)}{|m_+(E + i\varepsilon)(\omega) + m_-(E + i\varepsilon)(\omega)|^2}.$$

Hence,

$$\operatorname{Im} G_\omega(0, 0, E + i0+) > 0.$$

Since $G_\omega(0,0,\cdot)$ is a Herglotz function for a.e. $E \in K$ the limit $\operatorname{Im} G_\omega(0,0,E+i0+)$ exists and is finite. Hence (see also Section A.6),

$$\frac{d\varrho_{\omega,ac}}{d\lambda}(E) > 0.$$

Since K was arbitrary and I is σ-compact, there exists Ω_0 of full $\mathbb{P}$-measure such that for λ-a.e. $E \in I$ we have

$$\frac{d\varrho_{\omega,ac}}{d\lambda}(E) > 0.$$

For the second part, note that

$$\begin{aligned}
&-\frac{\operatorname{Re} w(E+i\varepsilon)}{\varepsilon} - \operatorname{Im} w'(E+i\varepsilon) \\
&= \mathbb{E}\Bigg(\Bigg(\frac{1}{\operatorname{Im} m_+(E+i\varepsilon)} + \frac{1}{\operatorname{Im} m_-(E+i\varepsilon)}\Bigg) \\
&\quad \frac{(\operatorname{Re} m_+(E+i\varepsilon) + \operatorname{Re} m_-(E+i\varepsilon))^2 + (\operatorname{Im} m_+(E+i\varepsilon) - \operatorname{Im} m_-(E+i\varepsilon))^2}{|m_+(E+i\varepsilon) + m_-(E+i\varepsilon)|^2}\Bigg).
\end{aligned}$$

Hence, by Fatou's Lemma, for λ-a.a. $E \in I$ we have

$$\begin{aligned}
0 = \mathbb{E}\Bigg(\Bigg(&\frac{1}{\operatorname{Im} m_+(E+i0+)} + \frac{1}{\operatorname{Im} m_-(E+i0+)}\Bigg) \\
&\Bigg(\frac{(\operatorname{Re} m_+(E+i0+) + \operatorname{Re} m_-(E+i0+))^2}{|m_+(E+i0+) + m_-(E+i0+)|^2} \\
&\quad + \frac{(\operatorname{Im} m_+(E+i0+) - \operatorname{Im} m_-(E+i0+))^2}{|m_+(E+i0+) + m_-(E+i0+)|^2}\Bigg)\Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{Re} m_+(E+i0+)(\omega) &= -\operatorname{Re} m_-(E+i0+)(\omega), \\
\operatorname{Im} m_+(E+i0+)(\omega) &= \operatorname{Im} m_-(E+i0+)(\omega)
\end{aligned}$$

for $\mathbb{P}$-a.a. $\omega \in \Omega$. //

The second claim in the theorem yields that for $\mathbb{P}$-a.a. $\omega \in \Omega$ the potential is reflectionless on I; cf. [47].

6.4.4 Corollary. *Let* $(\Omega, \alpha, \mathbb{P})$ *be ergodic and atomless. Then*

$$\Sigma_{ac} = \overline{\{E \in \mathbb{R};\ \gamma(E) = 0\}}^{ess}.$$

Proof. By Theorem 6.4.2 we have $\Sigma_{ac} \subseteq \overline{\{E \in \mathbb{R};\ \gamma(E) = 0\}}^{\mathrm{ess}}$. Conversely, let $E \in \overline{\{E \in \mathbb{R};\ \gamma(E) = 0\}}^{\mathrm{ess}}$. By Theorem 6.4.3, $E \in \operatorname{spt} \varrho_{\omega,ac}$ for $\mathbb{P}$-a.a. $\omega \in \Omega$. Hence, $\overline{\{E \in \mathbb{R};\ \gamma(E) = 0\}}^{\mathrm{ess}} \subseteq \Sigma_{ac}$. //

6.5. Measure dynamical systems

This section collects and combines the results obtained in the previous parts of this thesis. We will prove Cantor spectra for models where all transfer matrices to all energies are uniform, and also almost surely purely singular continuous spectra for operator families.

Then we describe a device to construct suitable families of operators by means of subshifts over finite alphabets, for which the theorems in this section can be applied.

6.5.1 Theorem (compare [29, Theorem 5.1]). *Let $(\Omega, \alpha, \mathbb{P})$ be ergodic and minimal having the s.f.d.p. (i.e. for every $\omega \in \Omega$: ω and $\omega(-(\cdot))$ has s.f.d.p., see Chapter 3) and assume that there exists $\omega \in \Omega$ which is not periodic. Then $\Sigma_{ac} = \varnothing$, where Σ_{ac} is the $\mathbb{P}$-a.s. constant absolutely continuous spectrum of (H_ω).*

Proof. Assume that $\{\omega \in \Omega;\ \sigma_{ac}(H_\omega) \neq \varnothing\}$ has positive $\mathbb{P}$-measure. By Theorem 3.2.2, the set $\{\omega \in \Omega;\ \omega \text{ or } \omega(-(\cdot)) \text{ is eventually periodic}\}$ has positive $\mathbb{P}$-measure. W.l.o.g. assume that ω is periodic for $t \geq t_0$ with period p. By closedness of Ω,

$$\tilde{\omega} := \lim_{t\to\infty} \alpha_t(\omega) = \lim_{t\to\infty} \omega(\cdot + t) \in \Omega$$

and $\tilde{\omega}$ is periodic with period p. For $\omega' \in \Omega$ there exists (t_n) in $\mathbb{R}$ such that $\alpha_{t_n}(\tilde{\omega}) \to \omega'$. Since $\tilde{\omega}$ is periodic and α is continuous, we arrive at

$$\alpha_p(\omega') = \alpha_p\left(\lim_{n\to\infty} \alpha_{t_n}(\tilde{\omega})\right) = \lim_{n\to\infty} \alpha_{t_n}\alpha_p(\tilde{\omega}) = \omega'.$$

So, every $\omega \in \Omega$ must be periodic with the same period, a contradiction. //

We say that a dynamical system (Ω, α) with ergodic measure $\mathbb{P}$ satisfies condition (K) if there exists (p_n) in $(0, \infty)$ with $p_n \to \infty$ such that

$$G_n := \left\{\omega \in \Omega;\ \mathbb{1}_{[0,p_n]}\omega = \mathbb{1}_{[0,p_n]}\alpha_{p_n}(\omega) = \mathbb{1}_{[0,p_n]}\alpha_{-p_n}(\omega)\right\} \quad (n \in \mathbb{N})$$

satisfies

$$\limsup_{n\to\infty} \mathbb{P}(G_n) > 0.$$

This condition is due to Kaminaga (for the discrete case), see [27].

6.5.2 Lemma (compare [29, Lemma 7.3]). *Let $(\Omega, \alpha, \mathbb{P})$ be ergodic satisfying (K). Then for $\mathbb{P}$-a.a. $\omega \in \Omega$ we have $\sigma_{pp}(H_\omega) = \varnothing$, i.e., H_ω does not have any eigenvalues $\mathbb{P}$-a.s.*

Proof. Let

$$\Omega_c := \{\omega \in \Omega;\ \sigma_{pp}(H_\omega) = \varnothing\}.$$

Note that Ω_c is α-invariant. Ergodicity implies $\mathbb{P}(\Omega_c) \in \{0, 1\}$. By Theorem 2.3.3 we have

$$G := \limsup_{n\to\infty} G_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} G_k \subseteq \Omega_c.$$

Hence,

$$\mathbb{P}(\Omega_c) \geq \mathbb{P}(G) = \mathbb{P}(\limsup_{n\to\infty} G_n) \geq \limsup_{n\to\infty} \mathbb{P}(G_n) > 0.$$

We conclude that $\mathbb{P}(\Omega_c) = 1$. //

6.5.3 Theorem. *Let $(\Omega, \alpha, \mathbb{P})$ be strictly ergodic having the s.f.d.p. and satisfying (K), and assume there exists $\omega \in \Omega$ which is not periodic. Then H_ω has purely singular continuous spectrum for $\mathbb{P}$-almost all $\omega \in \Omega$.*

Proof. By Theorem 6.5.1 we obtain $\Sigma_{ac} = \varnothing$. By Lemma 6.5.2, $\Sigma_{pp} = \varnothing$. Hence, for $\mathbb{P}$-a.a. $\omega \in \Omega$, H_ω has purely singular continuous spectrum. //

We now prove Cantor spectra for a large class of operators in case of atomless Ω. We call $C \subseteq \mathbb{R}$ a *Cantor set* if C is closed, nowhere dense and does not contain any isolated points.

6.5.4 Theorem. *Let $(\Omega, \alpha, \mathbb{P})$ be strictly ergodic, atomless, has the s.f.d.p. and assume that there exists $\omega \in \Omega$ which is not periodic. Furthermore, let T_E be uniform for all $E \in \mathbb{R}$. Then*

$$\Sigma = \{E \in \mathbb{R};\ \gamma(E) = 0\}$$

and Σ is a Cantor set of zero Lebesgue measure.

Proof. By Theorem 6.1.4, $\Sigma = \{E \in \mathbb{R};\ \gamma(E) = 0\}$. By Corollary 6.4.4 we have

$$\overline{\{E \in \mathbb{R};\ \gamma(E) = 0\}}^{\text{ess}} = \Sigma_{ac}.$$

Since $\Sigma_{ac} = \varnothing$ by Theorem 6.5.1 we infer $\lambda(\{E \in \mathbb{R};\ \gamma(E) = 0\}) = 0$.

We show that Σ does not contain isolated points. Indeed, assume that $E \in \Sigma$ was such an isolated point in the spectrum. Then E would be an eigenvalue of H_ω for $\mathbb{P}$-a.a. $\omega \in \Omega$. By [16, Corollary 8.4], the multiplicity of E would be 1, so E belongs to the discrete spectrum (the isolated eigenvalues of finite multiplicity) of H_ω for $\mathbb{P}$-a.a. $\omega \in \Omega$. By ergodicity there exists $\Sigma_{disc} \subseteq \mathbb{R}$ such that Σ_{disc} is the discrete spectrum of H_ω for $\mathbb{P}$-a.a. $\omega \in \Omega$, see [9, Remark V.2.5]. But $\Sigma_{disc} = \varnothing$ due to [9, Proposition V.2.8].

So, Σ is closed and every point in Σ is a limit point of Σ. Since $\lambda(\Sigma) = 0$, Σ is nowhere dense. We conclude that Σ is a Cantor set of zero Lebesgue measure. //

A similar theorem on Cantor spectra for Hölder continuous quasi-periodic potentials was stated in [26]. An analogous theorem for the discrete case was proven in [34].

At the end of this section we explain how to construct examples of subsets $\Omega \subseteq \mathcal{M}_{\text{loc,unif}}(\mathbb{R})$ such that the theory presented in this thesis can be applied for $(H_\omega)_{\omega\in\Omega}$.

Let A be a finite set of cardinality N, equipped with the discrete topology. A pair (X, τ) is a *subshift* over A if X is a closed subset of $A^{\mathbb{Z}}$, where $A^{\mathbb{Z}}$ is endowed with the product topology, and X is invariant under the shift $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, $\tau a(n) := a(n+1)$. Let $\nu_1, \ldots, \nu_N \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R})$ with compact support such that $\inf \operatorname{spt} \nu_j = 0$ for all $j \in \{1, \ldots, N\}$. For $j \in \{1, \ldots, N\}$ we define

$$l_j := \begin{cases} \sup \operatorname{spt} \nu_j & \text{if } \operatorname{spt} \nu_j \not\subseteq \{0\}\,, \\ 1 & \text{if } \operatorname{spt} \nu_j \subseteq \{0\}\,. \end{cases}$$

For $x \in X$ we define the measure $\omega_x \in \mathcal{M}_{\text{loc,unif}}(\mathbb{R})$ by

$$\omega_x := \sum_{n\in\mathbb{N}_0} \delta_{\sum_{k=0}^{n-1} l_{x(k)}} * \nu_{x(n)} + \sum_{n\in\mathbb{N}} \delta_{\sum_{k=-n}^{-1} -l_{x(k)}} * \nu_{x(-n)}.$$

Let

$$\Omega := \{\alpha_t(\omega_x);\ x \in X,\ t \in \mathbb{R}\}.$$

6.5.5 Proposition ([29, Proposition 4]). *(a) Assume that at most one of the measures ν_j is a multiple of Lebesgue measure. Then (Ω, α) has the s.f.d.p.*

(b) Any invariant probability measure $\mathbb{P}_X$ on (X, τ) induces a canonical invariant probability measure $\mathbb{P}$ on (Ω, α). If $\mathbb{P}_X$ is ergodic, then $\mathbb{P}$ is ergodic.

(c) If (X, τ) is uniquely ergodic, then (Ω, α) is uniquely ergodic.

(d) If (X, τ) is minimal, then (Ω, α) is minimal.

Condition (K) can also be derived from an analogous condition for subshifts, see [29].

Thus, we can construct various examples which can be treated by the theory developed in this thesis. One only has to construct suitable subshifts and choose certain measures.

6.6. Concluding remarks

There are several comments, remarks and outlooks to be made.

- In Chapter 3 one has to assure that both μ and the reflected measure $\mu(-(\cdot))$ have the s.f.d.p. in order to obtain absence of absolutely continuous spectrum. In [33] it was shown for almost periodic bounded potentials that the half-line operators have the same absolutely continuous spectrum. We are currently working on the corresponding generalization for measure-perturbed Schrödinger operators. This would allow us to get rid of the assumption on the reflected measure (although this is not a strong restriction).

  Also, the constancy of the absolutely continuous part of the spectrum for minimal ergodic models (even for measure perturbed ones) should follow by the methods developed in [33]. However, in our case this has not been worked out yet.

- We would like to prove Theorems 5.1.4 and 5.1.5 also for almost continuous (sub)additive processes (i.e., getting rid of continuity in Ω). Whether this is possible remains an open problem.

- In Lemma 6.3.5 we need to assume that Ω is atomless. If we could drop this assumption we could prove Kotani's Theorem without any further assumption on Ω. However, in case Ω is not atomless the m-functions are not continuous any more and due to this fact it would require a different proof.

- In the theorem concerning the Cantor spectra one requires that the transfer matrices are uniform. In the discrete case Boshernitzans condition is sufficient for uniformity, see for example [35]. As far as we know there is no analogon in the continuum case.

- In Chapter 5 we proved that Λ is continuous at all continuous cocycles $A \in \mathcal{C}$ with $\Lambda(A) = 0$. As we already stated there, in the discrete case continuity of Λ was shown at all uniform continuous cocycles. The generalization to the continuum case would be interesting.

- The definition of Delone measures of finite local complexity in Chapter 3 is not restricted to the one-dimensional case. It might be interesting to prove absence of absolutely continuous spectrum for such potentials in the higher dimensional case.

  This also leads to a much more general aim. We would like to develop a similar theory for the higher dimensional case. Measure perturbations can be dealt with also in higher dimensions if one restricts the class of measures (to absolutely continuous measures with respect to capacity), see [59]. Moreover, we belive that in the minimal case constancy of the spectrum as in Chapter 4 may be proven by the same method for a suitable class of potentials.

  Also, in the discrete case there is a multidimensional version of Gordon's theorem due to Damanik [13]. However, it heavily restricts the class of potentials. Since one also lacks the method of transfer matrices in higher dimensions it seems that one needs to develop new techniques in order to generalize the results to the higher dimensional case.

Appendix A

Appendix

The appendix collects some well-known results which are needed for the thesis.

A.1. Gronwall inequality

We state a measure-version of Gronwall's inequality, see also [18].

A.1 Lemma (Gronwall)**.** Let μ be a locally finite Borel measure on $[0,\infty)$, $u\colon [0,\infty) \to \mathbb{R}$ measurable and locally integrable with respect to μ, $a\colon [0,\infty) \to [0,\infty)$ measurable. Suppose, that

$$u(t) \leq a(t) + \int\limits_{[0,t)} u(s)\, d\mu(s) \quad (t \geq 0).$$

Then

$$u(t) \leq a(t) + \int\limits_{[0,t)} a(s) e^{\mu((s,t))}\, d\mu(s) \quad (t \geq 0).$$

Proof. (i) By induction on $n \in \mathbb{N}_0$ we show

$$u(t) \leq a(t) + \int\limits_{[0,t)} a(s) \sum_{k=0}^{n-1} \mu^{\otimes k}(A_k(s,t))\, d\mu(s) + R_n(t),$$

where

$$R_n(t) := \int\limits_{[0,t)} u(s) \mu^{\otimes n}(A_n(s,t))\, d\mu(s) \quad (t \geq 0)$$

is the remainder and

$$A_n(s,t) = \{(s_1,\ldots,s_n) \in (s,t)^n;\ s_1 < s_2 < \cdots < s_n\} \quad (n \geq 1)$$

is an n-dimensional simplex and $\mu^{\otimes 0}(A_0(s,t)) := 1$.

For $n = 0$ the inequality is just the assumption. For the step from n to $n+1$ inserting the inequality into the remainder gives

$$R_n(t) \le \int\limits_{[0,t)} a(s)\mu^{\otimes n}(A_n(s,t))\,d\mu(s) + \tilde{R}_n(t),$$

with

$$\tilde{R}_n(t) := \int\limits_{[0,t)} \Bigg(\int\limits_{[0,r)} u(s)\,d\mu(s) \Bigg) \mu^{\otimes n}(A_n(r,t))\,d\mu(r) \quad (t \ge 0).$$

By Fubini's Theorem,

$$\tilde{R}_n(t) = \int\limits_{[0,t)} u(s) \underbrace{\int\limits_{(s,t)} \mu^{\otimes n}(A_n(r,t))\,d\mu(r)}_{=\mu^{\otimes n+1}(A_{n+1}(s,t))} \,d\mu(s) = R_{n+1}(t) \quad (t \ge 0).$$

(ii) Let $k \in \mathbb{N}_0$ and $0 \le s < t$. Then

$$\mu^{\otimes k}(A_k(s,t)) \le \frac{\big(\mu((s,t))\big)^k}{k!}.$$

Indeed, for $k = 0$ this is trivial, thus consider $k \ge 1$. Let S_k be the set of all permutations of $\{1,\ldots,k\}$. For $\sigma \in S_k$ define

$$A_{k,\sigma}(s,t) := \Big\{ (s_1,\ldots,s_k) \in (s,t)^k;\ s_{\sigma(1)} < s_{\sigma(2)} < \cdots < s_{\sigma(k)} \Big\}.$$

For $\sigma, \sigma' \in S_k$, $\sigma \neq \sigma'$ we have $A_{k,\sigma}(s,t) \cap A_{k,\sigma'}(s,t) = \varnothing$. Furthermore,

$$\bigcup_{\sigma \in S_k} A_{k,\sigma}(s,t) \subseteq (s,t)^k.$$

Therefore,

$$\sum_{\sigma \in S_k} \mu^{\otimes k}(A_{k,\sigma}(s,t)) \le \big(\mu((s,t))\big)^k.$$

Since $\mu^{\otimes k}(A_{k,\sigma}(s,t)) = \mu^{\otimes k}(A_{k,\sigma'}(s,t))$ for $\sigma, \sigma' \in S_k$ and $|S_k| = k!$ we conclude

$$\mu^{\otimes k}(A_k(s,t)) = \mu^{\otimes k}(A_{k,I}(s,t)) \le \frac{\big(\mu((s,t))\big)^k}{k!}.$$

(iii) By (ii) we obtain

$$|R_n(t)| \le \frac{\big(\mu((0,t))\big)^n}{n!} \int\limits_{[0,t)} |u(s)|\,d\mu(s) \quad (t \ge 0, n \in \mathbb{N}).$$

Since u is locally integrable w.r.t. μ we have $R_n(t) \to 0$ as $n \to \infty$, for all $t \geq 0$. Again by (ii),

$$\sum_{k=0}^{n-1} \mu^{\otimes k}(A_k(s,t)) \leq \sum_{k=0}^{n-1} \frac{\big(\mu((s,t))\big)^k}{k!} \leq e^{\mu((s,t))} \quad (0 \leq s < t).$$

By (i) we conclude

$$u(t) \leq a(t) + \int\limits_{[0,t)} a(s) e^{\mu((s,t))}\, d\mu(s) \quad (t \geq 0). \qquad //$$

A.2. On sesquilinear forms and representation theorems

This section collects basic properties of sesquilinear forms and associated operators. All the definitions and statements are well-known and can be found for example in [30].

Throughout this section, let $\mathcal{H}$ be a Hilbert space.

Definition. Let $D \subseteq \mathcal{H}$ be a subspace, $\tau\colon D \times D \to \mathbb{K}$ be sesquilinear. Then τ is called a *form* with domain $D(\tau) := D$. The form τ is called *symmetric*, if

$$\tau(v,u) = \overline{\tau(u,v)} \quad (u, v \in D(\tau),$$

and *bounded from below*, if there exists $\gamma \in \mathbb{R}$ such that

$$\tau(u) := \tau(u,u) \geq -\gamma\,(u \,|\, u) \quad (u \in D(\tau)).$$

If τ is symmetric and bounded from below by γ, then

$$(u \,|\, v)_\tau := (\gamma + 1)\,(u \,|\, v) + \tau(u,v) \quad (u, v \in D(\tau))$$

defines an inner product on $D(\tau)$, with *form norm*

$$\|u\|_\tau := \Big((\gamma+1)\,\|u\|_{\mathcal{H}}^2 + \tau(u)\Big)^{1/2} \quad (u \in D(\tau)).$$

Then, τ is called *closed*, if $D_\tau := (D(\tau), \|\cdot\|_\tau)$ is complete (i.e., a Hilbert space).

A.2.1 Remark. If τ is bounded from below by γ, then τ is also bounded from below by γ' for all $\gamma' > \gamma$ and the norms $\|\cdot\|_\gamma$ and $\|\cdot\|_{\gamma'}$ are equivalent.

A.2.2 Lemma. *Let τ be symmetric and bounded from below. The following are equivalent:*

(a) τ is closed.

(b) If (u_n) in $D(\tau)$ with $u_n \to u$ in $\mathcal{H}$ and $\tau(u_n - u_m) \to 0$ $(m, n \to \infty)$, then $u \in D(\tau)$ and $\tau(u_n - u) \to 0$.

We now state the first representation theorem.

A.2.3 Theorem (first representation theorem). *Let τ be densely defined, symmetric, bounded from below and closed. Then then there exists a unique self-adjoint operator H on $\mathcal{H}$ such that H is bounded from below (by the same bound as τ), $D(H) \subseteq D(\tau)$ is dense in D_τ and*

$$\tau(u,v) = (Hu \,|\, v) \quad (u \in D(H), v \in D(\tau)).$$

We call H the operator *associated* with τ.

There is a second representation theorem precisely describing the form when starting with a self-adjoint lower-bounded operator. Since we do not need this fact we omit the statement.

Now, we focus on perturbations of closed forms. This leads to the celebrated KLMN-theorem.

A.2.4 Theorem. *Let τ_0 be densely defined, symmetric, bounded from below and closed. Let μ be another symmetric form with $D(\tau_0) \subseteq D(\mu)$ such that μ is* relatively form bounded with respect to τ, *i.e., there exist $a \in (0,1)$, $C > 0$ such that*

$$|\mu(u)| \leq a\tau_0(u) + C\,\|u\|_{\mathcal{H}}^2 \quad (u \in D(\tau_0)).$$

Then $\tau_\mu := \tau_0 + \mu$ with $D(\tau_\mu) := D(\tau)$ is densely defined, symmetric, bounded from below and closed.

A perturbation μ is called *infinitesimally form small with respect to* τ_0, if for all $a \in (0,1)$ there exists $C_a > 0$ such that μ is relatively form bounded with respect to τ_0 with parameters a and C_a.

A.3. Caccioppoli inequality, Combes-Thomas estimate, Shnol type arguments

In this section we collect some statements from [57] and [38]. We specialize to the case $\mathcal{H} = L_2(\mathbb{R})$.

The Combes-Thomas estimate can be found in [57]. Note that there are no measure-perturbations treated. However, the techniques rely on forms and form small perturbations, so the proof generalizes to our case without difficulty.

A.3.1 Proposition (Combes-Thomas estimate). *(a) Let $\omega \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, $r, s \in \mathbb{R}$, $r < s$, $(r,s) \subseteq \varrho(H_\omega)$ a spectral gap, $E \in (r,s)$, $A, B \subseteq \mathbb{R}$ measurable, $\delta :=$ $\mathrm{dist}(A,B)$. Then there exist $C, \eta > 0$ such that*

$$\left\| M_{\mathbb{1}_A}(H_\omega - E)^{-1} M_{\mathbb{1}_B} \right\| \leq C e^{-\eta\delta},$$

where M_f denotes the multiplication operator with the function f.

(b) If $\Omega \subseteq \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ is $\|\cdot\|_{\mathrm{loc}}$-bounded and $\sigma(H_\omega)$ is independent of $\omega \in \Omega$, then the constants C and η can be chosen independent of ω (i.e., the estimate in (a) holds uniformly on Ω).

A.3.2 Proposition. *Let τ_0 be the classical Dirichlet form in $L_2(\mathbb{R})$, i.e.,*

$$D(\tau_0) = W_2^1(\mathbb{R}),$$
$$\tau_0(u,v) = \int u'(t)\overline{v'(t)}\,dt,$$

with associated operator H_0.

(a) Define the intrinsic metric for τ_0 *by* $d_0 \colon \mathbb{R} \times \mathbb{R} \to [0, \infty]$,

$$d_0(t,s) := \sup \left\{ |u(t) - u(s)| \,;\ u \in D_{\mathrm{loc}}(\tau_0) \cap C(\mathbb{R}),\ |u'| \le 1 \text{ a.e.} \right\}.$$

Then d_0 *induces the original topology on* $\mathbb{R}$ *and* $d_0(t,s) = |t - s|$ $(s, t \in \mathbb{R})$.

(b) τ_0 *is ultracontractive, i.e., for* $t > 0$ *we have* $e^{-tH_0} \in L(L_2(\mathbb{R}), L_\infty(\mathbb{R}))$.

For $A \subseteq \mathbb{R}$ and $r > 0$ the r-neighborhood of A with respect ot d_0 as in the proposition is given by

$$A + B[0,r] = \{t \in \mathbb{R};\ d_0(t, A) \le r\}.$$

The Caccioppoli inequality estimates local L_2-norms of derivatives of solutions of a Schrödinger equation by the L_2-norm of the solution itself on a larger domain. The result even generalizes to the context of strongly local Dirichlet forms.

A.3.3 Proposition (Caccioppoli type inequality)**.** *Let* τ_0 *be the classical Dirichlet form in* $L_2(\mathbb{R})$*. Let* $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$, $E \in \mathbb{R}$, $r > 0$*. Then there exists* $C \ge 0$ *such that for any (local) solution* u *of* $H_\mu u = Eu$ *on* $A + B[0,r]$ *the inequality*

$$\int_A |u'(t)|^2 \, dt \le \frac{C}{r^2} \int_{A+B[0,r]} |u(t)|^2 \, dt$$

holds for any closed $A \subseteq \mathbb{R}$.

A.3.4 Proposition ($\frac{1}{2}$ Shnol)**.** *Let* τ_0 *be the classical Dirichlet form in* $L_2(\mathbb{R})$*. Let* $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$*. Let* u *be a nontrivial subexponentially bounded solution of* $H_\mu u = Eu$*. Then* $E \in \sigma(H_\mu)$.

A.3.5 Proposition ($\frac{1}{2}$ Shnol)**.** *Let* τ_0 *be the classical Dirichlet form in* $L_2(\mathbb{R})$*. Let* $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$*. Then for spectrally a.e.* $E \in \sigma(H_\mu)$ *there is a nontrivial subexponentially bounded solution of* $H_\mu u = Eu$.

A.4. Herglotz functions

We state basic properties of Herglotz functions, which can be found for example in [61].

Let $C^+ := \{z \in \mathbb{C};\ \operatorname{Im} z > 0\}$.

Let $f \colon \mathbb{C}^+ \to \mathbb{C}$. Then f is called a *Herglotz function* if f is holomorphic and $\operatorname{Im} f(z) \ge 0$ $(z \in \mathbb{C}^+)$.

A.4.1 Proposition. *Let* f *be a Herglotz function.*

(a) Then there exist $\alpha \in \mathbb{R}$, $\beta \ge 0$ *and a nonnegative measure* ϱ *on* $\mathbb{R}$ *with*

$$\int_{\mathbb{R}} \frac{d\varrho(t)}{1+t^2} < \infty,$$

such that

$$f(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1+t^2} \right) d\varrho(t).$$

(b) Assume that $\sup_{z\in\mathbb{C}^+} |f(z)\operatorname{Im} z| < \infty$*. Then*

$$f(z) = \int_{\mathbb{R}} \frac{1}{t-z}\, d\varrho(t)$$

for a nonnegative finite measure ϱ *on* $\mathbb{R}$*.*

The measure ϱ is called the *spectral measure* of the function f.

A.4.2 Proposition. *Let* ϱ *be a nonnegative finite measure on* $\mathbb{R}$*. For* $z \in \mathbb{C}\setminus\mathbb{R}$ *define*

$$f(z) := \int_{\mathbb{R}} \frac{1}{t-z}\, d\varrho(t).$$

(a) Then, for $t \in \mathbb{R}$*, we have*

$$\varrho((-\infty,t]) = \lim_{\delta\to 0+} \lim_{\varepsilon\to 0+} \frac{1}{\pi} \int_{-\infty}^{t+\delta} \operatorname{Im} f(E+i\varepsilon)\, dE.$$

(b) Let $\varrho = \varrho_{ac}+\varrho_{sc}+\varrho_{pp}$ *be the Lebesgue decomposition of* ϱ*. Then for* λ*-a.a.* $E \in \mathbb{R}$ *we have*

$$\frac{d\varrho_{ac}}{d\lambda}(E) = \frac{1}{\pi} \lim_{\varepsilon\to 0+} \operatorname{Im} f(E+i\varepsilon).$$

A.5. Spectral theorem

In this section we collect some statements on the spectral theorem. They can be found in literally every book on the spectral theory of unbounded self-adjoint operators, e.g. [61].

Let H be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$.

Definition. A family $(E(t))_{t\in\mathbb{R}}$ of self-adjoint projections in $\mathcal{H}$ is called a *spectral resolution* if the strong limits satisfy

$$\text{s-}\lim_{t\to-\infty} E(t) = 0, \quad \text{s-}\lim_{t\to\infty} E(t) = I,$$

E is monotone, i.e., $E(s) \le E(t)$ for $s \le t$, and E is strongly right continuous, i.e.,

$$\text{s-}\lim_{s\to t+} E(s) = E(t) \quad (t \in \mathbb{R}).$$

A.5.1 Theorem (Spectral Theorem)**.** *There is a unique spectral resolution* E *so that*

$$H = \int_{\mathbb{R}} t\, dE(t).$$

The proof of this theorem rests on the following observation. For $\xi \in \mathcal{H}$ and $z \in \mathbb{C}\backslash\mathbb{R}$ set

$$f_\xi(z) := \big((H-z)^{-1}\xi \,|\, \xi\big)\,.$$

Then f_ξ is a Herglotz function and $\sup_{z\in\mathbb{C}^+} |f_\xi(z)\operatorname{Im} z| \le \|\xi\|^2$. So,

$$f_\xi(z) = \int\limits_{\mathbb{R}} \frac{1}{t-z}\, d\varrho^\xi(t),$$

for some nonnegative finite measure ϱ^ξ. This measure is called *spectral measure for H in state ξ*.

We call $\xi \in \mathcal{H}$ a *maximal spectral vector* if ϱ^ψ is absolutely continuous with respect to ϱ^ξ for all $\psi \in \mathcal{H}$. Then ϱ^ξ is called *maximal spectral measure* for H.

A.5.2 Lemma. *There exists a maximal spectral vector for H.*

We now relate spectral properties of H with a maximal spectral measure. Let E be a spectral resolution for $\mathcal{H}$, $\xi \in \mathcal{H}$. Then $(E(\cdot)\xi \,|\, \xi)$ is the distribution function of the nonnegative finite measure ϱ^ξ on $\mathbb{R}$.

Define

$$\begin{aligned}
\mathcal{H}_{ac} &:= \Big\{\xi \in \mathcal{H};\ \varrho^\xi \text{ is absolutely continuous w.r.t. } \lambda\Big\}\,,\\
\mathcal{H}_{sc} &:= \Big\{\xi \in \mathcal{H};\ \varrho^\xi \text{ is singular continuous w.r.t. } \lambda\Big\}\,,\\
\mathcal{H}_{pp} &:= \Big\{\xi \in \mathcal{H};\ \varrho^\xi \text{ is a pure point measure}\Big\}\,,\\
\mathcal{H}_{s} &:= \Big\{\xi \in \mathcal{H};\ \varrho^\xi \text{ is singular w.r.t. } \lambda\Big\}\,,\\
\mathcal{H}_{c} &:= \Big\{\xi \in \mathcal{H};\ \varrho^\xi \text{ is a continuous measure}\Big\}\,.
\end{aligned}$$

Then these subspaces of $\mathcal{H}$ are closed, H-invariant and we have

$$\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}.$$

Let $\sigma_\bullet(H) := \sigma(H|_{\mathcal{H}_\bullet})$ for $\bullet \in \{ac, sc, pp, s, c\}$. Then we have

$$\sigma_{pp}(H) = \overline{\{E \in \mathbb{R};\ E \text{ is an eigenvalue of } H\}}.$$

Let ϱ be the maximal spectral measure for H and

$$\varrho = \varrho_{ac} + \varrho_{sc} + \varrho_{pp}$$

be the Lebesgue decomposition of ϱ. Then

$$\sigma_{ac}(H) = \sigma(\varrho_{ac}), \quad \sigma_{sc}(H) = \sigma(\varrho_{sc}), \quad \sigma_{pp}(H) = \sigma(\varrho_{pp})$$

are the absolutely continuous, the singular continuous and the pure point spectrum of H. Here,

$$\sigma(\varrho) := \{E \in \mathbb{R};\ \forall\, \varepsilon > 0 : \varrho((E-\varepsilon, E+\varepsilon)) > 0\}$$

is the set of *growth points* of a nonnegative measure μ and supports the measure.

A.6. Spectral theory for Sturm-Liouville operators

We briefly sketch the spectral theory of full-line Sturm-Liouville operators, see [9, Section III.1] and [16, Section 11].

Let $\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and $m_\pm$ be the m-functions of the half-line operators $H_\mu^\pm$. Let

$$M(z) := \frac{1}{m_+(z)+m_-(z)} \begin{pmatrix} -1 & \frac{1}{2}(m_-(z)-m_+(z)) \\ \frac{1}{2}(m_-(z)-m_+(z)) & m_-(z)m_+(z) \end{pmatrix} \quad (z \in \mathbb{C}\setminus\mathbb{R})$$

be the 2×2 Titchmarsh-Weyl matrix. Then M is a matrix Herglotz function. There exists a self-adjoint matrix M_0 and a symmetric matrix-valued measure Υ such that

$$M(z) = M_0 + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\Upsilon(t) \quad (z \in \mathbb{C}\setminus\mathbb{R}).$$

Then there is a unitary $\mathcal{F}\colon L_2(\mathbb{R}) \to L_2(\mathbb{R},\Upsilon)$ such that $H_\mu = \mathcal{F}^* M_{\mathrm{id}} \mathcal{F}$.

Let $\varrho := \operatorname{Tr}\Upsilon$. Then ϱ is a nonnegative measure and all components of Υ are absolutely continuous with respect to ϱ. So, each spectral measure of H_μ is absolutely continuous with respect to ϱ. On the other hand, for all $A \subseteq \mathbb{R}$ measurable we have

$$\varrho(A) = 0 \Longleftrightarrow \forall\, f \in L_2(\mathbb{R}) : \varrho^f(A) = \int_A d\,(E(t)f \mid f) = 0.$$

So, the spectral properties of H_μ are encoded in the measure ϱ.

The associated Herglotz function is given by

$$m(z) := \operatorname{Tr} M(z) = \frac{m_-(z)m_+(z)-1}{m_-(z)+m_+(z)} = G_\mu(0,0,z) + h(z),$$

where $h(z) = \frac{m_-(z)m_+(z)}{m_-(z)+m_+(z)}$. An essential support of ϱ_{ac} is given by the set of all $E \in \mathbb{R}$, such that

$$0 < \operatorname{Im} G_\mu(0,0,E+i0+) < \infty \quad \text{or} \quad 0 < \operatorname{Im} h(E+i0+) < \infty.$$

Since

$$\operatorname{Im} G_\mu(0,0,\cdot) = \frac{\operatorname{Im} m_+ + \operatorname{Im} m_-}{|m_+ + m_-|^2}, \quad \operatorname{Im} h = \frac{|m_+|^2 \operatorname{Im} m_- + |m_-|^2 \operatorname{Im} m_+}{|m_+ + m_-|^2},$$

and $|m_\pm(E+i0+)|^2$, $|m_+(E+i0+) + m_-(E+i0+)|$ are finite and nonzero for λ-a.a. $E \in \mathbb{R}$, an essential support of ϱ_{ac} is also given by

$$\{E \in \mathbb{R};\ 0 < \operatorname{Im} m_+(E+i0+) < \infty \text{ or } 0 < \operatorname{Im} m_-(E+i0+) < \infty\}.$$

Since $m_\pm$ are Herglotz functions, the limits $\operatorname{Im} m_\pm(E+i0+)$ exist and are finite for λ-a.a. $E \in \mathbb{R}$. Hence, we conclude

$$\operatorname{spt} \varrho_{ac} = \overline{\{E \in \mathbb{R};\ 0 < \operatorname{Im} G_\mu(0,0,E+i0+) < \infty\}}^{\mathrm{ess}}.$$

Theses

This thesis is concerned with spectral theory for one-dimensional continuum Schrödinger operators of the form $-\Delta + \mu$ in $L_2(\mathbb{R})$, where μ is a signed local measure. In order to define such operators one has to restrict the class of measures.

1. It turns out that if $\mathbb{1}_K\mu$ is a finite signed Radon measure for all compact $K \subseteq \mathbb{R}$ and, furthermore, if μ is uniformly locally bounded then a self-adjoint realization of $-\Delta+\mu$ can be defined in two different ways (which both can be found in the literature):

(a) via the form method interpreting μ as an infinitesimally form small perturbation of the classical Dirichlet form associated with $-\Delta$, or
(b) via a direct approach basically interpreting $-\Delta + \mu$ as the self-adjoint operator corresponding to a Sturm-Liouville differential expression.

2. The two self-adjoint realizations of $-\Delta+\mu$ obtained by the two methods coincide. Thus, we have two different ways to think of the self-adjoint operator in question.

Having established a self-adjoint realization we ask for spectral properties. In fact, we are interested in connections between geometric properties of the potential μ and measure-theoretic spectral types of the operator.

3. If the measure μ is close to periodic, i.e., it can be approximated by periodic measures on increasing intervals, then the corresponding Schrödinger operator does not have any pure point spectrum.

4. If the measure μ is not periodic and both μ and the reflected measure $\mu(-(\cdot))$ have the simple finite decomposition property then the corresponding operator does not have any absolutely continuous spectrum. The simple finite decomposition property states that the measure is built out of finitely many pieces and the decomposition is "not too difficult".

With these two observations at hand we can—deterministically—conclude purely singular continuous spectra for Schrödinger operators modelling quasicrystals.

We now want to deal with a whole family of potentials (and hence of operators) simultaneously, thus obtaining a random Schrödinger operator. To this end, assume that Ω consists of uniformly locally bounded signed local Radon measures such that Ω is bounded in the uniform-loc norm. Furthermore, let Ω be closed with respect to the vague topology and translation invariant.

5. The set Ω is vaguely compact and the vague topology on Ω is metrizable. The natural group action α of $\mathbb{R}$ on Ω by shifts is continuous. Therefore, we obtain a dynamical system (Ω, α).

We can now ask for connections between dynamical properties on the space of potentials and spectral properties for the corresponding family of operators.

6. If the dynamical system is minimal then all operators of the family have the same spectrum (as a set). In the same spirit, by well-known theory we conclude that ergodicity of the dynamical system (Ω, α) implies almost sure constancy of the spectrum with respect to an ergodic measure on Ω.

7. The solutions of the eigenvalue equation for the Schrödinger operator depend on the initial conditions for the solution and for the derivative at 0. The transfer matrix at $t \in \mathbb{R}$ maps the initial condition to the solution at t. The transfer matrices for the family of operators form a cocycle of volume-preserving transformations. For such cocycles one can define a Lyapunov exponent $\gamma(E)$ for an energy E by a limit. It describes the exponential growth rate of the norm of the transfer matrices as $t \to \pm\infty$. The transfer matrix at energy E is uniform if the limit is uniform on Ω.

8. We can also prove almost sure purely singular continuous spectrum for the operator family: Let (Ω, α) be strictly ergodic (i.e., uniquely ergodic and minimal) with (unique) ergodic probability measure $\mathbb{P}$ such that for every $\omega \in \Omega$ we have that ω and $\omega(-(\cdot))$ have the simple finite decomposition property, there exists $\omega \in \Omega$ wich is not periodic and Ω satisfies a certain condition (K). Then the corresponding Schrödinger operators have $\mathbb{P}$-almost surely purely singular continuous spectrum. This can be seen as a random version of the deterministic statement given above.

In the case where all potential in Ω are atomless we can conclude spectral properties in terms of the Lyapunov exponent:

8. If (Ω, α) is strictly ergodic and atomless then the common spectrum Σ of the operator family is given by

$$\Sigma = \{E \in \mathbb{R};\ \gamma(E) = 0\} \cup \{E \in \mathbb{R};\ \text{the transfer matrix at energy } E \text{ is not uniform}\}.$$

9. We also have an Ishii-Pastur-Kotani theorem for this case: Let $(\Omega, \alpha, \mathbb{P})$ be ergodic. Then an essential support of the absolutely continuous part of the spectrum is given by $\{E \in \mathbb{R};\ \gamma(E) = 0\}$.

10. Let (Ω, α) be strictly ergodic, atomless, having the simple finite decomposition property (for all ω and $\omega(-(\cdot))$) and assume there exists $\omega \in \Omega$ which is not periodic. Furthermore, assume that all transfer matrices at real energies are uniform. Then the common spectrum Σ of the operator family is given by

$$\Sigma = \{E \in \mathbb{R};\ \gamma(E) = 0\}$$

and Σ is a Cantor set of zero Lebesgue measure.

Bibliography

[1] R.A. Adams and J.J.F. Fournier: *Sobolev Spaces*. 2nd edition. Academic Press, Oxford, 2003.

[2] H. W. Alt: *Lineare Funktionalanalysis: Eine anwendungsorientiere Einführung*. 5. Auflage. Springer-Verlag, 2006.

[3] M. Baake, U. Grimm and R.V. Moody, *Die verborgene Ordnung der Quasikristalle*. In Spektrum der Wissenschaft, Februar 2002, 64–74 (2002).

[4] J. Bellisard, B. Iochum, E. Scoppola and D. Testard, *Spectral properties of one-dimensional quasicrystals*. Commun. Math. Phys. **125**, 527–543 (1989).

[5] J.F. Brasche, R. Figari and A. Teta, *Singular Schrödinger Operators as Limits of Point Interaction Hamiltonians*. Pot. An. **8**, 163–178 (1998).

[6] A. Ben Amor and C. Remling, *Direct and inverse spectral theory of one-dimensional Schrödinger operators with measures*. Integr. equ. oper. theory **52**, 395–417 (2005).

[7] P. Billingsley, *Probability and Measure*. 3rd edition. Wiley, New York, 1995.

[8] D. Buschmann, *A proof of the Ishii-Pastur theorem by the method of subordinacy*. Univ. Iagel. Acta Math. **34**, 29–34 (1997).

[9] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*. Birkhäuser Boston, 1990.

[10] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*. McGraw-Hill, New York, 1955.

[11] X. Dai, *Optimal state points of the subadditive ergodic theorem*. Nonlinearity **24**, 1565-1573 (2011).

[12] D. Damanik, *Gordon-type arguments in the spectral theory of one-dimensional quasicrystals.* in "Directions in Mathematical Quasicrystals", CRM Monogr. Ser. **13**, Amer. Math. Soc., Providence, RI, 277-305 (2000)

[13] D. Damanik, *A version of Gordon's theorem for multi-dimensional Schrödinger operators.* Trans. Amer. Math. Soc. **356**, 495–507 (2004).

[14] D. Damanik and D. Lenz, *A Condition of Boshernitzan and uniform convergence in the multiplicative ergodic Theorem.* Duke Math. J. **133**, 95–123 (2006).

[15] D. Damanik and G. Stolz, *A generalization of Gordon's theorem and applications to quasiperiodic Schrödinger operators.* Electron. J. Diff. Eqns. **55**, 1–8 (2000).

[16] J. Eckhardt and G. Teschl, *Sturm-Liouville operators with measure-valued coefficients.* arXiv-preprint 1105.3755v1.

[17] J. Elstrodt, *Maß- und Integrationstheorie.* 6th edition, Springer, Berlin, 2009.

[18] Ethier and Kurtz, Markov Processes: Characterization and Convergence. Wiley, 2005.

[19] L.C. Evans and R.F. Gariepy, *Measure Theory and fine properties of functions.* CRC Press, 1992.

[20] A. Furman, *On the multiplicative ergodic theorem for uniquely ergodic systems.* Ann. Inst. Henri Poincare: Prob. & Stat. **33**(6), 797–815 (1997).

[21] A. Gordon, On the point spectrum of the one-dimensional Schrödinger operator, Usp. Math. Nauk 31, 257–258 (1976).

[22] K. Gowrisankaran, *Measurability of functions in product spaces.* Proc. Amer. Math. Soc. **31**(2), 485–488 (1972).

[23] S. Jitomirskaya and B. Simon, *Operators with singular continuous spectrum: III. Almost periodic Schrödinger operators.* Commun. Math. Phys. **165**, 201–205 (1994).

[24] R. A. Johnson, *Exponential Dichotomy, rotation number, and linear differential operaotors with bounded coefficients.* J. Diff. Eqns. **61**(1), 54–78 (1986).

[25] R. A. Johnson, *Remarks on linear differential systems with measurable coefficients.* Proc. Amer. Math. Soc. **100**(3), 491–504 (1987).

[26] R. A. Johnson, *Cantor Spectrum for the Quasi-periodic Schrödinger Equation.* J. Differential Equations **91**, 88–110 (1991).

[27] M. Kaminaga, *Absence of point spectrum for a class of discrete Schrödingerg operators with quasiperiodic potential.* Forum Math. **8**, 63–69 (1996).

[28] S. Klassert, *Spektraltheoretische Untersuchungen von zufälligen Operatoren auf Delone-Mengen.* Dissertation Thesis, 2007.

[29] S. Klassert, D. Lenz and P. Stollmann, *Delone measures of finite local complexity and applications to spectral theory of one-dimensional continuum models of quasicrystals.* Discrete Contin. Dyn. Syst. **29**(4), 1553–1571 (2011).

[30] T. Kato, Perturbation Theory for linear operators. Springer, Berlin, 1995.

[31] S. Kotani, *Ljapunov Indices Determine Absolutely Continuous Spectra of Stationary Random One-dimensional Schrödinger Operators.* In: Stochastic analysis, 225–247, North-Holland, Amsterdam, 1984.

[32] S. Kotani, *Jacobi matrices with random potentials taking finitely many values.* Rev. Math. Phys. **1**, 129–133 (1989).

[33] Y. Last and B. Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators.* Invent. math. **135**, 329–367 (1999).

[34] D. Lenz, *Singular Spectrum of Lebesgue Measure Zero for One-Dimensional Quasicrystals.* Commun. Math. Phys. **227**, 119–130 (2002).

[35] D. Lenz, *Existence of non-uniform cocycles on uniquely ergodic systems.* Ann. Inst. Henri Poincare: Prob. & Stat. **40**, 197–206 (2004).

[36] D. Lenz, *Ergodic theory and discrete one-dimensional random Schrödinger operators: Uniform existence of the Lyapunov exponent.* Contemp. Math. **485**, 91–112 (2009).

[37] D. Lenz, *Existence of non-uniform cocycles on uniquely ergodic systems.* Ann. Inst. Henri Poincare: Prob. & Stat. **40**, 197–206 (2004).

[38] D. Lenz, P. Stollmann and I. Veselić, *Generalized eigenfunctions and spectral theory for strongly local Dirichlet forms.* In: Spectral theory and analysis 214, 83–106, Birkhäuser/Springer Basel AG, Basel, 2011.

[39] D. Lenz, Lecture Notes “Aperiodische Ordnung”, unpublished.

[40] R. Meise and D. Vogt, *Einführung in die Funktionalanalysis.* Vieweg, Braunschweig, 1992.

[41] D.R. Nelson, *Quasicrystals.* In Scientific American August 1986, 42–51 (1986).

[42] L.A. Pastur, *Spectra of random selfadjoint operators.* Russ. Math. Surv. **28**(1), 1–67 (1973).

[43] L.A. Pastur, *Spectral Properties of Disordered Systems in the One-Body Approximation.* Commun. Math. Phys. **75**, 179–196 (1980).

[44] G. K. Pedersen, *Analysis Now.* Springer, New York, 1989.

[45] M.S. Raghunathan, *A proof of Osedelec's multiplicative ergodic theorem.* Israel J. Math. **32**(4), 356–362 (1979).

[46] M. Reed and B. Simon, Methods of modern Mathematical Physics I: Functional Analysis. Academic Press, London, 1980.

[47] C. Remling, *The absolutely continuous spectrum of one-dimensional Schrödinger operators.* Math Phys Anal Geom **10**, 359–373 (2007).

[48] F. Riesz and B. Sz.-Nagy, *Vorlesungen über Funktionalanalysis.* 3. Auflage, VEB Deutscher Verlag der Wissenschaften, Berlin, 1973.

[49] R.J. Sacker and G.R. Sell, *Existence of Dichotomies and invariant splittings for linear differential systems I.* J. Diff. Eu. **15**, 429–458 (1974).

[50] R.J. Sacker and G.R. Sell, *Existence of Dichotomies and invariant splittings for linear differential systems II.* J. Diff. Eu. **22**, 478–496 (1976).

[51] R.J. Sacker and G.R. Sell, *A Spectral Theory for linear differential systems.* J. Diff. Eu. **27**, 320–358 (1978).

[52] C. Seifert, *Gordon type theorem for measure perturbation.* Electron. J. Diff. Eqns. **111**, 1–9 (2011).

[53] D. Shechtman, I. Blech, D. Gratias and J.W. Cahn, *Metallic phase with long-range orientational order and no translational symmetry.* Phys. Rev. Let. **53**, 1951–1953 (1984).

[54] B. Simon, *Schrödinger Semigrous.* Bull. Amer. Math. Soc. (N.S.) **7**(3), 447–526 (1982).

[55] J. Stark, U. Feudel, P.A. Glendinning and A. Pikovsky, *Rotation number for quasi-periodically forced monotone circle maps.* Dyn. Syst. **17**(1), 1–28 (2002).

[56] J. M. Steele, *Kingman's subadditive ergodic theorem.* An. Inst. Henri Poincaré **25**(1), 93–98 (1989).

[57] P. Stollmann, *Caught by disorder. Bound states in random media.* Progress in Mathematical Physics 20, Birkhäuser Boston, Boston, 2001.

[58] P. Stollmann, *A dual characterization of length spaces with application to Dirichlet metric spaces.* Studia Math. **198**(3), 221–233 (2010).

[59] P. Stollmann and J. Voigt, *Perturbation of Dirichlet forms by measures.* Pot. An. **5**, 109–138 (1996).

[60] R. Sturman and J Stark, *Semi-uniform ergodic theorems and applications to forced systems.* Nonlinearity **13**, 113-143 (2000).

[61] G. Teschl, *Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators.* Graduate Studies in Mathematics 99, AMS, Providence, 2009.

[62] J. Weidmann, *Spectral theory of Ordinary Differential operators.* Lecture Notes in Mathematics 1258, Springer, Berlin, 1987.

Statutory declaration

I assure that this thesis is a result of my personal work and that no other than the indicated aids have been used for its completion. Furthermore, I assure that all quotations and statements that have been inferred literally or in a general manner from published or unpublished writings are marked as such.

Chemnitz, June 28, 2012

Christian Seifert