Ramsey numbers for bipartite graphs with small bandwidth☆

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Abstract

We estimate Ramsey numbers for bipartite graphs with small bandwidth and bounded maximum degree. In particular we determine asymptotically the two and three color Ramsey numbers for grid graphs. More generally, we determine the two color Ramsey number for bipartite graphs with small bandwidth and bounded maximum degree and the three color Ramsey number for such graphs with the additional assumption that both color classes have almost the same size.

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1. Introduction

For graphs $G_1, G_2, \ldots, G_r$, the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then

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at least one H_i $(1 \leq i \leq r)$ contains a subgraph isomorphic to G_i. The existence of such a positive integer follows from Ramsey's theorem. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number of the graphs $G_1, G_2, \ldots, G_r$. Determining $R(G_1, G_2, \ldots, G_r)$ for general graphs appears to be a difficult problem (see e.g. [9] or [19]). For $r = 2$, a well-known theorem of Gerencsér and Gyárfás [8] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor,$$

where P_n denotes the path with $n \geq 2$ vertices. In [13] more general trees were considered. For a tree T, we write t_1 and t_2, with $t_2 \geq t_1$, for the sizes of the vertex classes of T as a bipartite graph. Note that $R(T,T) \geq 2t_1 + t_2 - 1$, since the following edge-coloring of $K_{2t_1+t_2-2}$ has no monochromatic copy of T. Partition the vertices into two classes V_1 and V_2 such that $|V_1| = t_1 - 1$ and $|V_2| = t_1 + t_2 - 1$, then use color "red" for all edges inside the two classes and use color "blue" for all edges between the classes. Furthermore, a similar edge-coloring of K_{2t_2-2} with two classes both of size $t_2 - 1$ shows that $R(T,T) \geq 2t_2$, where here the color we put inside the class with t_1 vertices is arbitrary. Thus

$$R(T,T) \geq \max\{2t_1 + t_2, 2t_2\} - 1. \tag{1}$$

Haxell, Łuczak and Tingley provided in [13] an asymptotically matching upper bound for trees T with $\Delta(T) = o(t_2)$.

We partially extend this to bipartite graphs with small bandwidth and a more restrictive maximum degree condition. A graph $H = (W, E_H)$ is said to have *bandwidth* at most b, if there exists a labelling of the vertices by numbers $1, \ldots, n$ such that for every edge $ij \in E_H$ we have $|i - j| \leq b$. We focus our attention on the following class of bipartite graphs.

Definition 1.1. *A bipartite graph H is called a (β, Δ)-graph if it has bandwidth at most $\beta|V(H)|$ and maximum degree at most Δ. Furthermore, we say that H is a balanced (β, Δ)-graph if it has a legal 2-coloring $\chi \colon V(H) \to [2]$ such that $\big||\chi^{-1}(1)| - |\chi^{-1}(2)|\big| \leq \beta|\chi^{-1}(2)|$.*

For example, it was shown in [5] that sufficiently large planar graphs with maximum degree at most Δ are (β, Δ)-graphs for any fixed $\beta > 0$. Our first theorem is an analogue of the result in [13] for (β, Δ)-graphs.

Theorem 1.2. *For every $\gamma > 0$ and natural number Δ, there exist a constant $\beta > 0$ and natural number n_0 such that for every (β, Δ)-graph H on $n \geq n_0$ vertices with a legal 2-coloring $\chi : V(H) \to [2]$ where $t_1 = |\chi^{-1}(1)|$ and $t_2 = |\chi^{-1}(2)|$, with $t_1 \leq t_2$, we have*

$$R(H,H) \leq (1 + \gamma)\max\{2t_1 + t_2, 2t_2\}.$$

For more recent results on the Ramsey number of graphs of higher chromatic number and sublinear bandwidth, we refer the reader to the work of Allen, Brightwell and Skokan [1].

For $r \geq 3$ less is known about Ramsey numbers. Let T be a tree and consider t_1 and t_2, with $t_1 \leq t_2$, the sizes of the vertex classes of T as a bipartite graph. For $r = 3$ the following construction gives a lower bound for $R(T, T, T)$. Partition the vertices of T in four classes, one special class V_0 with $|V_0| = t_1$ and three classes V_1, V_2 and V_3 of size $t_2 - 1$. The color for edges inside V_0 is arbitrary. Use color i inside the classes V_i and color i between V_i and V_0 for $1 \leq i \leq 3$. Finally, use color $k \in [3] \setminus \{i, j\}$ for edges between the classes V_i and V_j for $1 \leq i < j \leq 3$. It is easy to check that this coloring yields no monochromatic copy of T. Thus

$$R(T, T, T) \geq t_1 + 3t_2 - 2. \tag{2}$$

Proving a conjecture of Faudree and Schelp [6], it was shown in [10] that this construction is optimal for large paths, i.e., for sufficiently large n we have

$$R(P_n, P_n, P_n) = \begin{cases} 2n - 1 & \text{for odd } n, \\ 2n - 2 & \text{for even } n. \end{cases}$$

Asymptotically this was also proved independently by Figaj and Łuczak [7]. Benevides and Skokan [2] proved that $R(C_n, C_n, C_n) = 2n$ for sufficiently large even n. Our second result extend this asymptotically to balanced (β, Δ)-graphs.

Theorem 1.3. *For every $\gamma > 0$ and every natural number Δ, there exist a constant $\beta > 0$ and natural number n_0 such that for every balanced (β, Δ)-graph H on $n \geq n_0$ vertices we have*

$$R(H, H, H) \leq (2 + \gamma)n.$$

In particular, Theorems 1.2 and 1.3 give the asymptotics for two and three color Ramsey numbers of grid graphs. The 2-dimensional grid graph $G_{a,b}$ is the graph with vertex set $V = [a] \times [b]$ and there is an edge between two vertices if they are equal in one coordinate and consecutive in another. Note that any grid graph $G_{a,b}$ on ab vertices has bandwidth at most $\sqrt{ab}$ and satisfies $\Delta(G) \leq 4$. Moreover, $G_{a,b}$ is a balanced $(\beta, 4)$-graph for any fixed $\beta > 0$ and sufficiently large ab. Consequently, Theorems 1.2 and 1.3 combined with (1) and (2) give the following corollary.

Corollary 1.4. *For grid graphs $G_{a,b}$ we have*

$$R(G_{a,b}, G_{a,b}) = \big(3/2 + o(1)\big)ab$$

and

$$R(G_{a,b}, G_{a,b}, G_{a,b}) = \big(2 + o(1)\big)ab,$$

where $o(1)$ tends to 0 as $ab \to \infty$.

We remark that similar bounds follow for bipartite planar graphs with bounded degree and grids of higher dimension.

This paper is organized as follows. We first give the necessary tools in Section 2 and then present a detailed proof of Theorem 1.3 in Section 3. The proof of Theorem 1.2 is very similar and here we only present an outline, in Section 4.

2. Auxiliary results

The main purpose of this section is to present the tools for the proof of Theorem 1.3. A main tool in the proof is Szemerédi's Regularity Lemma [22]. We discuss the Regularity Method in Section 2.1. In Sections 2.2 and 2.3 we give some results that allows us to make use of the regularity method.

2.1. The Regularity Method

Given an graph G on n vertices, the *density* of G is given by $d_G = e(G)/\binom{n}{2}$. Furthermore, if $A, B \subset V(G)$ are non-empty and disjoint, we denote by $e_G(A, B)$ the number of edges of G with one endpoint in A and the other in B and

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}$$

is the *density* of G between A and B.

The bipartite graph $G = (A, B; E)$ is called *ε-regular* if for all $X \subset A$, $Y \subset B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$ we have

$$|d_G(X, Y) - d_G(A, B)| < \varepsilon.$$

Furthermore, we say that G is *(ε, d)-regular* if it is ε-regular and $d_G(A, B) \geq d$. An ε-regular bipartite graph $(A, B; E)$ is called *(ε, d)-super-regular* if we have $\deg_G(a) > d|B|$ for all $a \in A$ and $\deg_G(b) > d|A|$ for all $b \in B$.

For a graph $G = (V, E)$, a partition $(V_i)_{i\in[s]}$ of V is said to be *(ε, d)-regular* (resp. *super-regular*) *on a graph R* with vertex set contained in $[s]$ if the bipartite subgraph of G induced by the pair $\{V_i, V_j\}$ is (ε, d)-regular (resp. super-regular) whenever $ij \in E(R)$. We say that a graph R on vertex set $[s]$ is the *(ε, d)-reduced graph* of $(V_i)_{i\in[s]}$ (or of G) if ij is an edge of R if and only if the bipartite graph defined by the pair $\{V_i, V_j\}$ is (ε, d)-regular in G.

The proof of Theorem 1.3 is based on the following three color version of the Regularity Lemma.

Lemma 2.1 (Regularity Lemma)**.** *For every $\varepsilon > 0$ and every integer $k_0 > 0$ there is a positive integer $K_0(\varepsilon, k_0)$ such that for $n \geq K_0$ the following holds. For all graphs G_1, G_2 and G_3 with $V(G_1) = V(G_2) = V(G_3) = V$, $|V| = n$, there is a partition of V into $k+1$ classes $V = V_0 + V_1 + V_2 + \cdots + V_k$ such that*

(i) $k_0 \leq k \leq K_0$,

(ii) $|V_1| = |V_2| = \cdots = |V_k|$,

(iii) $|V_0| < \varepsilon n$,

(iv) apart from at most $\varepsilon\binom{k}{2}$ *exceptional pairs, the pairs* $\{V_i, V_j\}$ *are* ε*-regular in* G_1*,* G_2 *and* G_3*.*

For extensive surveys on the Regularity Lemma and its applications see [17, 16]. A key component of the regularity method is the Blow-up Lemma [14] (see also [15, 20, 21] for alternative proofs). This lemma guarantees that bipartite spanning subgraphs of bounded degree can be embedded into super-regular pairs. In fact, the statement is more general and allows the embedding of r-chromatic graphs into the union of r vertex classes that form $\binom{r}{2}$ super-regular pairs.

Here we will use a version of the Blow-up lemma that allows us to embed graphs H of bounded-degree in a graph G when G and H have "compatible" partitions, in the sense explained in the definition below. In our proof we will embed H in parts, considering a partition of a monochromatic subgraph G of K_N with corresponding reduced graph containing a tree T that contains a "large" matching M, where the bipartite graphs of G corresponding to the matching edges are super-regular pairs.

Definition 2.2. *Let* $H = (W, E_H)$ *be a graph. Let* $T = ([s], E_T)$ *be a tree and* $M = ([s], E_M)$ *be a subgraph of* T *where* E_M *is a matching. Given a partition* $(W_i)_{i\in[s]}$ *of* W*, let* U_i*, for* $i \in [s]$*, be the set of vertices in* W_i*, with neighbours in some* W_j *with* $ij \in E_T \setminus E_M$*. Set* $U = \bigcup U_i$ *and* $U'_i = N_H(U) \cap (W_i \setminus U)$*.*

We say that $(W_i)_{i\in[s]}$ *is* (ε, T, M)*-compatible with a vertex partition* $(V_i)_{i\in[s]}$ *of a graph* $G = (V, E)$ *if the following holds.*

(i) $xy \in E_H$ *for* $x \in W_i$ *and* $y \in W_j$ *implies* $ij \in E_T$ *for all* $i, j \in [s]$*, in other words, the mapping from* W *to* $[s]$ *given by the partition* $(W_i)_{i\in[s]}$ *is a homomorphism from* H *to* T*,*

(ii) $|W_i| \le |V_i|$ *for all* $i \in [s]$*,*

(iii) $|U_i| \le \varepsilon|V_i|$ *for all* $i \in [s]$*,*

(iv) $|U'_i|, |U'_j| \le \varepsilon \min\{|V_i|, |V_j|\}$ *for all* $ij \in E_M$*.*

We remark that for connected graphs H and for every vertex i of T which is not covered by M we have $U_i = W_i$ and $U'_i = \emptyset$.

The following corollary of the Blow-up Lemma (see [3]) asserts that in the setup of Definition 2.2 graphs H of bounded degree can be embedded into G, if G admits a partition being sufficiently regular on T and super-regular on M.

Lemma 2.3 (Embedding Lemma [3, 4])**.** *For all* $d, \Delta > 0$ *there is a constant* $\varepsilon = \varepsilon(d, \Delta) > 0$ *such that the following holds. Let* $G = (V, E)$ *be an* N*-vertex graph that has a partition* $(V_i)_{i\in[s]}$ *of* V *with* (ε, d)*-reduced graph* T *on* $[s]$ *which is* (ε, d)*-super-regular on a graph* $M \subset T$*. Further, let* $H = (W, E_H)$ *be an* n*-vertex graph with maximum degree* $\Delta(H) \le \Delta$ *and* $n \le N$ *that has a vertex partition* $(W_i)_{i\in[s]}$ *of* W *which is* (ε, T, M)*-compatible with* $(V_i)_{i\in[s]}$*. Then* $H \subset G$*.*

We close this section with two simple facts. They follow easily by the definitions of regular and super-regular pairs.

Fact 2.4. *Let $B = (V_1, V_2; E)$ be an ε-regular bipartite graph and let $V_1' \subset V_1$ and $V_2' \subset V_2$ with $|V_1'| \geq \alpha|V_1|$ and $|V_2'| \geq \alpha|V_2|$ for some $\alpha > \varepsilon$. Then the graph $B' = \big(V_1', V_2'; E_B(V_1', V_2')\big)$ is ε'-regular such that $|d_B(V_1, V_2) - d_{B'}(V_1', V_2')| < \varepsilon$, where $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$.*

Fact 2.5. *Consider a graph $G = (V, E)$ with an (ε, d)-regular partition $(V_i)_{i\in[s]}$ of V with $|V_i| = m$ for $1 \leq i \leq s$. Let T be a tree on vertex set $[s]$ contained in the corresponding (ε, d)-reduced graph of $(V_i)_{i\in[s]}$ and let M be a matching contained in T. Then for each vertex i of M, the associated set V_i in G contains a subset V_i' of size $(1-\varepsilon r)m$ such that for every edge ij of M the bipartite graph $(V_i', V_j'; E_G(V_i', V_j'))$ is $(\varepsilon/(1 - \varepsilon r), d - (1 + r)\varepsilon)$-super-regular.*

2.2. Regular blow-up of a tree

In this section we show that for any coloring of $E(K_N)$ there exists a dense, regular, monochromatic subgraph of K_N with some structural properties that allow us to embed H into this subgraph. Here the notion of a connected matching in the reduced graph (originating in [18], see also [7, 10, 11, 12]) plays a central role. A *connected matching* in a graph R is a matching M such that all edges of M are in the same connected component of R. The following lemma, proved in [7], states that in a 3-colored almost complete graph we can always find a connected matching that covers almost half of the vertices and it is contained in a monochromatic tree.

Lemma 2.6. *For every $\delta > 0$ there exist an $\varepsilon_0 > 0$ and a natural number k_0 such that for every $\varepsilon < \varepsilon_0$ and $k \geq k_0$ and for every 3-edge colored graph R on k vertices with density at least $(1 - \varepsilon)$ there exists a matching M with at least $(1 - \delta)k/4$ edges in R that is contained in a monochromatic tree $T \subset R$.*

This lemma can be found in a stronger structural form in [10]. In fact, there it is proved that either there is a monochromatic connected matching covering more than half of the vertices, or the graph R is close to one of two extremal cases. It is not hard to see that in both extremal cases there is a monochromatic connected matching M of size at least $(1 - \delta)|V(R)|/4$. We will also make use of the following simple fact.

Fact 2.7. *If a tree T contains a matching M with ℓ edges then the vertices covered by the matching can be labelled in such way that $E_M = \{x_iy_i : i = 1, \dots, \ell\}$ and x_i and x_j are at an even distance in T for all $1 \leq i < j \leq \ell$.*

Indeed, consider a legal two-coloring $\chi\colon V(T) \to [2]$. Label those endpoints of the matching edges with x_i that are in $\chi^{-1}(1)$ and label the other endpoints by y_i. Clearly, the distance in T between any x_i and x_j is even, since they belong to the same color class.

Given a coloring $c\colon E(K_n) \to [3]$, we denote by K_n^1 the spanning subgraph of K_n such that $ij \in K_n^1$ if and only if $c(ij) = 1$.

Lemma 2.8. *For every $\gamma > 0$ there exists an ε_0 such that for every $\varepsilon \leq \varepsilon_0$, there exists a natural number K_0 such that for all $N = (2 + \gamma)n \geq K_0$ and* *for every coloring $c\colon E(K_N) \to [3]$, there exist a color (say color 1), integers ℓ, ℓ', k with $\ell, \ell' \le k \le K_0$ and $\ell \ge (1-\gamma/4)k/4$, a tree T on vertex set $\{x_1,\dots,x_\ell,y_1,\dots,y_\ell,z_1,\dots,z_{\ell'}\}$ containing a matching M with edge set $E_M = \{x_iy_i\colon i=1,\dots,\ell\}$ with an even distance in T between any x_i and x_j for all i and j, such that there exists a partition $(V_i)_{i\in[k]}$ of $V(K_N)$ such that K_N^1 is $(\varepsilon,1/3)$-regular on T and $|V_1| = \ldots = |V_k| \ge (1-\varepsilon)N/k$.*

Proof. Fix $\gamma > 0$ and set $\delta = \gamma/4$. Let ε_0 and k_0 be the constants obtained by Lemma 2.6 applied with δ. Fix $\varepsilon < \varepsilon_0$ and let K_0 be obtained by an application of the Regularity Lemma (Lemma 2.1) with parameters ε and k_0. Finally let $N = (2+\gamma)n \ge K_0$ be given.

Consider an arbitrary 3-coloring $\chi_{K_N}\colon E(K_N) \to [3]$ of the edges of K_N and spanning subgraphs G_1, G_2 and G_3 of K_N where $ij \in G_s$ if and only if $\chi_{K_N}(ij) = s$, for $s = 1,2,3$. Owing to the Regularity Lemma, there is a partition $V_0, V_1, \dots, V_k$ of the vertices of K_N such that $|V_i| = m \ge (1-\varepsilon)N/k$ for $1 \le i \le k$ and more than $(1-\varepsilon)\binom{k}{2}$ pairs $\{V_i, V_j\}$ for $1 \le i < j \le k$ are ε-regular G_1, G_2 and G_3, where $k_0 \le k \le K_0$.

We define the following reduced graph: let R be the graph with vertex set $[k]$ where $ij \in E(R)$ if and only if $\{V_i, V_j\}$ is ε-regular in G_1, G_2 and G_3. Thus, $|E(R)| \ge (1-\varepsilon)\binom{k}{2}$. Therefore, we know that R is a graph on k vertices with density at least $(1-\varepsilon)$. Now we define a coloring $\chi_R\colon E(R) \to [3]$ of the edges of R such that $\chi_R(i,j) = s$ if $s \in [3]$ is the biggest integer such that $|E_{G_s}(V_i,V_j)| \ge |E_{G_r}(V_i,V_j)|$ for $1 \le r \le 3$, i.e., the edge ij receives one of the colors that appears in most edges of $E_{K_N}(V_i,V_j)$ with respect to the coloring χ_{K_N} of $E(K_N)$. Clearly, if $\chi_R(ij) = s$, then $|E_{G_s}(V_i,V_j)| \ge |V_i||V_j|/3$.

Since $k \ge k_0$ and the density of R is at least $(1-\varepsilon)$, by Lemma 2.6, we know that R contains a monochromatic tree T that contains a matching M of size $\ell \ge (1-\delta)k/4$. Without loss of generality we may assume that the edges of T are colored with color 1. By Fact 2.7 we can label $M = (\{x_i, x_j\})_i$ such that x_i and x_j are at even distance in T for $1 \le i < j \le \ell$.

Let $\{z_1, \dots, z_{\ell'}\}$ be the vertices of T that are not covered by edges of the matching M. Since all the edges of T are present in R we know that, for all $ij \in E(T)$, the pairs $\{V_i, V_j\}$ are ε-regular in G_1 with $|E_{G_1}(V_i,V_j)| \ge |V_i||V_j|/3$. Thus we are done, since we can define G as the graph composed by the classes V_i for every $i \in V(T)$ and with edge set $E_{G_1}(V_i,V_j)$ between every pair. □

2.3. Balanced intervals

By definition, given $\beta > 0$ and a natural Δ, a balanced (β,Δ)-graph H has a 2-coloring of its vertices that uses both colors similarly often *in total*, but this does not have to be true *locally*. In this section we show how to balance H so that the two colors appear in approximately the same number of vertices also locally.

Given a graph $H = (W,E)$ with $W = \{w_1,\dots,w_n\}$, let $\chi\colon W \to [2]$ be a 2-coloring. Define the function C_i such that if $W' \subset W$ then, for $i = 1,2$, we have $C_i(W') = |\chi^{-1}(i) \cap W'|$. We say that χ is a *β-balanced* coloring of W if $1-\beta \le C_1(W)/C_2(W) \le 1+\beta$. A subset $I \subset W$ is called *interval* if there exists

$p < q$ such that $I = \{w_p, w_{p+1}, \ldots, w_q\}$. Finally, let ℓ' and $\hat{\ell}$ be positive integers with $\ell' \leq \hat{\ell}$ and let $\sigma\colon [\ell'] \to [\hat{\ell}]$ be an injection. Consider a partition of W in a set of intervals $\mathcal{I} = \{I_1, \ldots, I_{\hat{\ell}}\}$. We define $C_i(\mathcal{I}, \sigma, a) = \sum_{j=1}^{a} C_i(I_{\sigma(j)})$ for $i = 1, 2$. If it is clear what partition we are considering then we write $C_i(\sigma, a)$ for simplicity.

Given a graph $H = (W, E)$, let $c\colon W \to [2]$ be a coloring of W such that H is globally balanced. Roughly speaking, the next lemma states that every partition of W in intervals of almost the same size can be rearranged in some way that, after the rearrangement, if we remove the "last" intervals, then, in the subgraph of H induced by the remaining vertices, the difference between the number of vertices w with $c(w) = 1$ and those w with $c(w) = 2$ is "small".

Lemma 2.9. *For every integer $\hat{\ell} \geq 1$ there exists n_0 such that if $H = (W, E)$ is a graph with $W = \{w_1, \ldots, w_n\}$ with $n \geq n_0$, then every β-balanced 2-coloring χ of W with $\beta \leq 2/\hat{\ell}$, and every partition of W in intervals $I_1, \ldots, I_{\hat{\ell}}$ with sizes $|I_1| \leq \ldots \leq |I_{\hat{\ell}}| \leq |I_1| + 1$ there exists a permutation $\sigma\colon [\hat{\ell}] \to [\hat{\ell}]$ such that for every $1 \leq i \leq \hat{\ell}$ we have*

$$|C_1(\sigma, i) - C_2(\sigma, i)| \leq \frac{n}{\hat{\ell}} + 1.$$

Proof. Fix $\hat{\ell} \geq 1$ and set $n_0 = 2\hat{\ell}^3$. Let $H = (W, E)$ be a graph such that $W = \{w_1, \ldots, w_n\}$ with $n \geq n_0$. Fix a β-balanced coloring χ of W and a partition of W in intervals $I_1, \ldots, I_{\hat{\ell}}$ with $|I_1| \leq \ldots \leq |I_{\hat{\ell}}| \leq |I_1| + 1$ where $\beta \leq 2/\hat{\ell}$.

Let us construct the permutation σ iteratively. We can take any integer on $[\hat{\ell}]$ to be $\sigma(1)$, as long as the size of the intervals is at most $n/\hat{\ell}+1$. Now suppose we have defined $\sigma(1), \ldots, \sigma(i)$ in such a way that $|C_1(\sigma, i) - C_2(\sigma, i)| \leq n/\hat{\ell}+1$, where $i \leq \hat{\ell} - 1$.

If $C_1(\sigma, i) = C_2(\sigma, i)$, then clearly $\sigma(i+1)$ can be defined as being any of the remaining integers on $[\hat{\ell}]$. So, w.l.o.g. assume that $C_1(\sigma, i) = C_2(\sigma, i) + k$, with $1 \leq k \leq n/\hat{\ell} + 1$. But since $C_1(\sigma, i) + C_2(\sigma, i) \leq i(n/\hat{\ell} + 1)$, we can conclude that

$$C_2(\sigma, i) \leq \frac{in}{2\hat{\ell}} - \frac{k - i}{2}. \tag{3}$$

We will prove that there exists some $r \in [\hat{\ell}] \setminus \bigcup_{j=1}^{i} \sigma(j)$ with $C_2(I_r) \geq k/2$. Suppose by contradiction that $C_2(I_r) < k/2$ for all integers $r \in [\hat{\ell}] \setminus \bigcup_{j=1}^{i} \sigma(j)$.

This fact together with (3) implies the following.

$$\begin{aligned} C_2(W) &\le C_2(\sigma,i) + (\hat{\ell}-i)\frac{k}{2} \\ &= \frac{in}{2\hat{\ell}} + (\hat{\ell}-i-1)\frac{k}{2} + \frac{i}{2} \\ &\le \left(\frac{\hat{\ell}-1}{\hat{\ell}}\right)\frac{n}{2} + \frac{\hat{\ell}}{2} \\ &= \left(1 - \left(\frac{1}{\hat{\ell}} - \frac{\hat{\ell}}{n}\right)\right)\frac{n}{2}, \end{aligned} \tag{4}$$

where the last inequality holds as long as $k \le n/\hat{\ell}+1$ and $i \le \hat{\ell}-1$.

Since $C_1(W) + C_2(W) = n$, using (4) we know that

$$\begin{aligned} \frac{C_1(W)}{C_2(W)} &= \frac{n}{C_2(W)} - 1 \\ &\ge 1 + \frac{2(n-\hat{\ell}^2)}{n(\hat{\ell}-1)+\hat{\ell}^2} \\ &> 1+\beta, \end{aligned}$$

where the last inequality follows by the choice of n_0, as long as $\beta \le 2/\hat{\ell}$. But this contradicts the β-balancedness of the coloring χ of W. Therefore, we know that there exists $r \in [\hat{\ell}] \setminus \bigcup_{j=1}^{i} \sigma(j)$ with $C_2(I_r) \ge k/2$. Set $\sigma(i+1) = r$. Then

$$\begin{aligned} C_1(\sigma,i+1) &= C_1(\sigma,i) + C_1(I_r) \\ &\le (C_2(\sigma,i)+k) + \left(\frac{n}{\hat{\ell}} + 1 - \frac{k}{2}\right) \\ &= \left(C_2(\sigma,i) + \frac{k}{2}\right) + \frac{n}{\hat{\ell}} + 1 \\ &\le C_2(\sigma,i+1) + \frac{n}{\hat{\ell}} + 1. \end{aligned}$$

From the above inequality, since $C_1(\sigma,i+1) \ge C_2(\sigma,i+1) - (n/\hat{\ell}+1)$, we conclude that $|C_1(\sigma,i+1) - C_2(\sigma,i+1)| \le n/\hat{\ell}+1$. □

Let $H = (W,E)$ be a graph with $W = \{w_1,\dots,w_n\}$ and let $\chi\colon W \to [2]$ be a coloring of W. Consider a partition of W in a set of intervals $\mathcal{I} = \{I_1,\dots,I_{\hat{\ell}}\}$. We define $C_i(I,\sigma,a,b) = \sum_{j=a}^{b} C_i(I_{\sigma(j)})$ for $i=1,2$. If it is clear what partition we are considering then we write $C_i(\sigma,a,b)$ for simplicity.

Corollary 2.10. *For every integer $\hat{\ell} \ge 1$ there exists n_0 such that if $H = (W,E)$ is a graph with $W = \{w_1,\dots,w_n\}$ with $n \ge n_0$, then every β-balanced 2-coloring χ of W with $\beta \le 2/\hat{\ell}$, and every partition of W in intervals $I_1,\dots,I_{\hat{\ell}}$ with sizes*

$|I_1| \leq \ldots \leq |I_{\hat\ell}| \leq |I_1| + 1$ *there exists a permutation* $\sigma\colon [\hat\ell] \to [\hat\ell]$ *such that for every integers* $1 \leq a < b \leq \hat\ell$,

$$|C_1(\sigma, a, b) - C_2(\sigma, a, b)| \leq 2\left(\frac{n}{\hat\ell} + 1\right). \tag{5}$$

Proof. Fix $\hat\ell \geq 1$ and let n_0 be obtained by Lemma 2.9 applied with $\hat\ell$. Let $H = (W, E)$ be a graph with $W = \{w_1, \ldots, w_n\}$ with $n \geq n_0$. Now fix a β- balanced 2-coloring χ of W and a partition of W in intervals $I_1, \ldots, I_{\hat\ell}$ with $|I_1| \leq \ldots \leq |I_{\hat\ell}| \leq |I_1| + 1$, where $\beta \leq 2/\hat\ell$.

Let σ be the permutation given by Lemma 2.9. Fix integers $1 \leq a < b \leq \hat\ell$ and suppose w.l.o.g. that $C_1(\sigma, a, b) \geq C_2(\sigma, a, b)$. Therefore

$$\begin{aligned} C_1(\sigma, a, b) &= C_1(\sigma, b) - C_1(\sigma, a-1) \\ &\leq (C_2(\sigma, b) + n/\hat\ell + 1) - (C_2(\sigma, a-1) - (n/\hat\ell + 1)) \\ &= C_2(\sigma, a, b) + 2(n/\hat\ell + 1). \end{aligned}$$

□

The next result, the main result of this subsection, guarantees the local balancedness that we need.

Lemma 2.11. *For every* $\xi > 0$ *and every integer* $\hat\ell \geq 1$ *there exists* n_0 *such that if* $H = (W, E)$ *is a graph with* $W = \{w_1, \ldots, w_n\}$ *with* $n \geq n_0$*, then every* β*-balanced* 2*-coloring* χ *of* W *with* $\beta \leq 2/\hat\ell$*, and every partition of* W *in intervals* $I_1, \ldots, I_{\hat\ell}$ *with* $|I_1| \leq \ldots \leq |I_{\hat\ell}| \leq |I_1| + 1$ *there exists a permutation* $\sigma\colon [\hat\ell] \to [\hat\ell]$ *such that for every integers* $1 \leq a < b \leq \hat\ell$ *with* $b - a \geq 7/\xi$*, we have*

$$|C_1(\sigma, a, b) - C_2(\sigma, a, b)| \leq \xi C_2(\sigma, a, b),$$

Proof. Fix $\xi > 0$, $\hat\ell \geq 1$ and let n_0 be obtained by Corollary 2.10 applied with $\hat\ell$. Let $H = (W, E)$ be a graph with $W = \{w_1, \ldots, w_n\}$ with $n \geq \max\{n_0, (4/3)\hat\ell\}$ and fix a β-balanced 2-coloring χ of W and a partition of W in intervals $I_1, \ldots, I_{\hat\ell}$ with $|I_1| \leq \ldots \leq |I_{\hat\ell}| \leq |I_1| + 1$ where $\beta \leq 2/\hat\ell$.

Let σ be the permutation given by Corolary 2.10. Fix integers $1 \leq a < b \leq \hat\ell$ such that $b - a > 7/\xi$. Note that, by Corollary 2.10,

$$|C_1(\sigma, a, b) - C_2(\sigma, a, b)| \leq 2(n/\hat\ell + 1). \tag{6}$$

The above inequality and the fact that $C_1(\sigma, a, b) + C_2(\sigma, a, b) \geq (b - a)(n/\hat\ell)$ implies

$$C_2(\sigma, a, b) \geq \left(\frac{b-a}{2}\right)\frac{n}{\hat\ell} - (n/\hat\ell + 1).$$

By the choice of a, b and n_0, we have

$$C_2(\sigma, a, b) \geq (2/\xi)(n/\hat{\ell} + 1). \tag{7}$$

Putting inequalities (6) and (7) together we conclude the proof. $\square$

3. Proof of the main result

Before go into the details of the proof of Theorem 1.3 we give some brief overview discussing the main ideas of the proof and explaining how to connect the results of Section 2.

Overview of the proof of Theorem 1.3

For every $\gamma > 0$ and sufficiently large n, given an arbitrary edge coloring of K_N for $N = (2+\gamma)n$ we want to prove that if H is a (β, Δ)-balanced graph on n vertices, then we always find a monochromatic copy of H in K_N.

The strategy to prove Theorem 1.3 is to apply the Embedding Lemma (Lemma 2.3) to find the desired copy of H in K_N. In order to do this we use Lemma 2.8 to find a monochromatic subgraph G of K_N composed by sufficiently dense regular pairs. So, using Facts 2.4 and 2.5 it is easy to see that deleting some vertices of G we can find a monochromatic graph $G' \subset G$ which has a regular partition containing super-regular pairs covering $(1 + o(1))n$ vertices.

In the second part of the proof we carefully construct a partition of $V(H)$ and we make use of Lemma 2.11 to show that this partition is compatible with the partition of G'. Then, we can apply the Embedding Lemma to find the monochromatic copy of H, concluding the proof.

Proof of Theorem 1.3

Let $\gamma > 0$ and $\Delta \geq 1$ be given. Lemma 2.8 applied with γ gives ε_0. Next we apply Lemma 2.3 with $d = 1/4$ and Δ to get ε_1. Set

$$\varepsilon = \min\{\varepsilon_0, \varepsilon_1/2, \gamma/19\}.$$

Since $\varepsilon \leq \varepsilon_0$, Lemma 2.8 gives to us a natural number K_0. Fix $\xi = \gamma/304$ and let n_0 be obtained by an application of Lemma 2.11 with parameters ξ and K_0. Set

$$\beta = \varepsilon\xi(1 + 2\xi)/36\Delta^2 K_0^2.$$

Let $H = (W, E_H)$ be a balanced (β, Δ)-graph on n vertices. Now put $N = \lfloor (2+\gamma)n \rfloor$, where $N \geq \max\{n_0, K_0\}$. Consider an arbitrary coloring $\chi_{K_N} \colon E(K_N) \to [3]$ of the edges of K_N. We want to show that every such coloring yields a monochromatic copy of H.

Partitioning the vertices of K_N.

Next we find a monochromatic and sufficiently regular subgraph G' of K_N. By Lemma 2.8, there are a color (say color 1), integers ℓ, ℓ', k with $\ell, \ell' \leq k \leq K_0$

and $\ell \geq (1-\gamma/4)k/4$, a tree T on vertex set $\{x_1,\ldots,x_\ell,y_1,\ldots,y_\ell,z_1,\ldots,z_{\ell'}\}$ containing a matching M with edge set $E_M = \{x_iy_i\colon i=1,\ldots,\ell\}$ with an even distance in T between any x_i and x_j for all i and j, such that there exists a partition $(V_i)_{i\in[k]}$ of $V=V(K_N)$ such that K_N^1 is $(\varepsilon,1/3)$-regular on T and $|V_1|=\ldots=|V_k|=m$, where $m\geq(1-\varepsilon)N/k$. Let G_T be the subgraph of K_N^1 induced by the classes in $(V_i)_{i\in[k]}$ corresponding to the vertices of T.

In order to apply the Embedding Lemma, we need the classes of G_T that correspond to the matching edges to form super-regular pairs and the other pairs of classes should be sufficiently regular. We can ensure this by deleting some vertices of G_T. In fact, applying Fact 2.5 and, after that, Fact 2.4, it is easy to see that we find a subgraph $G'\subset G_T$ with classes $A_1,\ldots,A_\ell$, $B_1,\ldots,B_\ell$, $C_1,\ldots,C_{\ell'}$ of size at least $(1-\varepsilon)m$ corresponding, respectively, to the vertices $x_1,\ldots,x_\ell,y_1,\ldots,y_\ell,z_1,\ldots,z_{\ell'}$ of the tree T, such that the bipartite graphs induced by A_i and B_i are $(2\varepsilon,1/3-\varepsilon)$-super-regular and the bipartite graphs induced by all the other pairs are $(2\varepsilon,1/3-\varepsilon)$-regular. Furthermore, let $D_{\min}$ be the set with the smallest cardinality among the sets in $A_1,\ldots,A_\ell$, $B_1,\ldots,B_\ell$, $C_1,\ldots,C_{\ell'}$. Since $\varepsilon\leq\gamma/19$, $m\geq(1-\varepsilon)N/k$ and $\ell\geq(1-\gamma/4)k/4$, one can see that

$$|D_{\min}|\geq(1+\gamma/152)n/2\ell. \tag{8}$$

Partitioning the vertices of H.

Now it is time to construct a partition of W ready for the application of Lemma 2.3. Since H is a balanced (β,Δ)-graph, there exists a coloring $\chi_H\colon V(H)\to[2]$ such that $\big||\chi^{-1}(1)|-|\chi^{-1}(2)|\big|\leq\beta|\chi^{-1}(2)|$.

Let $w_1,\ldots,w_n$ be the ordering of W such that $|i-j|\leq\beta$ for every $w_iw_j\in E_H$ and let $\hat\ell$ be the smallest integer dividing n with $\hat\ell\geq(7K_0/\xi)+\ell\geq\ell(7/\xi+1)$. Consider the partition of $V(H)$ in intervals $I_1,\ldots,I_{\hat\ell}$ with $|I_1|=\ldots=|I_{\hat\ell}|=n/\hat\ell$ taking this ordering into account, i.e., $I_i=w_{(i-1)n/\hat\ell+1},\ldots,w_{in/\hat\ell}$ for $i=1,\ldots,\hat\ell$. By Lemma 2.11, since $\beta\leq2/\hat\ell$, there exists a permutation $\sigma\colon[\hat\ell]\to[\hat\ell]$ such that

$$|C_1(\sigma,a,b)-C_2(\sigma,a,b)|\leq\xi C_2(\sigma,a,b)$$

for all integers $1\leq a<b\leq\hat\ell$ with $b-a\geq7/\xi$. Define $a_i=(i-1)\hat\ell/\ell+1$ and $b_i=i\hat\ell/\ell$ and consider the blocks $J_i=\{I_{\sigma(a_i)},I_{\sigma(a_i+1)},\ldots,I_{\sigma(b_i)}\}$ for $i=1,\ldots,\ell$. We write $C_1(J_i)$ for $C_1(\sigma,a_i,b_i)$ and $C_2(J_i)$ for $C_2(\sigma,a_i,b_i)$. Thus, for $i=1,\ldots,\ell$, since $b_i-a_i=\hat\ell/\ell+1\geq7/\xi$, we have, by Lemma 2.11,

$$|C_1(J_i)-C_2(J_i)|\leq\xi C_2(J_i), \tag{9}$$

Recall we have found a tree T on vertex set $\{x_1,\ldots,x_\ell,y_1,\ldots,y_\ell,z_1,\ldots,z_{\ell'}\}$ containing matching edges $E_M=\{x_iy_i\colon i=1,\ldots,\ell\}$ such that the distance in T between any x_i and x_j for all i and j is even. Our partition of W will be composed by classes $X_1,\ldots,X_\ell,Y_1,\ldots,Y_\ell,Z_1,\ldots,Z_{\ell'}$ corresponding, respectively, to $x_1,\ldots,x_\ell,y_1,\ldots,y_\ell,z_1,\ldots,z_{\ell'}$.

For every $i = 1, \dots, \ell$, we will put most of the vertices of J_i in the classes X_i and Y_i, depending on the color they received from χ_H. The remaining vertices will be distributed in order to make it possible to "walk" between the matching classes.

We divide each interval I_i in two parts. The first one, called *link* of I_i, is denoted by L_i. The links are responsible to make the connections between the matching classes. For the last interval, we set $L_{\hat{\ell}} = \emptyset$. For $1 \leq i \leq \hat{\ell} - 1$, if I_i and I_{i+1} are in the same block J_r, then $L_i = \emptyset$.

Suppose that $I_i \in J_r$ and $I_{i+1} \in J_s$ with $r \neq s$ and $1 \leq i \leq \hat{\ell}-1$. Let $P_T(r,s)$ be the path of T between x_r and x_s and consider the path $P_T^{\text{int}}(r,s) \subset P_T(r,s)$ obtained by excluding the vertices of the set $\{x_r, y_r, x_s, y_s\}$ from $P_T(r,s)$, i.e., $P_T^{\text{int}}(r,s)$ is the "internal" part of the path of T that one should use to reach x_s from x_r. For simplicity set $t_{r,s} = |P_T^{\text{int}}(r,s)|$. In this case, we divide the $(t_{r,s}+1)\beta n$ last vertices of I_i in $t_{r,s}+1$ "pieces" of size βn, respecting their sequence in the interval, where the j-th piece is denoted by $L_i(j)$ for $1 \leq j \leq t_{r,s}+1$, that is,

$$L_i(j) = w_{(i-(t_{r,s}+2-j)\beta\hat{\ell})n/\hat{\ell}+1}, \dots, w_{(i-(t_{r,s}+1-j)\beta\hat{\ell})n/\hat{\ell}}.$$

We put $L_i = \{L_i(1), \dots, L_i(t_{r,s}), L_i(t_{r,s}+1)\}$.

Since we have described the links, we can now define the main part of the intervals. We define $\text{KE}_i = I_i \setminus L_i$ as the *kernel* of the interval I_i, which will be placed on the matching classes X_i and Y_i.

We have to construct the clusters that will compose the partition of H. Initially, let each cluster in $\{X_1, \dots, X_\ell, Y_1, \dots, Y_\ell, Z_1, \dots, Z_{\ell'}\}$ be empty. Consider the block J_i for every $1 \leq i \leq \ell$. For each interval $I_p \in J_i$ we include in X_i all the vertices w of the kernel KE_p with $\chi_H(w) = 1$ and we include in Y_i all the vertices w of KE_p with $\chi_H(w) = 2$.

The next step is to accommodate all the links. Consider the interval I_i for $1 \leq i \leq \hat{\ell} - 1$ and assume that I_i is in J_r and I_{i+1} is in J_s with $r \neq s$, otherwise the link we are looking for is empty and there is nothing to do. Denote the internal path $P_T^{\text{int}}(r,s)$ of $P_T(r,s)$ by $\{u_1, \dots, u_{t_{r,s}}\}$ and let u_0 and $u_{t_{r,s}+1}$ be, respectively, the vertices of T connected to u_1 and $u_{t_{r,s}}$ in $P_T(r,s)$.

Now we will show how it is possible to "walk" between the matching classes. note that u_0 can be either x_r or y_r. Without loss we assume that $u_0 = x_r$. For $1 \leq j \leq t_{r,s}+1$, we put the vertices w of $L_i(j)$ with $\chi_H(w) = 1$ in the corresponding class of u_{j-1} if j is even, and in the corresponding class of u_j if j is odd. For those w with $\chi_H(w) = 2$, we do the other way around, i.e., we put them in the corresponding class of u_j if j is even, and in the corresponding class of u_{j-1} if j is odd. Since x_i and x_j are at an even distance for all $1 \leq i < j \leq \ell$ and the links have size βn, we know that there is no edges inside the clusters and if there is an edge between two clusters, then the corresponding edge is present in T.

Applying the Embedding Lemma.

Here we will show that the vertex partition of W is $(2\varepsilon_1, T, M)$-compatible with the partition of $V(G')$ we constructed before. Thus, we can apply the Embedding Lemma to find the desired monochromatic copy of H in K_N.

The first step is to bound by above the size of each cluster in the partition $\{X_1, \ldots, X_\ell, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_{\ell'}\}$ of W. Note that, for every $1 \leq i \leq \ell$, we have $C_1(J_i) + C_2(J_i) = n/\ell$. Using this fact and (9) one can easily obtain that, for every $1 \leq i \leq \ell$,

$$(1-\xi)\frac{n}{2\ell} \leq C_1(J_i), C_2(J_i) \leq (1+\xi)\frac{n}{2\ell}. \tag{10}$$

By the construction, every X_i (Y_i) is composed only by vertices v with $\chi(v) = 1$ ($\chi(v) = 2$). Furthermore, these vertices can come from the kernels and at most two pieces of each link. Then,

$$\begin{aligned} |X_i|, |Y_i| &\leq (1+\xi)\frac{n}{2\ell} + 2\hat{\ell}\beta n \\ &= \left(1 + \xi + 4\ell\hat{\ell}\beta\right)\frac{n}{2\ell} \\ &\leq |D_{\min}|, \end{aligned} \tag{11}$$

where the last inequality follows by inequality (8) and the choice of ξ, β and $\hat{\ell}$.

For the clusters Z_i, for $1 \leq i \leq \ell'$, we know that they are composed only by vertices in at most two pieces of each link. Thus,

$$\begin{aligned} |Z_i| &\leq 2\hat{\ell}\beta n \\ &= (4\ell\hat{\ell}\beta)\frac{n}{2\ell} \\ &\leq \frac{\varepsilon}{\Delta^2}|D_{\min}|, \end{aligned} \tag{12}$$

where the last inequality follows by inequality (8) and the choice of β and $\hat{\ell}$.

Now we can check that the partitions of W and $V(G')$ are compatible. Based on Definition 2.2 we define the sets U_j and U'_j for $1 \leq j \leq 2\ell + \ell'$ with respect to the partition $\{X_1, \ldots, X_\ell, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_{\ell'}\}$ of W. Define $W_j = X_j$ if $1 \leq j \leq \ell$, $W_j = Y_{j-\ell}$ if $\ell + 1 \leq j \leq 2\ell$, and $W_j = Z_{j-2\ell}$ if $2\ell + 1 \leq j \leq 2\ell + \ell'$. Then, we will verify that the four conditions of Definition 2.2 hold:

(i) By the construction of the partition of W, if there is an edge between two clusters, then the corresponding edge is present in T.

(ii) By (8), we know that every set D in the partition $\{A_1, \ldots, A_\ell, B_1, \ldots, B_\ell, C_1, \ldots, C_{\ell'}\}$ of $V(G')$ has size $|D| \geq (1+\gamma/152)n/2\ell$. So, inequalities (11) and (12) show that condition (ii) holds.

(iii) Fix $1 \leq j \leq 2\ell + \ell'$. Define U_j as the set of vertices of W_j with neighbours in some W_k with $j \neq k$ and $\{j, k\} \notin M$. We divide in two cases:

  (a) $2\ell+1 \leq j \leq 2\ell+\ell'$: We have $U_j = Z_{j-2\ell}$. By (12), $|U_j| \leq \varepsilon|D_{\min}|/\Delta$.

(b) $1 \leq j \leq 2\ell$: In this case, U_j is composed only by neighbours of vertices in exactly one set of $\{Z_1, \ldots, Z_{\ell'}\}$. Thus, since Δ if the maximum degree of H, by (12), we conclude that $|U_j| \leq \varepsilon|D_{\min}|/\Delta$.

Thus, for every $j = 1, \ldots, 2\ell + \ell'$ we have

$$|U_j| \leq \frac{\varepsilon}{\Delta}|D_{\min}|, \tag{13}$$

which shows that condition (iii) holds.

(iv) Define the set $U'_j = N_H(U) \cap (W_j \setminus U)$, where $U = \bigcup_{i=1}^{2\ell+\ell'} U_i$. Consider the following cases.

(a) $2\ell + 1 \leq j \leq 2\ell + \ell'$: Note that since every vertex of $Z_{j-2\ell}$ belongs to U_j, we have $U'_j = \emptyset$. Thus, it is obvious that $|U'_j| \leq |D_{\min}|$.

(b) $\ell + 1 \leq j \leq 2\ell$: Here, $U'_j \subset W_j = Y_{j-\ell}$. Then, U'_j is composed only by neighbours of $U_{j-\ell} \subset X_{j-\ell}$. Then, using (13), we have $|U'_j| \leq \Delta|U_{j-\ell}| \leq \varepsilon|D_{\min}|$.

(c) $1 \leq i \leq \ell$: This case is analogous to case (b).

Since we proved that the four conditions of Definition 2.2 hold, the partition $\{X_1, \ldots, X_\ell, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_{\ell'}\}$ of W is $(2\varepsilon, T, M)$-compatible (then, it is clearly (ε_1, T, M)-compatible) with $\{A_1, \ldots, A_\ell, B_1, \ldots, B_\ell, C_1, \ldots, C_{\ell'}\}$, which is a partition of $V(G')$. Then, by Lemma 2.3, we conclude that $H \subset G'$. This finishes the proof, since G' is a monochromatic subgraph of K_N. □

4. Sketch of the proof of Theorem 1.2

We want to prove that for every $\gamma > 0$ and natural number Δ, there exists a constant $\beta > 0$ such that for every sufficiently large (β, Δ)-graph H with a legal 2-coloring $\chi_H : V(H) \to [2]$ where $t_1 = |\chi_H^{-1}(1)|$ and $t_2 = |\chi_H^{-1}(2)|$, with $t_1 \leq t_2$, we can find a monochromatic copy of H in every edge coloring of $E(K_N)$ with $N = (1 + \gamma)\max\{2t_1 + t_2, 2t_2\}$. Let H be such a graph and assume $2t_1 \geq t_2$.

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.3. Here we also embed H in parts, considering a partition of a monochromatic subgraph G of K_N. The partition we need is composed by a special class W and classes $X_1, Y_1, \ldots, X_m, Y_m$ corresponding to a "large" matching M with matching edges $E_M = \{x_i, y_i : i = 1, \ldots, m\}$ such that the pairs $\{X_i, Y_i\}$ are super-regular and the pairs $\{X_i, W\}$ are regular, for $i = 1, \ldots, m$.

The problem in the preparation of the host monochromatic graph G is the fact that H is not as balanced as it is in the setup of Theorem 1.3. So, in order to embed H in G we need that $|Y_i|/|X_i| = t_2/t_1$. Fortunately, by a result from [13], since $t_2/t_1 \leq 1/2$ in the case we are considering, we can find such a monochromatic graph G. Using Fact 2.5 we can easily make the matching pairs super-regular.

Now we have to prepare the graph H for the embedding. We consider the ordering of its vertices respecting the bandwidth condition and divide the set of vertices into intervals. Thus, we can find a permutation of such intervals such that blocks of intervals fit into the super-regular pairs of G. Then, using few vertices we can "walk" from one super-regular pair to another as done in the proof of Theorem 1.3 and we are done.

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