Coloring d-Embeddable k-Uniform Hypergraphs

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Abstract

We extend the scenario of the Four Color Theorem in the following way. Let $\mathcal{H}_{d,k}$ be the set of all k-uniform hypergraphs that can be linearly embedded into $\mathbb{R}^d$. We investigate lower and upper bounds on the maximum (weak and strong) chromatic number of hypergraphs in $\mathcal{H}_{d,k}$. For example, we can prove that for $d \geq 3$ there are hypergraphs in $\mathcal{H}_{2d-3,d}$ on n vertices whose weak chromatic number is $\Omega(\log n / \log\log n)$, whereas the weak chromatic number for n-vertex hypergraphs in $\mathcal{H}_{d,d}$ is bounded by $\mathcal{O}(n^{(d-2)/(d-1)})$ for $d \geq 3$.

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Extended Abstract

The Four Color Theorem [1, 2] has been one of the driving forces in Discrete Mathematics and its theme has inspired many variations. For example, the chromatic number of graphs that are embedabble into a surface of fixed genus has been intensively studied by Heawood [7], Ringel and Youngs [11], and many others.

Here, we consider k-uniform hypergraphs that are embeddable into $\mathbb{R}^d$ in such a way that their edges do not intersect (see Definition 1 below). For $k = d = 2$ the problem specializes to graph planarity. For $k = 2$ and $d \geq 3$ it is not a very interesting question because for any $n \in \mathbb{N}$ the vertices of the complete graph K_n can be embedded into $\mathbb{R}^3$ using arbitrary points on the moment curve $t \mapsto (t, t^2, t^3)$.

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As a consequence, we focus our attention on hypergraphs, which are in general not embeddable into any specific dimension. Some properties of these hypergraphs (or more generally simplicial complexes) have been investigated (see e. g. [4, 8, 9, 15]), but to our surprise, we have not been able to find any bounds on their (vertex-)chromatic number. However, Grünbaum and Sarkaria (see [6, 12]) have differently generalized the concept of graph colorings to simplicial complexes by coloring faces. They also bound this face-chromatic number subject to embeddability constraints.

We now quickly recall and introduce some useful notation. The pair $H = (V, E)$ is a *k-uniform hypergraph* if the vertex set V is a finite set and the edge set E consists of k-element subsets of V.

Let H be a k-uniform hypergraph. A function $\kappa : V(H) \to \{1, \dots, c\}$ is said to be a *strong c-coloring* if for all $e \in E(H)$ the property $|\kappa(e)| = k$ holds. The function κ is said to be a *weak c-coloring* if $|\kappa(e)| > 1$ for all $e \in E(H)$. The *strong and weak chromatic number* of H is denoted by $\chi^{\mathsf{s}}(H)$ and $\chi^{\mathsf{w}}(H)$, respectively. For graphs, weak and strong colorings are equivalent.

We next define what we mean when we say that a hypergraph is embedabble into $\mathbb{R}^d$. Here, *aff* denotes the affine hull of a set of points and *conv* the convex hull.

Definition 1 (*d*-embeddings)
Let H be a k-uniform hypergraph and $d \in \mathbb{N}$. A (linear) *embedding* of H into $\mathbb{R}^d$ is a function $\varphi : V(H) \to \mathbb{R}^d$, where $\varphi(A)$ for $A \subseteq V(H)$ is to be interpreted pointwise, such that $\dim \operatorname{aff} \varphi(e) = k - 1$ for all $e \in E(H)$ and $\operatorname{conv} \varphi(e_1 \cap e_2) = \operatorname{conv} \varphi(e_1) \cap \operatorname{conv} \varphi(e_2)$ for all $e_1, e_2 \in E(H)$.

A k-uniform hypergraph H is said to be *d-embeddable* if there exists an embedding of H into $\mathbb{R}^d$. Also, we denote by $\mathcal{H}_{d,k}$ the set of all d-embeddable k-uniform hypergraphs. By Fáry's theorem (see [5]), we have that the $k = d = 2$ case of this notion of embeddability coincides with the classical concept of planarity.

Our main results are summarized in the Tables 1 and 2, which contain upper or lower bounds for the maximum weak chromatic number of a d-embeddable k-uniform hypergraph on n vertices. All results which only follow non-trivially from prior knowledge are indexed with a number of the theorem in this extended abstract. On the other hand, the trivial entries in the tables are direct consequences of the Menger-Nöbeling Theorem (see [9, p. 295] and [10]) which characterizes for which d all k-uniform hypergraphs are d-embeddable. The main results for the maximum strong chromatic number are Proposition 2 and Theorem 3.

For $d, k, n \in \mathbb{N}$ we define $\chi^{\mathsf{s}}_{d,k}(n) = \max\{\chi^{\mathsf{s}}(H) : H \in \mathcal{H}_{d,k}, |V(H)| = n\}$ to be the maximum strong chromatic number of a d-embeddable k-uniform hypergraph on n vertices. The maximum weak chromatic number $\chi^{\mathsf{w}}_{d,k}(n)$ is defined analogously.

Proposition 2
For large n, $d \geq 3$, and $d+1 \geq k$ we have that $\chi^{\mathsf{s}}_{d,k}(n) \geq \left\lfloor \sqrt{n-d+3} \right\rfloor + d - 3$.

$d \diagdown k$	3	4	5	6	7
1	1	1	1	1	1
2	2	1	1	1	1
3	$\Omega\left(\frac{\log n}{\log\log n}\right)_{\langle 6\rangle}$	1	1	1	1
4	$\Omega\left(\frac{\log n}{\log\log n}\right)_{\langle 6\rangle}$	1	1	1	1
5	$\lceil n/2\rceil$	$\Omega\left(\frac{\log n}{\log\log n}\right)_{\langle 7\rangle}$	1	1	1
6	$\lceil n/2\rceil$	$\Omega\left(\frac{\log n}{\log\log n}\right)_{\langle 7\rangle}$	1	1	1
7	$\lceil n/2\rceil$	$\lceil n/3\rceil$	$\Omega\left(\frac{\log n}{\log\log n}\right)_{\langle 7\rangle}$	1	1
8	$\lceil n/2\rceil$	$\lceil n/3\rceil$	$\Omega\left(\frac{\log n}{\log\log n}\right)_{\langle 7\rangle}$	1	1

Table 1: Currently known *lower bounds* for the maximum weak chromatic number of a d-embeddable k-uniform hypergraph on n vertices as $n \to \infty$. The number in chevrons indicates the theorem number where we prove this bound.

Theorem 3
For large n, $d \geq 3$, and $d \geq k$ we have that $\chi^{\mathsf{s}}_{d,k}(n) = n$.

This bound is shown by constructing a sequence of hypergraphs in $\mathcal{H}_{d,k}$, which have strong chromatic number equal to the number of their vertices. Thus, except for the cases where $k = d + 1$, the maximum strong coloring problem was solved. In particular, we have shown that an unbounded number of colors can be necessary for any strong coloring of a d-embeddable hypergraph if $d > 2$.

Theorem 4
Let $d \geq 3$. Then one has

$$\chi^{\mathsf{w}}_{d,d}(n) \leq \left\lceil \left(\frac{6ed}{(d-1)!}\right)^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}} \right\rceil = \mathcal{O}\left(\left(\frac{n}{d}\right)^{\frac{d-2}{d-1}}\right).$$

Theorem 5
Let $d \geq l \geq 3$. Then one has

$$\chi^{\mathsf{w}}_{2d-l,d}(n) \leq \left\lceil (ed)^{\frac{1}{d-1}} n^{1-\frac{3^{l-1-d}}{d-1}} \right\rceil = \mathcal{O}\left(n^{1-\frac{3^{l-1-d}}{d-1}}\right).$$

These two results also holds for piecewise linear embeddings (for a definition e. g. see [8]). To prove them, we first limit the number of edges (relative to the number of vertices) that a hypergraph in $\mathcal{H}_{d,d}$ and $\mathcal{H}_{2d-l,d}$ can have. Then, an easy application of the Lovász Local Lemma [3, 14] yields the existence of a weak c-coloring if c is as high as requested.

$d \diagdown k$	3	4	5	6	7
1	1	1	1	1	1
2	2	1	1	1	1
3	$\mathcal{O}(n^{\frac{1}{2}})_{\langle 4 \rangle}$	$\mathcal{O}(n^{\frac{1}{2}})_{\langle 4 \rangle}$	1	1	1
4	$\lceil n/2 \rceil$	$\mathcal{O}(n^{\frac{2}{3}})_{\langle 4 \rangle}$	$\mathcal{O}(n^{\frac{2}{3}})_{\langle 4 \rangle}$	1	1
5	$\lceil n/2 \rceil$	$\mathcal{O}(n^{\frac{26}{27}})_{\langle 5 \rangle}$	$\mathcal{O}(n^{\frac{3}{4}})_{\langle 4 \rangle}$	$\mathcal{O}(n^{\frac{3}{4}})_{\langle 4 \rangle}$	1
6	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\mathcal{O}(n^{\frac{35}{36}})_{\langle 5 \rangle}$	$\mathcal{O}(n^{\frac{4}{5}})_{\langle 4 \rangle}$	$\mathcal{O}(n^{\frac{4}{5}})_{\langle 4 \rangle}$
7	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\mathcal{O}(n^{\frac{107}{108}})_{\langle 5 \rangle}$	$\mathcal{O}(n^{\frac{44}{45}})_{\langle 5 \rangle}$	$\mathcal{O}(n^{\frac{5}{6}})_{\langle 4 \rangle}$
8	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\lceil n/4 \rceil$	$\mathcal{O}(n^{\frac{134}{135}})_{\langle 5 \rangle}$	$\mathcal{O}(n^{\frac{53}{54}})_{\langle 5 \rangle}$

Table 2: Currently known *upper bounds* for the maximum weak chromatic number of a d-embeddable k-uniform hypergraph on n vertices as $n \to \infty$. The number in chevrons indicates the theorem number where we prove this bound.

Theorem 6
As $n \to \infty$ one has $\chi^{\mathsf{w}}_{3,3}(n) = \Omega\left(\frac{\log n}{\log \log n}\right)$.

Sketch of proof. We inductively construct a sequence of 3-uniform, 3-embeddable hypergraphs H_m which are weakly m-chromatic. Each new H_m consists of several copies of H_{m-1} and a few additional vertices (see Figure 1). The vertices are then arranged on the moment curve and the embedabbility is proven using a theorem by Shephard [13].

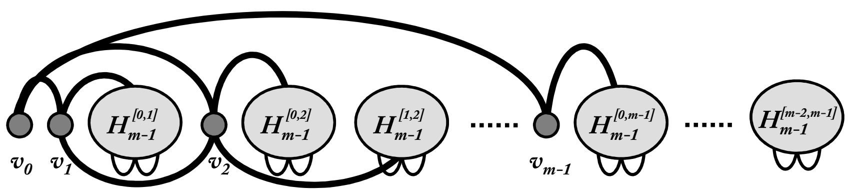

Figure 1: Construction of H_m.

Note that by monotonicity also $\chi^{\mathsf{w}}_{4,3}(n) = \Omega\left(\frac{\log n}{\log \log n}\right)$. Furthermore, it is possible to generalize this result for higher dimensions as follows.

Theorem 7
Let $d \geq 3$. Then, as $n \to \infty$ one has $\chi^{\mathsf{w}}_{2d-3,d}(n) = \Omega\left(\frac{\log n}{\log \log n}\right)$.

Note that in general there are several other notions of embeddability, the most popular being piecewise linear embeddings and general topological embeddings. A short and comprehensive introduction is given in Section 1 in [8]. Since piecewise linear and topological embeddings are more general than linear embeddings, all lower bounds for chromatic numbers can easily be transferred. Furthermore, we prove all our results on upper bounds for piecewise linear embeddings.

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References

[1] K. Appel and W. Haken, *Every planar map is four colorable. Part I: Discharging*, Illinois Journal of Mathematics **21** (1977), no. 3, 429–490.

[2] K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable. Part II: Reducibility*, Illinois Journal of Mathematics **21** (1977), no. 3, 491–567.

[3] P. Erdős and L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, Infinite and Finite Sets (to Paul Erdős on his 60th birthday), Vol. II, North-Holland (1975), 609–627.

[4] A. Flores, *Über n-dimensionale Komplexe, die im $\mathbb{R}_{2n+1}$ absolut selbstverschlungen sind*, Ergebnisse Eines Mathematischen Kolloquiums **6** (1934), 4–7.

[5] I. Fáry, *On straight-line representation of planar graphs*, Acta Scientiarum Mathematicarum Szeged **11** (1948), 229–233.

[6] B. Grünbaum, *Higher-dimensional analogs of the four-color problem and some inequalities for simplicial complexes*, Journal of Combinatorial Theory **8** (1970), no. 2, 147–153.

[7] J. C. Heawood, *Map-colour theorem*, The Quarterly Journal of Pure and Applied Mathematics **24** (1890), 332–338.

[8] J. Matoušek, M. Tancer, and U. Wagner, *Hardness of Embedding Simplicial Complexes in $\mathbb{R}^d$*, Journal of the European Mathematical Society **13** (2011), 259–295.

[9] K. Menger, *Dimensionstheorie*, Teubner, Leipzig, 1928.

[10] G. Nöbeling, *Über eine n-dimensionale Universalmenge im $\mathbb{R}_{2n+1}$*, Mathematische Annalen **104** (1931), no. 1, 71–80.

[11] G. Ringel and J. W. T. Youngs, *Solution of the Heawood map-coloring problem*, Proceedings of the National Academy of Sciences **60** (1968), no. 2, 438–445.

[12] K. S. Sarkaria, *Heawood inequalities*, Journal of Combinatorial Theory, Series A **46** (1987), no. 1, 50–78.

[13] G. C. Shephard, *A Theorem on Cyclic Polytopes*, Israel Journal of Mathematics **6** (1968), no. 4, 368–372.

[14] J. Spencer, *Asymptotic lower bounds for Ramsey functions*, Discrete Mathematics **20** (1977), 69–76.

[15] E. R. van Kampen, *Komplexe in euklidischen Räumen*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **9** (1933), no. 1, 72–78, corrections ibidem pp. 152–153.