

Coloring d -Embeddable k -Uniform Hypergraphs

Carl Georg Heise^{1,*}, Konstantinos Panagiotou², Oleg Pikhurko^{3,†}, and Anusch Taraz^{1,‡}

¹Zentrum Mathematik, Technische Universität München, Germany, {cgh, taraz}@ma.tum.de

²Mathematisches Institut, Ludwig-Maximilians-Universität München, Germany, kpanagio@math.lmu.de

³Mathematics Institute, University of Warwick, Coventry, UK, o.pikhurko@warwick.ac.uk

September 24, 2012

Abstract

This paper extends the scenario of the Four Color Theorem in the following way. Let $\mathcal{H}_{d,k}$ be the set of all k -uniform hypergraphs that can be (linearly) embedded into \mathbb{R}^d . We investigate lower and upper bounds on the maximum (weak and strong) chromatic number of hypergraphs in $\mathcal{H}_{d,k}$. For example, we can prove that for $d \geq 3$ there are hypergraphs in $\mathcal{H}_{2d-3,d}$ on n vertices whose weak chromatic number is $\Omega(\log n / \log \log n)$, whereas the weak chromatic number for n -vertex hypergraphs in $\mathcal{H}_{d,d}$ is bounded by $\mathcal{O}(n^{(d-2)/(d-1)})$ for $d \geq 3$.

1 Introduction

The Four Color Theorem [AH77, AHK77] asserts that every graph that is embeddable in the plane has chromatic number at most four. This question has been one of the driving forces in Discrete Mathematics and its theme has inspired many variations. For example, the chromatic number of graphs that are embeddable into a surface of fixed genus has been intensively studied by Heawood [Hea90], Ringel and Youngs [RY68], and many others.

*Partially supported by the ENB graduate program TopMath and DFG grant GR 993/10-1. The author gratefully acknowledges the support of the TUM Graduate School's Thematic Graduate Center TopMath at the Technische Universität München

†Partially supported by the National Science Foundation, Grant DMS-1100215, and the Alexander von Humboldt Foundation.

‡Partially supported by DFG grant TA 309/2-2.

In this paper, we consider graphs and hypergraphs that are embeddable into \mathbb{R}^d for $d \geq 3$ in such a way that their edges do not intersect (see Definition 1 below). For graphs, however, this is not a very interesting question because for any $n \in \mathbb{N}$ the vertices of the complete graph K_n can be embedded into \mathbb{R}^3 using the embedding

$$\varphi(v_i) = (i, i^2, i^3) \quad \forall i \in \{1, \dots, n\}. \quad (1)$$

It is a well known property of the moment curve $t \mapsto (t, t^2, t^3)$ that any two edges between four distinct vertices do not intersect. E. g., this follows trivially from Corollary 11 in the case of $k = 2$ and $d = 3$.

As a consequence, we now focus our attention on hypergraphs, which are in general not embeddable into any specific dimension. Some properties of these hypergraphs (or more generally simplicial complexes) have been investigated (see e. g. [Men28, MTW11, vK33, Flo34]), but to our surprise, we have not been able to find any bounds on their chromatic number.

Before we can state our main results, we quickly recall and introduce some useful notation. We say that $H = (V, E)$ is a k -uniform hypergraph if the vertex set V is a finite set and the edge set E consists of k -element subsets of V , i. e. $E \subseteq \binom{V}{k}$. For any hypergraph H , we denote by $V(H)$ the vertex set of H and by $E(H)$ its edge set. We define

$$K_n^{(k)} := \left(\{1, 2, \dots, n\}, \binom{\{1, 2, \dots, n\}}{k} \right)$$

and call any hypergraph isomorphic to $K_n^{(k)}$ a *complete k -uniform hypergraph of order n* .

Let H be a k -uniform hypergraph. A function $\kappa : V(H) \rightarrow \{1, \dots, c\}$ is said to be a *strong c -coloring* if for all $e \in E(H)$ the property $|\kappa(e)| = k$ holds. The function κ is said to be a *weak c -coloring* if $|\kappa(e)| > 1$ for all $e \in E(H)$. The *strong/weak chromatic number* of H is defined as the minimum $c \in \mathbb{N}$ such that there exists a strong/weak coloring of H with c colors. The chromatic number of H is denoted by $\chi^s(H)$ and $\chi^w(H)$ respectively. Obviously, for graphs, weak and strong colorings are equivalent.

We next define what we mean when we say that a hypergraph is embeddable into \mathbb{R}^d . Here, aff denotes the affine hull of a set of points and conv the convex hull.

Definition 1 (d -embeddings)

Let H be a k -uniform hypergraph and $d \in \mathbb{N}$. A (linear) *embedding* of H into \mathbb{R}^d is a function $\varphi : V(H) \rightarrow \mathbb{R}^d$, where $\varphi(A)$ for $A \subseteq V(H)$ is to be interpreted pointwise, such that

- $\dim \text{aff } \varphi(e) = k - 1$ for all $e \in E(H)$ and
- $\text{conv } \varphi(e_1 \cap e_2) = \text{conv } \varphi(e_1) \cap \text{conv } \varphi(e_2)$ for all $e_1, e_2 \in E(H)$

The first property is needed to exclude functions mapping the vertices of one edge to affinely non-independent points. The second guarantees that the embedded edges only intersect in the convex hull of their common vertices. Note that the inclusion from left to right always holds. A k -uniform hypergraph H is said to be *d -embeddable* if there exists an embedding of H into \mathbb{R}^d . Also, we denote by $\mathcal{H}_{d,k}$ the set of all d -embeddable k -uniform hypergraphs.

One can easily see that our definition of 2-embeddability coincides with the classical concept of planarity [Fár48]. Note that in general there are several other notions of embeddability. The most popular thereof are piecewise linear embeddings and general topological embeddings. A short and comprehensive introduction is given in Section 1 in [MTW11]. We have decided to focus on linear embeddings, as they lead to a very accessible type of geometry and, at least in theory, the decision problem of whether a given k -uniform hypergraph is d -embeddable is decidable and in PSPACE [Ren92]. The aforementioned three types of embeddings have shown to be equivalent only in the less than 3-dimensional case (see e.g. [Bre83, BG00]). Since piecewise linear and topological embeddings are more general than linear embeddings, all lower bounds for chromatic numbers can easily be transferred. Furthermore, we prove all our results on upper bounds for piecewise linear embeddings.

We can now give a summary of our main results in the following Tables 1.1, 1.2, and 1.3, which contain upper or lower bounds for the maximum weak respectively strong chromatic number of a d -embeddable k -uniform hypergraph on n vertices. All results which only follow non-trivially from prior knowledge are indexed with a theorem number from which they can be derived.

$d \setminus k$	2	3	4	5	6	7
1	2	1	1	1	1	1
2	4	4	1	1	1	1
3	n	$n_{(19)}$	$\Omega(\sqrt{n})_{(17)}$	1	1	1
4	n	$n_{(20)}$	$n_{(20)}$	$\Omega(\sqrt{n})_{(18)}$	1	1
5	n	n	$n_{(20)}$	$n_{(20)}$	$\Omega(\sqrt{n})_{(18)}$	1
6	n	n	$n_{(20)}$	$n_{(20)}$	$n_{(20)}$	$\Omega(\sqrt{n})_{(18)}$
7	n	n	n	$n_{(20)}$	$n_{(20)}$	$n_{(20)}$

Table 1.1: Currently known *values* for the maximum strong chromatic number of a d -embeddable k -uniform hypergraph on n vertices as $n \rightarrow \infty$. The number in chevrons indicates the theorem number where we prove this bound.

To conclude the introduction here is a rough outline for the rest of the paper. In Section 2 the general concept of embedding hypergraphs into d -dimensional space is discussed and we also show the embeddability of certain structures needed later on, hereby extensively using known properties of the moment curve $t \mapsto (t, t^2, t^3, \dots, t^d)$. Section 3 then applies this to the strong coloring problem of hypergraphs and, finally, Section 4 presents our current level of knowledge for the apparently more difficult problem of weakly coloring hypergraphs.

2 Embeddability

The first part of this section gives insight into the structure of neighborhoods of single vertices in a hypergraph $H \in \mathcal{H}_{d,k}$. We will later use them to prove upper bounds on the number of edges in our hypergraphs. This will then yield upper bounds on the weak chromatic number. However, we must first take a small technical detour into piecewise linear embeddings. As our hypergraphs are finite and of fixed uniformity we give a slightly simplified definition.

$d \setminus k$	2	3	4	5	6	7
1	2	1	1	1	1	1
2	4	2	1	1	1	1
3	n	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(25)}$	1	1	1	1
4	n	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(25)}$	1	1	1	1
5	n	$\lceil n/2 \rceil$	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(26)}$	1	1	1
6	n	$\lceil n/2 \rceil$	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(26)}$	1	1	1
7	n	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(26)}$	1	1
8	n	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\Omega\left(\frac{\log n}{\log \log n}\right)_{(26)}$	1	1

Table 1.2: Currently known *lower bounds* for the maximum weak chromatic number of a d -embeddable k -uniform hypergraph on n vertices as $n \rightarrow \infty$. The number in chevrons indicates the theorem number where we prove this bound.

$d \setminus k$	2	3	4	5	6	7
1	2	1	1	1	1	1
2	4	2	1	1	1	1
3	n	$\mathcal{O}(n^{1/2})_{(23)}$	$\mathcal{O}(n^{1/2})_{(23)}$	1	1	1
4	n	$\lceil n/2 \rceil$	$\mathcal{O}(n^{2/3})_{(23)}$	$\mathcal{O}(n^{2/3})_{(23)}$	1	1
5	n	$\lceil n/2 \rceil$	$\mathcal{O}(n^{26/27})_{(24)}$	$\mathcal{O}(n^{3/4})_{(23)}$	$\mathcal{O}(n^{3/4})_{(23)}$	1
6	n	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\mathcal{O}(n^{35/36})_{(24)}$	$\mathcal{O}(n^{4/5})_{(23)}$	$\mathcal{O}(n^{4/5})_{(23)}$
7	n	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\mathcal{O}(n^{107/108})_{(24)}$	$\mathcal{O}(n^{44/45})_{(24)}$	$\mathcal{O}(n^{5/6})_{(23)}$
8	n	$\lceil n/2 \rceil$	$\lceil n/3 \rceil$	$\lceil n/4 \rceil$	$\mathcal{O}(n^{134/135})_{(24)}$	$\mathcal{O}(n^{53/54})_{(24)}$

Table 1.3: Currently known *upper bounds* for the maximum weak chromatic number of a d -embeddable k -uniform hypergraph on n vertices as $n \rightarrow \infty$. The number in chevrons indicates the theorem number where we prove this bound.

Definition 2 (Piecewise linear d -embeddings)

Let H be a k -uniform hypergraph and $d \in \mathbb{N}$. By the Menger-Nöbeling Theorem (see [Men28, p. 295] and [Nöb31]) $H \in \mathcal{H}_{D,k}$ for $D \geq 2k - 1$. So, let $\varphi : V(H) \rightarrow \mathbb{R}^D$ be such a linear embedding and define $\varphi(H) = \bigcup_{e \in E(H)} \text{conv } \varphi(e)$.

A *piecewise linear embedding* of H into \mathbb{R}^d is a homeomorphism $\psi : \varphi(H) \rightarrow \mathbb{R}^d$ such that there exists a subdivision K of $\varphi(H)$ (seen as a geometric simplicial complex) such that ψ is affine on all elements of K . We say H to be *piecewise linearly d -embeddable* if there exists an embedding of H into \mathbb{R}^d and we denote by $\mathcal{H}_{d,k}^{\text{PL}}$ the set of all piecewise linearly d -embeddable k -uniform hypergraphs.

Definition 3 (Neighborhoods)

For a k -uniform hypergraph H and a vertex $v \in V(H)$ we say the *neighborhood* of v is $N_H(v) = \{w \in V(H) : w \neq v \text{ and there is an edge in } E(H) \text{ incident with } w \text{ and } v\}$. We define the *neighborhood hypergraph* (or *link*) of $v \in V(H)$ to be the induced $(k - 1)$ -uniform hypergraph

$$\text{NH}_H(v) = (N_H(v), \{e \setminus \{v\} : e \in E(H), v \in e\}).$$

The *degree* $\deg_H(v) = \deg(v)$ is the number of edges in $E(H)$ incident with v .

Lemma 4

For a hypergraph $H \in \mathcal{H}_{d,k}^{\text{PL}}$ on n vertices, $d \geq k \geq 2$, and for any vertex v we have that $\text{NH}_v(H) \in \mathcal{H}_{d-1,k-1}^{\text{PL}}$.

Proof. Let $d \geq k \geq 2$, $H \in \mathcal{H}_{d,k}^{\text{PL}}$, $v \in V(H)$, and $V_v = N_H(v)$ nonempty. Then there exist $\varphi : V(H) \rightarrow \mathbb{R}^{2k-1}$ a linear embedding and $\psi : \varphi(H) \rightarrow \mathbb{R}^d$ a piecewise linear embedding of H for some subdivision K of $\varphi(H)$. Without restriction assume that $\varphi(v) = \mathbf{0}_{2k-1}$ and $\psi(\mathbf{0}_{2k-1}) = \mathbf{0}_d$.

Let $H_v = (V_v \cup \{v\}, \{e \in E(H) : v \in e\})$ be the sub-hypergraph of H of all edges containing v . Obviously, $\psi|_{\varphi(H_v)}$ (the restriction of ψ onto $\varphi(H_v)$) is a piecewise linear embedding of H_v for some subdivision $K_v \subseteq K$. Let $K_v^1 = \{e \in K_v : \mathbf{0}_{2k-1} \in e\}$. Then there exists an $\varepsilon > 0$ such that

$$\varepsilon \cdot \varphi(H_v) \subseteq \bigcup_{e \in K_v^1} e,$$

i. e. all points in $\varepsilon \cdot \varphi(H_v)$ are so close to $\mathbf{0}_{2k-1}$ that they lie completely in elements of K_v that contain the origin.

Then $\varphi' : V_v \cup \{v\} \rightarrow \mathbb{R}^{2k-1}$, $w \mapsto \varepsilon \cdot \varphi(w)$ is a linear and thus $\psi|_{\varphi'(H_v)}$ a piecewise linear embedding of H_v for the subdivision $K_v^2 = \{e \cap \varphi'(H_v) : e \in K_v^1\}$. We now claim that there exists a 1-dimensional subspace S of \mathbb{R}^d such that $\psi(\varphi'(H_v)) \cap S = \{\mathbf{0}_d\}$. This is true as each simplex $\psi(e)$, $e \in K_v^2$, has dimension less or equal $k - 1$ and hence “forbids” only a null set of possible lines through the origin when $d \geq k$. Without loss of generality, let $S = \mathbb{R} \times \{\mathbf{0}_{d-1}\}$.

Let $V_{K_v^2} \supseteq \varphi'(V_v)$ be the set of all subdivision points of K_v^2 without $\mathbf{0}_d$. Let $\pi_{d-1} : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ and $\pi_1 : \mathbb{R}^d \rightarrow \mathbb{R}$ be the natural projections onto the last $d - 1$ coordinates and the first coordinate respectively. Further, put $\delta = \min\{\|\pi_{d-1}(\psi(w))\| : w \in V_{K_v^2}\} > 0$ and $\Delta = \max\{|\pi_1(\psi(w))| :$

$w \in V_{K_v^2}$. We take a regular $(d-1)$ -simplex $T \subseteq \mathbb{R}^{d-1}$ centered at the origin with sides of length δ and set $C = \partial \operatorname{conv}(\{-2\Delta\} \times T) \cup \{(2\Delta, \mathbf{0}_{d-1})\}$. Thus, C is the boundary of a stretched d -simplex. Due to our choice of δ and Δ , all $\psi(w)$ for $w \in V_{K_v^2}$ lie outside of C and for all $e \in K_v^2$ the intersection $\psi(e) \cap C$ completely lies in $C' := C \setminus (\{-2\Delta\} \times T)$, is connected, and the union of finitely many at most $(k-2)$ -dimensional simplices.

Thus, there exists a subdivision K_v^3 of K_v^2 such that for all $e \in K_v^3$ with dimension $k-1$ we have that $\psi(e) \cap C'$ is a $(k-2)$ -dimensional simplex and still $\mathbf{0}_d \in e$. We denote the set of subdivision points *without* $\mathbf{0}_d$ by $V_{K_v^3} \supseteq V_{K_v^2}$. Now, one can find a retraction $\varrho : \psi(\varphi'(H_v)) \rightarrow \psi(\varphi'(H_v))$ that maps each $\psi(w)$, $w \in V_{K_v^3}$, to the intersection point of the line segment $[\mathbf{0}_d, \psi(w)]$ with C' , such that ϱ is linear on all $\psi(e)$ for $e \in K_v^3$.

Finally, set $\hat{K} = \{\operatorname{conv}(e \cap V_{K_v^3}) : e \in K_v^3\}$ which is now a subdivision of $\varphi'(\operatorname{NH}_H(v)) \subseteq \varphi'(H_v)$. Then the image of $\varrho \circ (\psi|_{\hat{K}})$ lies completely in C' . Thus,

$$\hat{\psi} = \pi_{d-1} \circ \varrho \circ (\psi|_{\hat{K}}) \quad \square$$

is a piecewise linear embedding of $\operatorname{NH}_H(v)$ into \mathbb{R}^{d-1} for the subdivision \hat{K} and $\operatorname{NH}_H(v) \in \mathcal{H}_{d-1, k-1}^{\text{PL}}$.

Lemma 5

- a) For a hypergraph $H \in \mathcal{H}_{k,k}^{\text{PL}}$ on n vertices, $k \geq 2$, we have that $|E(H)| \leq \frac{6n^{k-1} - 12n^{k-2}}{k!}$.
- b) For a hypergraph $H \in \mathcal{H}_{k+1, k+1}^{\text{PL}}$ on n vertices, $k \geq 2$, and for any vertex v we have that $\deg_H(v) \leq \frac{6n^{k-1} - 12n^{k-2}}{k!}$.

Proof. If $k = 2$, then (a) is equivalent to the fact that for G planar $|E(G)| \leq 3n - 6$. Given that (a) is true for some $k \geq 2$, we show that (b) holds for k as well. Let $H \in \mathcal{H}_{k+1, k+1}^{\text{PL}}$, v one of the n vertices. By Lemma 4, $\operatorname{NH}_H(v) \in \mathcal{H}_{k,k}^{\text{PL}}$. By (a), $|E(\operatorname{NH}_H(v))| \leq \frac{6n^{k-1} - 12n^{k-2}}{k!}$ which implies $\deg_H(v) \leq \frac{6n^{k-1} - 12n^{k-2}}{k!}$.

Given that (b) is true for some $k \geq 2$, we show that (a) holds for $k+1$. Let $H \in \mathcal{H}_{k+1, k+1}^{\text{PL}}$. Since (b) is true for every vertex v_i , we have

$$|E(H)| = \frac{\sum_{i=1}^n \deg_H(v_i)}{k+1} \leq \frac{n(6n^{k-1} - 12n^{k-2})}{(k+1)k!} = \frac{6n^k - 12n^{k-1}}{(k+1)!}. \quad \square$$

Corollary 6

For a hypergraph $H \in \mathcal{H}_{k,k}^{\text{PL}}$ on n vertices, $k \geq 3$, and for any edge $e \in E(H)$ there exist at most $k(6n^{k-2} - 12n^{k-3})/(k-1)! - 1$ other edges adjacent to it.

Proof. This follows from Lemma 5, since every edge has exactly k vertices and each of them has at most degree $\frac{6n^{k-2} - 12n^{k-3}}{(k-1)!}$. As e itself counts for the degree as well, one can subtract 1. \square

We need to bound the number of edges in a d -embeddable hypergraph to prove upper bounds for the chromatic number. The following proposition will also help to do this. Note that there exist much stronger conjectured bounds (see [Kal02, Conjecture 27] and [Gun09, Conjecture 1.4.4]).

Proposition 7 (Gundert [Gun09, Proposition 3.3.5])

Let $k \geq 2$. For a k -uniform hypergraph on n vertices that is topologically embeddable into \mathbb{R}^{2k-2} , we have that $|E(H)| < n^{k-3^{1-k}}$.

Corollary 8

Using Lemma 4 and Proposition 7, we inductively obtain that for $H \in \mathcal{H}_{2k-l,k}^{\text{PL}}$ on n vertices, $k \geq l \geq 2$, we have that $|E(H)| < n^{k-3^{l-1-k}}$.

Corollary 9

For a hypergraph $H \in \mathcal{H}_{2k-l,k}^{\text{PL}}$ on n vertices, $k \geq l \geq 3$, and for any edge $e \in E(H)$ there exist at most $kn^{k-1-3^{l-1-k}} - 1$ other edges adjacent to it.

Proof. This fact follows analogously to Corollary 6 from Corollary 8. \square

In order to find lower bounds for the chromatic number of hypergraphs later on, we need to be able to prove embeddability. The following theorem from Shephard will turn out to be very useful when embedding vertices of a hypergraph on the moment curve.

Theorem 10 (Shephard [She68])

Let $W = \{w_1, \dots, w_m\} \subseteq \mathbb{R}^d$ be distinct points on the moment curve in that order and $P = \text{conv } W$. We say a q -element subset $\{w_{i_1}, w_{i_2}, \dots, w_{i_q}\} \subseteq W$ where $i_1 < i_2 < \dots < i_q$ is called contiguous if $i_q - i_1 = q - 1$. Then $U \subseteq W$ is the set of vertices of a $(k-1)$ -face of P iff $|U| = k$ and for some $t \geq 0$

$$U = Y_S \cup X_1 \cup \dots \cup X_t \cup Y_E,$$

where all X_i , Y_S , and Y_E are contiguous sets, $Y_S = \emptyset$ or $w_1 \in Y_S$, $Y_E = \emptyset$ or $w_m \in Y_E$, and at most $d - k$ sets X_i have odd cardinality.

Shephard's Theorem thus says that the absolute position of points on the moment curve is irrelevant and only their relative order is important. Furthermore, note that all points in W are vertices of P . The following corollary helps in proving that two given edges of a hypergraph intersect properly.

Corollary 11

In the setting of Theorem 10 assume that $W = U_1 \cup U_2$ where U_1 and U_2 are embedded edges of a k -uniform hypergraph. Then these edges do not intersect in a way forbidden by Definition 1, if at least one of them is a face of $P = \text{conv } W$. It thus suffices to show that for one $j \in \{1, 2\}$

$$U_j = Y_S \cup X_1 \cup \dots \cup X_t \cup Y_E$$

holds where at most $d - k$ of the contiguous sets X_i have odd cardinality.

In the $k = d = 3$ case Corollary 11 allows zero odd sets X_i . Thus, we can easily classify all possible configurations for two edges.

Corollary 12

Given a 3-uniform hypergraph H and $\varphi : V(H) \rightarrow \mathbb{R}^3$. Then φ is an embedding of H if φ maps all vertices one-to-one on the moment curve and, for each pair of edges e and f sharing at most one vertex, the order of the points $\varphi(e \cup f)$ on the moment curve has one of Configurations 1–10 shown in Figure 2.1. The relative order of edges with two common vertices is irrelevant.

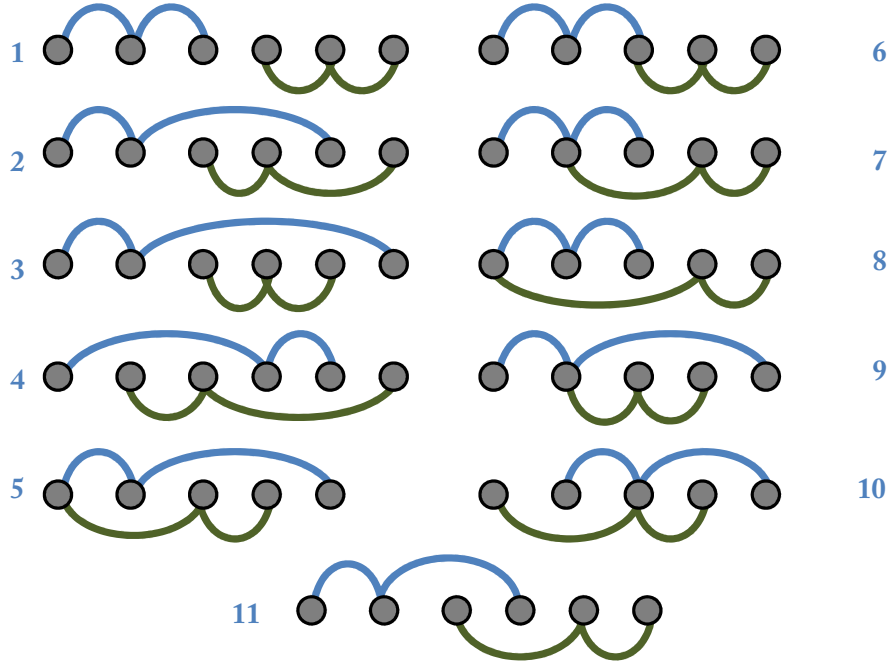


Figure 2.1: Possible configurations for two edges e and f in \mathbb{R}^3 . The vertices of e are marked on the top, those of f marked on the bottom. Equivalent cases, one being the reverse of the other, are only displayed once.

However, there is one more possible configuration which is not covered by Shephard’s Theorem as both edges are not faces of the polytope of their vertices.

Lemma 13

Given a 3-uniform hypergraph H and $\varphi : V(H) \rightarrow \mathbb{R}^3$. Then φ is an embedding of H if φ maps all vertices one-to-one on the moment curve and for each pair of edges e and f sharing at most one vertex, the order of the points $\varphi(e \cup f)$ on the moment curve has one of the configurations shown in Figure 2.1. The relative order of edges with two common vertices is irrelevant.

Proof. Having in mind Corollary 12, it is sufficient to prove the following: For $x_{0,0} < x_{1,0} < x_{0,1} < x_{2,0} < x_{1,1} < x_{2,1} \in \mathbb{R}$, $\psi : \mathbb{R} \rightarrow \mathbb{R}^3$, $\psi(x) = (x, x^2, x^3)$ the moment curve, and $D_i = \{x_{0,i}, x_{1,i}, x_{2,i}\}$ we have that $\text{conv} \psi(D_0) \cap \text{conv} \psi(D_1) = \emptyset$. Assume otherwise. Note that if two triangles intersect in \mathbb{R}^3 the intersection points must contain at least one point of the border of at least one of the triangles. Thus, without loss of generality, $\text{conv}\{\psi(x_{j_1,0}), \psi(x_{j_2,0})\} \cap \text{conv} \psi(D_1) \neq \emptyset$. However, by Theorem 10 we know that $\text{conv}\{\psi(x_{j_1,0}), \psi(x_{j_2,0})\}$ is a face of the polytope $P = \text{conv}(\{\psi(x_{j_1,0}), \psi(x_{j_2,0})\} \cup \psi(D_1))$ which is a contradiction. \square

3 Strong colorings

For $d, k, n \in \mathbb{N}$ we define

$$\chi_{d,k}^s(n) = \max\{\chi^s(H) : H \in \mathcal{H}_{d,k}, |V(H)| = n\}$$

to be the maximum strong chromatic number of a d -embeddable k -uniform hypergraph on n vertices.

Clearly, $\chi_{d,k}^s(n)$ is monotonically increasing in n and in d and monotonically decreasing in k if the other parameters remain fixed. Furthermore, it is not difficult to establish some kind of strict simultaneous monotonicity as follows:

Lemma 14

For $d, k, n \in \mathbb{N}$, we have

$$\chi_{d+1,k+1}^s(n+1) \geq \chi_{d,k}^s(n) + 1.$$

Proof. Let $H \in \mathcal{H}_{d,k}$ be such that $c = \chi^s(H) = \chi_{d,k}^s(n)$. Let φ be an embedding of H into \mathbb{R}^d . Set $V' = V(H) \cup \{v'\}$ where $v' \notin V(H)$. Furthermore, set $E' = \{e \cup \{v'\} : e \in E(H)\}$ and consider the embedding

$$\varphi' : V' \rightarrow \mathbb{R}^{d+1}, \quad v \mapsto \begin{cases} (\varphi(v), 0) & \text{if } v \in V(H) \\ (\mathbf{0}_d, 1) & \text{if } v = v' \end{cases}.$$

Then, $H' = (V', E')$ is in $\mathcal{H}_{d+1,k+1}$ using the embedding φ' . Assume that H' has a strong coloring with at most c colors. By the construction of E' , this would yield a coloring with less than c colors for H , which contradicts the choice of c . \square

We now give lower bounds on $\chi_{d,k}^s(n)$, which we essentially derive by retreating to graphs.

Definition 15 (The shadow of a hypergraph)

Let H be a k -uniform hypergraph, $k \geq 2$. Then we call, following [GGL95, §7.1],

$$\mathcal{S}(H) = (V(H), \{\{v_1, v_2\} : \{v_1, v_2\} \subseteq e \text{ for some } e \in E(H)\})$$

the (*second*) shadow of H .

For a k -uniform hypergraph H , we have

$$\chi^s(H) = \chi(\mathcal{S}(H)), \tag{2}$$

where $\chi(G)$ is the classical chromatic number of a graph G .

Remark 16

a) The Four Color Theorem and Equation (2) imply that $\chi_{2,k}^s(n) \leq 4$ for $k \in \{2, 3\}$. Obviously, there are graphs and hypergraphs for which this bound is sharp.

b) For $d \geq 2k - 1$, we have $\chi_{d,k}^s(n) = n$ as $K_n^{(k)}$ is $(2k - 1)$ -embeddable for all $k \in \mathbb{N}$ by the Menger-Nöbeling Theorem (see [Men28, p. 295] and [Nöb31]) and $\chi^s(K_n^{(k)}) = n$.

c) For $d \leq k - 2$, we know $\chi_{d,k}^s(n) = 1$ as $H \in \mathcal{H}_{d,k}$ cannot have any edge.

Theorem 17

For $n \geq 4$ we have $\chi_{3,4}^s(n) \geq \lfloor \sqrt{n} \rfloor$.

Proof. It is sufficient that for $m \geq 4$ we can find a hypergraph $H \in \mathcal{H}_{3,4}$ on m^2 vertices with strong chromatic number larger or equal m . Let $m \geq 4$ and let $G = K_m$ be the complete graph on m vertices. We can use the embedding φ from Equation (1) to embed G into \mathbb{R}^3 .

Note that G has only finitely many edges. So for every embedded edge $e = \{u, v\}$ of G there exists a small open convex set $C_e \subseteq \mathbb{R}^3$ such that $(\text{conv } \varphi(e)) \setminus \varphi(e) \subseteq C_e$, $\varphi(e) \subseteq \partial C_e$, and $(C_e \cap \text{conv } \varphi(f)) \subseteq \{\varphi(u), \varphi(v)\}$ for all $f \neq e \in E(G)$.

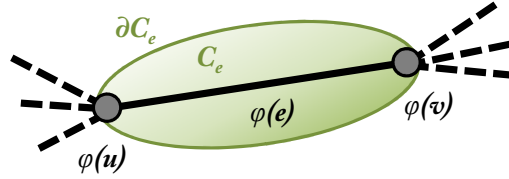


Figure 3.1: Construction of C_e .

Now for each edge $e \in E(G)$ define two new vertices v'_e and v''_e . Further, let w'_e and $w''_e \in \mathbb{R}^3$ be two arbitrary points in C_e such that $\{w'_e, w''_e, \varphi(u), \varphi(v)\}$ are in general position and thus form the edges of a tetrahedron in C_e . We want to define the hypergraph H using the vertex set $V = V(G) \cup \{v'_e, v''_e : e \in E(G)\}$ thus adding the newly created vertices. We set $E = \{e \cup \{v'_e, v''_e\} : e \in E(G)\}$ and

$$\varphi' : V \rightarrow \mathbb{R}^3, \quad v \mapsto \begin{cases} \varphi(v) & \text{if } v \in V(G) \\ w'_e & \text{if } v = v'_e \text{ for some } e. \\ w''_e & \text{if } v = v''_e \text{ for some } e \end{cases}$$

Then φ' is an embedding of H as each edge forms a tetrahedron inside its corresponding set C_e and intersects with other tetrahedrons at most at its two endpoints. Hence, $H \in \mathcal{H}_{3,4}$. Obviously, $|V| = m^2$ and $\chi^s(H) = \chi(\mathcal{S}(H)) \geq \chi(G) = \chi(K_m) = m$. \square

Corollary 18

Together with Lemma 14 we obtain for large n and $d \geq 3$ that

$$\chi_{d,d+1}^s(n) \geq \lfloor \sqrt{n - d + 3} \rfloor + d - 3.$$

Using monotonicity the same holds for all $d \geq 3, k \leq d + 1$ and $\chi_{d,k}^s(n)$.

Theorem 19

For $n \geq 1$ we have $\chi_{3,3}^s(n) = n$.

Proof. First consider $n \in \mathbb{N}$ odd, $n = 2m + 1$. Set $V_m = \{v_0, \dots, v_m, w_1, \dots, w_m\}$, $E_m = \{\{v_i, v_j, w_j\} : 0 \leq i < j \leq m\} \cup \{\{w_i, v_j, w_j\} : 1 \leq i < j \leq m\}$ and look at the 3-uniform hypergraph $F_m = (V_m, E_m)$. It has $2m + 1$ vertices and one can easily see that $\chi^s(F_m) = \chi(\mathcal{S}(F_m)) = \chi(K_{2m+1}) = n$.

It is now left to show that F_m is 3-embeddable for all m . Define $\varphi(x) = (x, x^2, x^3)$ for $x \in \mathbb{R}$. Set $\psi_m(v_0) = \varphi(1)$, $\psi_m(v_i) = \varphi(2i)$, and $\psi_m(w_i) = \varphi(2i + 1)$ for all $i \in \{1, \dots, m\}$.

ψ_m is an embedding of F_m : We use induction on m and for $m = 0$ or $m = 1$ this is trivial. Now assume that this is true for some m . Take any two edges $e_1 \neq e_2 \in E_{m+1}$. Observe that $\psi_{m+1}|_{V_m} = \psi_m$. If $e_1, e_2 \in E_m$, then they intersect according to Definition 1 because ψ_m was an embedding of F_m . If both edges are in $E_{m+1} \setminus E_m$ then they share the two vertices v_{m+1} and w_{m+1} . Thus, they also intersect according to Definition 1.

When $e_1 \in E_{m+1} \setminus E_m$ and $e_2 \in E_m$, we have to distinguish the several different cases of Corollary 12. If e_1 and e_2 share one vertex, we have one of Cases 6–8 and we are done. So, assume e_1 and e_2 to be disjoint. Let $0 \leq i, j, k \leq m$ be pairwise distinct. If $e_1 = \{v_i, v_{m+1}, w_{m+1}\}$ and $e_2 = \{w_i, v_k, w_k\}$, then $i < k$ and we have Case 3. If $e_1 = \{w_i, v_{m+1}, w_{m+1}\}$ and $e_2 = \{v_i, v_k, w_k\}$, we have Case 2. The last possibility is that $e_1 = \{[v_i \text{ or } w_i], v_{m+1}, w_{m+1}\}$ and $e_2 = \{[v_j \text{ or } w_j], v_k, w_k\}$. However, then we are in Case 1 if $i > k > j$, in Case 2 if $k > i > j$, and in Case 3 if $k > j > i$. Hence, ψ_{m+1} is an embedding of F_{m+1} .

Now, let n be even, $n = 2m$. Note that by monotonicity we already have $\chi_{3,3}^5(2m) \geq \chi_{3,3}^5(2m - 1) \geq 2m - 1$. Take F_{m-1} and add one more vertex to obtain $V'_m = V_{m-1} \cup \{v'_0\}$. Furthermore, set $E'_m = E_{m-1} \cup \{\{v'_0, v_i, w_i\} : 1 \leq i \leq m\} \cup \{\{v'_0, v_0, w_{m-1}\}\}$ and define $F'_m = (V'_m, E'_m)$. This hypergraph has $2m$ vertices and $\chi^s(F'_m) = \chi(\mathcal{S}(F'_m)) = \chi(K_{2m}) = n$. We claim that it is 3-embeddable for all m via the embedding $\psi'_m(v) = \psi_{m-1}(v)$ if $v \neq v'_0$ and $\psi'_m(v'_0) = \varphi(0)$.

As before, take any two edges $e_1 \neq e_2 \in E'_m$. If both edges are in E_{m-1} , both edges are of the form $\{v'_0, v_i, w_i\}$, or both edges share two vertices, then they obviously intersect according to Definition 1. If $e_1 = \{v'_0, v_0, w_{m-1}\}$, we have Case 3 if the edges are disjoint and one of Cases 5, 8, or 9 if not. Let $0 \leq i, j, k \leq m - 1$ be pairwise distinct and let $e_1 = \{v'_0, v_i, w_i\}$. If $e_2 = \{v_0, v_j, w_j\}$, we then have Case 3 if $j < i$ and Case 2 otherwise. If $e_2 = \{[v_j \text{ or } w_j], v_k, w_k\}$, we are in Case 1 if $k > j > i$, Case 2 if $k > i > j$, and Case 3 if $i > k > j$. Finally, if $e_2 = \{v_i, v_k, w_k\}$, we have Case 7 and if $e_2 = \{w_i, v_k, w_k\}$, Case 6. Hence, ψ'_m is an embedding of F'_m . \square

Corollary 20

Together with Lemma 14 we obtain that

$$\chi_{d,d}^s(n) = n.$$

By monotonicity, $\chi_{d,k}^s(n) = n$ holds for all $d \geq 3, k \leq d$.

Thus, except for the cases where $k = d + 1$, the maximum strong coloring problem was solved. In particular, we have shown that an unbounded number of colors can be necessary for any strong coloring of a d -embeddable hypergraph if $d > 2$.

4 Weak colorings

For $d, k, n \in \mathbb{N}$ we define

$$\chi_{d,k}^w(n) = \max\{\chi^w(H) : H \in \mathcal{H}_{d,k}, |V(H)| = n\}$$

to be the maximum weak chromatic number of a d -embeddable k -uniform hypergraph on n vertices.

In this section, we give lower and upper bounds on $\chi_{d,k}^w(n)$. Obviously, $\chi_{d,k}^w(n)$ is monotonically increasing in n and in d and monotonically decreasing in k if the other parameters remain fixed. Note that an equivalent of Lemma 14 is not true for weak colorings as the existence of one vertex incident with all edges automatically implies the existence of a weak 2-coloring.

Remark 21

a) For $k = 2$, the results in Tables 1.2 and 1.3 are the same as in Table 1.1 as weak and strong chromatic numbers coincide.

b) For $d \geq 2k - 1$, we have $\chi_{d,k}^w(n) = \Theta(n)$ as $K_n^{(k)}$ is $(2k - 1)$ -embeddable for all $k \in \mathbb{N}$ by the Menger-Nöbeling Theorem (see [Men28, p. 295] and [Nöb31]) and $\chi^w(K_n^{(k)}) = \lceil n/(k - 1) \rceil$.

c) For $d \leq k - 2$, we again know $\chi_{d,k}^w(n) = 1$ as $H \in \mathcal{H}_{d,k}$ cannot have any edge.

Proposition 22

For all $n \geq 3$ we have $\chi_{2,3}^w(n) \leq 2$. (This bound is obviously sharp.)

Proof. Let $H \in \mathcal{H}_{2,3}$ and $V = V(H)$. Then $G = \mathcal{S}(H)$ is a planar graph, thus $\chi(G) \leq 4$. Let $\kappa : V \rightarrow \{1, 2, 3, 4\}$ be a 4-coloring of G . Define

$$\kappa' : V \rightarrow \{1, 2\}, v \mapsto (\kappa(v) \bmod 2) + 1.$$

In any triangle $\{u, v, w\}$ of H under the coloring κ these vertices have exactly three different colors. Therefore, under the coloring κ' at least one vertex with color 1 and one vertex with color 2 exists. Thus κ' is a valid 2-coloring of H . \square

Theorem 23

Let $d \geq 3$. Then one has

$$\chi_{d,d}^w(n) \leq \left\lceil \left(\frac{6ed}{(d-1)!} \right)^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}} \right\rceil = \mathcal{O} \left(\left(\frac{n}{d} \right)^{\frac{d-2}{d-1}} \right).$$

This result also holds for piecewise linear embeddings.

Proof. Let $H \in \mathcal{H}_{d,d}^{\text{PL}} \supseteq \mathcal{H}_{d,d}$. By Corollary 6 we know that every edge is at most adjacent to $\Delta = d(6n^{d-2} - 12n^{d-3})/(d-1)! - 1$ other edges.

We want to apply the Lovász Local Lemma [EL75, Spe77] to bound the weak chromatic number of H . Let $c \in \mathbb{N}$. In any c -coloring of the vertices of H an edge is called bad if it is monochromatic and good if not. In any uniformly random c -coloring the probability for one edge to be bad is

$p = \frac{1}{c^{d-1}}$. Moreover, note that the events whether one edge and a set of edges not adjacent to the first edge are bad are independent. Thus each event whether an edge is bad depends on at most Δ other such events.

The Lovász Local Lemma guarantees us that there is positive probability that all edges are good if $e \cdot p \cdot (\Delta + 1) \leq 1$. This implies that H is weakly c -colorable. Note that

$$e \cdot p \cdot (\Delta + 1) \leq 1 \Leftrightarrow \frac{ed(6n^{d-2} - 12n^{d-3})}{(d-1)!} \leq c^{d-1}.$$

Choosing an integer

$$c \geq \left(\frac{6ed}{(d-1)!} \right)^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}} \geq \left(\frac{ed(6n^{d-2} - 12n^{d-3})}{(d-1)!} \right)^{\frac{1}{d-1}},$$

the hypergraph H is c -colorable and $\chi^w(H) \leq c$. \square

Theorem 24

Let $d \geq l \geq 3$. Then one has

$$\chi_{2d-l,d}^w(n) \leq \left\lceil (ed)^{\frac{1}{d-1}} n^{1-\frac{3^{l-1}-d}{d-1}} \right\rceil = \mathcal{O} \left(n^{1-\frac{3^{l-1}-d}{d-1}} \right).$$

This result also holds for piecewise linear embeddings.

Proof. By Corollary 9 we know that every edge is at most adjacent to $\Delta = dn^{d-1-3^{l-1}-d} - 1$ other edges. The rest of the proof is now analogous to the proof of Theorem 23. \square

By monotonicity, the upper bounds presented here also hold if the uniformity of the hypergraph is larger than stated in Theorems 23 and 24. In the remaining part of this section, we now consider lower bounds for the chromatic number of hypergraphs.

Theorem 25

As $n \rightarrow \infty$ one has

$$\chi_{3,3}^w(n) = \Omega \left(\frac{\log n}{\log \log n} \right).$$

Proof. We first define a sequence of hypergraphs H_m for $m \geq 2$ such that $\chi^w(H_m) \geq m$. Set $H_2 = K_3^{(3)}$ which has 3 vertices. Define H_m for $m > 2$ iteratively, assuming $\chi^w(H_{m-1}) \geq m-1$. Take m new vertices $\{v_0, \dots, v_{m-1}\}$ and $m(m-1)/2$ disjoint copies of H_{m-1} , labeled $H_{m-1}^{[0,1]}, \dots, H_{m-1}^{[m-2,m-1]}$.

The edges of H_m shall be all former edges of all $H_{m-1}^{[i,j]}$ together with all edges of the form $\{v_i, v_j, w\}$ where $i < j$ and $w \in H_{m-1}^{[i,j]}$. Assume H_m is weakly $(m-1)$ -colorable. Given such a coloring, one color must occur twice in $\{v_0, \dots, v_{m-1}\}$. Say, these are the vertices v_{i_1} and v_{i_2} where $i_1 < i_2$. This color cannot occur anymore in the coloring of $H_{m-1}^{[i_1,i_2]}$. Thus, $H_{m-1}^{[i_1,i_2]}$ must be weakly $(m-2)$ -colorable. This is a contradiction and H_m is at least (and obviously exactly) weakly m -chromatic.

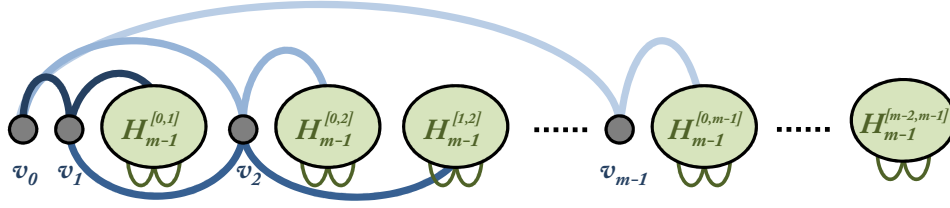


Figure 4.1: Construction of H_m .

We now claim that $H_m \in \mathcal{H}_{3,3}$ for all $m \geq 2$. Therefore we give a function $f_m : V(H_m) \rightarrow \{1, \dots, n_m\}$ where n_m is the number of vertices of H_m . This function defines the order in which the vertices of H_m will be arranged on the moment curve $t \mapsto (t, t^2, t^3)$. Lemma 13 on possible configurations then guarantees that H_m is embeddable via arbitrary points on the moment curve. Note that the absolute position of vertices on the moment curve is not important, only their relative order.

The hypergraph $H_2 = K_3^{(3)}$ can be embedded into \mathbb{R}^3 via any three points on the moment curve, so $f_2 : V(H_2) \rightarrow \{0, 1, 2\}$ can be chosen arbitrarily. Assume that f_{m-1} has already been defined and that the vertices of H_{m-1} arranged in that order on the moment curve form an embedding. Look at the vertices of H_m as given before. We define $f_m(v_j) = n_{m-1} \cdot j(j-1)/2 + j$ for $0 \leq j \leq m-1$ and for any $w \in H_{m-1}^{[i,j]}$ with $i < j$ we set $f_m(w) = n_{m-1} \cdot (j(j-1)/2 + i) + j + f_{m-1}(w)$. This gives exactly the order shown in Figure 4.1.

Now, arrange the vertices of H_m on the moment curve in that order and pick any two edges e_1 and e_2 . By Lemma 13 we can assume that they do not share two vertices. If both edges are from the same subhypergraph $H_{m-1}^{[i,j]}$ then they can only intersect according to Definition 1 as their relative order reflects that of f_{m-1} . If they originate from distinct subhypergraphs $H_{m-1}^{[i_1, j_1]}$ and $H_{m-1}^{[i_2, j_2]}$, they are of Case 1 in Figure 2.1. Next, assume that one of them is of the form $\{v_{i_1}, v_{j_1}, v\}$ where $v \in H_{m-1}^{[i_1, j_1]}$ and that the other is from some $H_{m-1}^{[i_2, j_2]}$. Then, by definition, $i_1 < j_1$ and $i_2 < j_2$ and all the possible cases of Lemma 13 are listed in Table 4.1.

Case number	Relative order of i_1, i_2, j_1, j_2
1	$j_1 < j_2, j_1 = j_2$ and $i_1 < i_2, i_1 > j_2$
3	$j_1 > j_2$ and $i_1 \leq j_2, j_1 = j_2$ and $i_1 > i_2$
1, 2, 3, 6–8, or 11	$j_1 = j_2$ and $i_1 = i_2$

Table 4.1: Possible cases if one edge is newly constructed and one is an old edge.

Finally, take $e_1 = \{v_{i_1}, v_{j_1}, v\}$ and $e_2 = \{v_{i_2}, v_{j_2}, w\}$ (again $i_1 < j_1$ and $i_2 < j_2$). We then have one of the cases listed in Table 4.2. Thus, the order given by f_m provides an embedding of H_m .

To estimate n_m , we use the following recursion

$$n_2 = 3,$$

$$n_m = m + n_{m-1} \cdot m(m-1)/2 \quad \text{for } m > 2.$$

Case number	Relative order of i_1, i_2, j_1, j_2
1	$j_1 < i_2, i_1 > j_2$
2	$i_2 < i_1 < j_2 < j_1, i_1 < i_2 < j_1 < j_2$
3	$i_1 < i_2 < j_2 < j_1, i_2 < i_1 < j_1 < j_2$
7	$j_1 = i_2, i_1 = j_2$
8	$i_1 = i_2$ and $j_1 \neq j_2$
10	$i_2 \neq i_1$ and $j_1 = j_2$
two shared vertices	$i_1 = i_2$ and $j_1 = j_2$

Table 4.2: Possible cases if both edges are newly constructed.

This can be bounded by $n_m \leq m^{2m} =: \hat{n}_m$. Then

$$\frac{\log \hat{n}_m}{\log \log \hat{n}_m} = 2m \cdot \frac{\log m}{\log(2m \log m)} = \mathcal{O}(m)$$

and we finally get that $m = \Omega\left(\frac{\log \hat{n}_m}{\log \log \hat{n}_m}\right) \geq \Omega\left(\frac{\log n_m}{\log \log n_m}\right)$. □

Note that by monotonicity also $\chi_{4,3}^w(n) = \Omega\left(\frac{\log n}{\log \log n}\right)$. Figure 4.2 gives examples for an embedding of H_3 and H_4 .

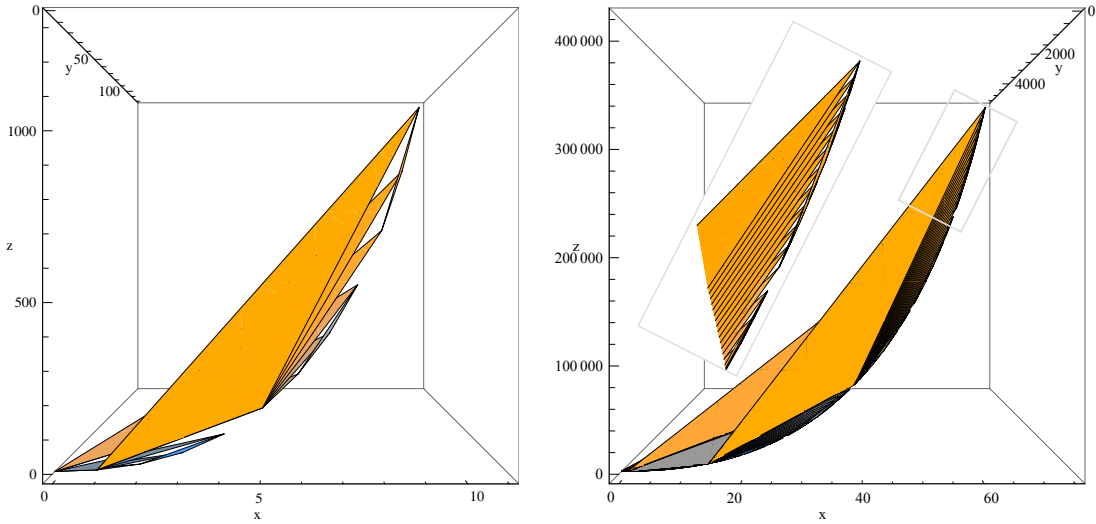


Figure 4.2: Examples for an embedding of H_3 and H_4 .

Theorem 26

Let $d \geq 3$. Then, as $n \rightarrow \infty$ one has

$$\chi_{2d-3,d}^w(n) = \Omega\left(\frac{\log n}{\log \log n}\right).$$

Proof. Induction over d . The case $d = 3$ was shown in Theorem 25. Let $d > 3$. Suppose we have constructed a family $(H_m^{d-1})_{m \in \mathbb{N}}$ of hypergraphs in $\mathcal{H}_{2d-5,d-1}$ such that $\chi^w(H_m^{d-1}) \geq m$.

Let $H_2^d = K_d^{(d)}$. The hypergraph H_2^d has d vertices, one edge, and is weakly 2-colorable. Define H_m^d for $m > 2$ iteratively, given that $\chi^w(H_{m-1}^d) \geq m - 1$. Therefor take one copy of H_{m-1}^d and one copy of $(d - 1)$ -uniform H_m^{d-1} .

The edges of H_m^d shall be all edges of H_{m-1}^d and all edges of the form $(\{v\} \cup e)$ for $v \in V(H_{m-1}^d)$ and $e \in E(H_m^{d-1})$. Assume that there exists a weak $(m - 1)$ -coloring of H_m^d . Then there has to be at least one monochromatic edge $e \in E(H_m^{d-1})$. No vertex of H_{m-1}^d can be colored with this color, so its edges must be weakly $(m - 2)$ -colored. This is a contradiction and thus $\chi^w(H_m^d) \geq m$.

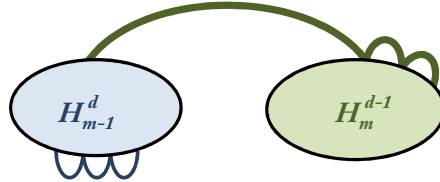


Figure 4.3: Construction of H_m^d .

We now claim that $H_m^d \in \mathcal{H}_{2d-3,d}$ for all $m \geq 2$. As in the proof of Theorem 25, we give a function $f_m^d : V(H_m^d) \rightarrow \{1, \dots, n_m^d\}$ where n_m^d is the number of vertices of H_m^d . This defines the order in which the vertices of H_m^d will be arranged on the moment curve $t \mapsto (t, \dots, t^{2d-3})$. We then use Corollary 11 to prove that H_m^d is embeddable via arbitrary points on the moment curve. As before, the absolute position of vertices on the moment curve is not important. For a fixed uniformity d and dimension $2d - 3$, Corollary 11 guarantees that if for two given edges the vertices of at least one edge have at most $d - 3$ odd contiguous subsets, they intersect properly according to Definition 1.

If $d = 3$ we can set $f_m^3 = f_m$ for all $m \geq 2$, where f_m is as in the proof of Theorem 25. For $d > 3$ we have by assumption that there exists a corresponding family of functions

$$(f_m^{d-1} : V(H_m^{d-1}) \rightarrow \{1, \dots, n_m^{d-1}\})_m$$

such that the vertices of H_m^{d-1} arranged in that order on the moment curve form an embedding. We then have to give a family of functions f_m^d for d .

H_2^d can be embedded into \mathbb{R}^{2d-3} via any d points on the moment curve, so $f_2^d : V(H_2^d) \rightarrow \{1, \dots, d\}$ can be chosen arbitrarily. Assume that f_{m-1}^d has already been defined and gives an embedding of H_{m-1}^d . We define $f_m^d(v) = f_{m-1}^d(v)$ for $v \in V(H_{m-1}^d)$ and for any $w \in V(H_m^{d-1})$ we set $f_m^d(w) = n_{m-1}^d + f_m^{d-1}(w)$. This is also shown in Figure 4.3.

Arrange the vertices of H_m^d on the moment curve in that order and pick any two edges g_1 and g_2 . If both edges are from the subhypergraph H_{m-1}^d then they intersect in accordance to Definition 1 as their relative order reflects that of f_{m-1}^d . If one edge is from H_{m-1}^d and the other of the form $(\{v\} \cup e)$ where $v \in V(H_{m-1}^d)$ and $e \in E(H_m^{d-1})$, both edges have at most one odd contiguous subset (except from the first and last one), which is no problem for $d > 3$.

Finally, we look at the case $g_1 = (\{v_1\} \cup e_1)$ and $g_2 = (\{v_2\} \cup e_2)$. Then their vertex sets have at most one more odd contiguous subset than the edges e_1 and e_2 had in the ordering of f_m^{d-1} . The last number, by assumption, was bounded from above by $(d - 1) - 3$ for at least on e_i ,

$i \in \{1, 2\}$. So at least one g_i has at most $d - 3$ odd contiguous subsets. Thus, the order given by f_m^d provides an embedding of H_m^d .

Note that there is one small exception when $d = 4$. Here, e_1 and e_2 could be in the relative position of Case 11 in Figure 2.1 and consequently have more than $(d - 1) - 3 = 0$ odd contiguous subsets. However, this is no problem as in all possible extensions to g_1 and g_2 at least one of the edges continues to have only one odd contiguous subset (see Figure 4.4).

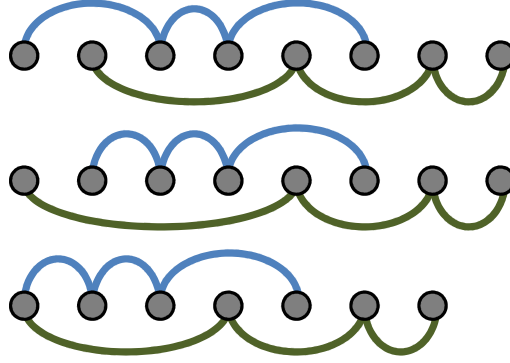


Figure 4.4: All possible 4-uniform extensions of Case 11 in Figure 2.1 as occurring in the construction of H_m^4 .

To bound the number of vertices of H_m^d we use

$$\begin{aligned} n_2^d &= d, \\ n_m^d &= n_{m-1}^d + n_m^{d-1} \quad k > 2. \end{aligned}$$

Iteratively, we get that $n_m^d = d + \sum_{r=3}^m n_r^{d-1} \leq m \cdot n_m^{d-1} \leq \dots \leq m^{d-3} \cdot \hat{n}_m = m^{2m+d-3}$. Now for large m and fixed d ,

$$\frac{\log n_m^d}{\log \log n_m^d} \leq (2m + d - 3) \cdot \frac{\log(m)}{\log((2m + d - 3)\log(m))} \leq 3m \cdot \frac{\log(m)}{\log(m \log(m))} = \mathcal{O}(m).$$

Hence, $m = \Omega\left(\frac{\log n_m^d}{\log \log n_m^d}\right)$. □

Note that by monotonicity also $\chi_{2d-2,d}^w(n) = \Omega\left(\frac{\log n}{\log \log n}\right)$.

References

- [AH77] K. Appel and W. Haken, *Every planar map is four colorable. Part I: Discharging*, Illinois Journal of Mathematics **21** (1977), no. 3, 429–490.
- [AHK77] K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable. Part II: Reducibility*, Illinois Journal of Mathematics **21** (1977), no. 3, 491–567.

- [BG00] J. Bokowski and A. Guedes de Oliveira, *On the Generation of Oriented Matroids*, *Discrete and Computational Geometry* **24** (2000), 197–208.
- [Bre83] U. Brehm, *A nonpolyhedral triangulated Möbius strip*, *Proceedings of the American Mathematical Society* **89** (1983), no. 3, 519–522.
- [EL75] P. Erdős and L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, *Infinite and Finite Sets (to Paul Erdős on his 60th birthday)*, Vol. II, North-Holland (1975), 609–627.
- [Flo34] A. Flores, *Über n -dimensionale Komplexe, die im \mathbb{R}_{2n+1} absolut selbstverschlungen sind*, *Ergebnisse Eines Mathematischen Kolloquiums* **6** (1934), 4–7.
- [Fár48] I. Fáry, *On straight-line representation of planar graphs*, *Acta Scientiarum Mathematicarum Szeged* **11** (1948), 229–233.
- [GGL95] R. Graham, M. Grötschel, and L. Lovász (eds.), *Handbook of Combinatorics, Vol. 1*, North-Holland, 1995.
- [Gun09] A. Gundert, *On The Complexity of Embeddable Simplicial Complexes*, Diplomarbeit, Technische Universität Berlin, 2009.
- [Hea90] J. C. Heawood, *Map-colour theorem*, *The Quarterly Journal of Pure and Applied Mathematics* **24** (1890), 332–338.
- [Kal02] G. Kalai, *Algebraic Shifting*, *Computational Commutative Algebra and Combinatorics*, *Advanced Studies in Pure Mathematics* **33** (2002), 121–163, Mathematical Society of Japan.
- [Men28] K. Menger, *Dimensionstheorie*, Teubner, Leipzig, 1928.
- [MTW11] J. Matoušek, M. Tancer, and U. Wagner, *Hardness of Embedding Simplicial Complexes in \mathbb{R}^d* , *Journal of the European Mathematical Society* **13** (2011), 259–295.
- [Nöb31] G. Nöbeling, *Über eine n -dimensionale Universalmenge im \mathbb{R}_{2n+1}* , *Mathematische Annalen* **104** (1931), no. 1, 71–80.
- [Ren92] J. Renegar, *On the computational complexity and geometry of the first-order theory of the reals. I, II, III*, *Journal of Symbolic Computation* **13** (1992), no. 3, 255–299, 301–327, 329–352.
- [RY68] G. Ringel and J. W. T. Youngs, *Solution of the Heawood map-coloring problem*, *Proceedings of the National Academy of Sciences* **60** (1968), no. 2, 438–445.
- [She68] G. C. Shephard, *A Theorem on Cyclic Polytopes*, *Israel Journal of Mathematics* **6** (1968), no. 4, 368–372.
- [Spe77] J. Spencer, *Asymptotic lower bounds for Ramsey functions*, *Discrete Mathematics* **20** (1977), 69–76.
- [vK33] E. R. van Kampen, *Komplexe in euklidischen Räumen*, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* **9** (1933), no. 1, 72–78, corrections *ibidem* pp. 152–153.