On the Structure of Graphs with Large Minimum Bisection

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Abstract

Bounded degree trees and bounded degree planar graphs on n vertices are known to admit bisections of width $\mathcal{O}(\log n)$ and $\mathcal{O}(\sqrt{n})$, respectively. We investigate the structure of graphs that meet this bound. In particular, we show that such a tree must have diameter $\mathcal{O}(n/\log n)$ and such a planar graph must have tree width $\Omega(\sqrt{n})$. To show the result for trees, we derive an inequality that relates the width of a minimum bisection with the diameter of a tree.

1 Introduction and Results

A *bisection* of a graph G is a partition of its vertex set into two sets L and R of sizes differing by at most one. The *width* of a bisection is defined to be the number of edges with one vertex in L and one vertex in R, and the minimum width of a bisection in G is denoted by $\mathrm{MinBis}(G)$. Determining a bisection of minimum width is a famous optimization problem that is (unlike the Minimum Cut Problem) known to be NP-hard [GJS].

In this paper, we will concentrate on minimum bisections of trees and planar graphs. Jansen et al. showed that dynamic programming gives an algorithm with running time $\mathcal{O}(2^t n^3)$ for an arbitrary graph on n vertices when a tree decomposition of width t is provided as input [JKLS]. Thus, the problem becomes polynomially tractable for graphs of constant tree width. On the other hand, it is open whether finding a minimum bisection for a planar graph is in P or is NP-hard. Currently, the best known approximation algorithm achieves an approximation ratio of $\mathcal{O}(\log n)$ for arbitrary graphs on n vertices [Räc], and nothing better has

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been established for planar graphs. Further, Berman and Karpinski showed that the Minimum Bisection Problem restricted to 3-regular graphs is as hard to approximate as its general version [BK]. Moreover, Arora et al. presented a PTAS for finding a minimum bisection in graphs with a linear minimum degree [AKK]. Thus, we focus on graphs that have small (in fact, constant) maximum degree and still a large minimum bisection width. If T is a tree on n vertices and maximum degree d, one can show that owing to the existence of a *separating vertex* (i.e. a vertex whose removal leaves no connected component of size greater than $n/2$), we always have $\mathrm{MinBis}(T) \leq d \cdot \log_2 n$. Similarly, using the Planar Separator Theorem [AST], this idea can be generalized to give an $\mathcal{O}(\sqrt{n})$ upper bound on $\mathrm{MinBis}(G)$ for planar graphs G with bounded maximum degree. Both bounds are tight up to a constant factor as one can show that the perfect ternary tree on n vertices has minimum bisection width $\Omega(\log n)$ and the square grid on n vertices has minimum bisection width $\Omega(\sqrt{n})$.

Our aim is to investigate the structure of bounded degree graphs that have a large minimum bisection, in other words, trees T with $\mathrm{MinBis}(T) = \Omega(\log n)$ and planar graphs G with $\mathrm{MinBis}(G) = \Omega(\sqrt{n})$. For example, we will show that in a bounded degree tree with minimum bisection of order $\Omega(\log n)$, the length of any path must be bounded by $\mathcal{O}(n/\log n)$. More generally, we establish the following inequality:

Theorem 1 *Let T be a tree on n vertices and denote by $\Delta(T)$ its maximum degree and by $\mathrm{diam}(T)$ its diameter. Then*

$$\mathrm{MinBis}(T) \leq \frac{8\Delta(T)n}{\mathrm{diam}(T)}.$$

In the case of planar graphs, we can no longer use the diameter to control the minimum bisection. For example, a graph consisting of a square grid on $\frac{3}{4}n$ vertices connected to a path on $\frac{n}{4}$ vertices has linear diameter but does not allow a bisection of constant width. However, we can show that a planar graph with a minimum bisection width close to the upper bound $\mathcal{O}(\sqrt{n})$ must indeed be far away from a tree-like structure.

Theorem 2 *For every $d \in \mathbb{N}$ and every $c > 0$, there is a $\gamma = \gamma(c) > 0$ such that for all planar graphs G on n vertices with $\Delta(G) \leq d$, we have*

$$\mathrm{MinBis}(G) \geq c\sqrt{n} \qquad \Rightarrow \qquad \mathrm{tw}(G) \geq \gamma\sqrt{n}.$$

Using Theorem 6.2 of Robertson, Seymour, and Thomas in [RST], this immediately implies that there is a constant $\gamma' = \gamma'(c) > 0$ such that every such planar graph contains a square grid on $\gamma' n$ vertices as minor.

2 Trees

Although the statement in Theorem 1 looks like an elementary inequality, its proof is somewhat lengthy and we need to introduce a few more definitions to be able to sketch it. First, we define the *relative diameter* of a graph to be

$$\operatorname{diam}^*(G) := \frac{1}{|V(G)|} \sum_{G'\colon \text{component of } G} (\operatorname{diam}(G') + 1).$$

Observe that, in a tree, there is only one component to consider and thus the relative diameter of a tree denotes the fraction of the vertices in a longest path of the tree, but we will need this parameter also for graphs that may not be connected.

Moreover, we need to take a more general approach and consider partitions where the size of the classes can be specified by an input parameter m. Furthermore, we denote by $e_G(V_1, \ldots, V_k)$ the number of edges in G that have their vertices in two different sets $V_i \neq V_j$ and define $[n] = \{1, 2, 3, \ldots, n\}$ for $n \in \mathbb{N}$. The following theorem is the driving engine for the proof of Theorem 1.

Theorem 3 *For all trees T on $n \geq 3$ vertices and for all $m \in [n]$, the vertex set of T can be partitioned into three classes $L \cup R \cup S$ such that one of the following two options occurs:*

(i) $S = \emptyset$ *and* $|L| = m$ *and* $e_T(L, R, S) \leq 2$.

(ii) $S \neq \emptyset$ *and* $|L| \leq m \leq |L| + |S|$ *and* $e_T(L, R, S) \leq \frac{2}{\operatorname{diam}^*(T)} \Delta(T)$ *and* $\operatorname{diam}^*(T[S]) \geq 2 \operatorname{diam}^*(T)$.

This result states that we can either find a partition into two sets L and R with exactly the right cardinality by cutting very few edges, or there is a partition with an additional set S, such that the set L is smaller and the set $L \cup S$ is larger than the required size m, as well as the additional feature that the relative diameter of $T[S]$ is at least twice as large as that of T. Using Theorem 3 recursively for the graph $T[S]$, the relative diameter can therefore be doubled in each round, until it exceeds $1/2$, at which point Option *(ii)* in Theorem 3 is no longer feasible, which will then prove Theorem 1.

We conclude this section with a few words about how to prove Theorem 3. Consider a longest path P in T and denote by x_0 and y_0 its first and last vertex. For each vertex $z \in V(P)$, let T_z be the component of $T - E(P)$ that contains z and call z the root of T_z. We label the vertices of T with $1, 2, \ldots, n$ so that x_0 receives label 1; for each $z \in V(P)$, the vertices of T_z receive consecutive labels and z receives the largest label among those; for all $z, z' \in V(P)$ with $z \neq z'$, if x_0 is closer to z than to z', then the label of z is smaller than the label of z'. Given this labeling, we now define for every $x \in [n]$ the vertex $f(x) := x + m$. If for some vertex $x \in V(P)$ the vertex $f(x)$ lies also in P, then we are done by choosing $L := \{x + 1, \ldots, x + m\}$ and $R := [n] \setminus L$, which satisfies all requirements of Option *(i)* in Theorem 3. Otherwise, one can show that there exists a vertex $z \in V(P)$ such that all vertices of P that are mapped into $V(T_z)$ and the vertex sets of all trees that are mapped completely into T_z by f form a set S such that the condition on $\operatorname{diam}^*(T[S])$ is satisfied. Furthermore, this vertex

z has the property that the set L, which consists of the vertex sets of all trees whose roots have labels strictly between $z - m$ and z and of some additional vertices from T_z, satisfies the remaining conditions of Option *(ii)* in Theorem 3.

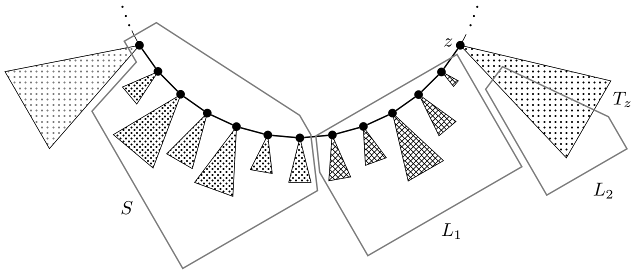

Figure 1: Construction of S and $L := L_1 \cup L_2$.

3 Planar Graphs

To sketch the proof for Theorem 2, consider its contrapositive: for all $d \in \mathbb{N}$ and for all $c > 0$, we choose $\gamma > 0$ so small that $\gamma \left(2 \log_2 \frac{3}{\gamma\sqrt{2}} + 7\right) d \leq c$. We claim that then, for every planar graph G on n vertices with $\Delta(G) \leq d$, the following holds:

$$\mathrm{tw}(G) + 1 \leq \gamma\sqrt{n} \qquad \Rightarrow \qquad \mathrm{MinBis}(G) \leq c\sqrt{n}. \tag{1}$$

To explain how we find a bisection of sufficiently small width, let us assume that, for some $0 < \delta < 1$, we can find δ-*separators* of a given size in the graph G and its subgraphs, i.e. a vertex subset whose removal leaves connected components of size at most δn. Having removed such a separator S' from the graph, we can assign the vertices of all but one component of $G - S'$ to L and R, so that $|L| \leq \lceil \frac{n}{2} \rceil$ and $|R| \leq \lfloor \frac{n}{2} \rfloor$. We then continue recursively with the remaining component, which has size at most δn. At the end, we denote by S the union of the various separators and distribute the vertices in S to the sets L and R in such a way that L and R form a bisection of G. It is easy to see that $e_G(L, R) \leq \Delta(G) \cdot |S|$ and it only remains to find a bound on $|S|$.

In each round of a first phase, we use a cluster from an optimal tree decomposition of G that can serve as a $1/2$-separator and, by assumption, has size at most $\gamma\sqrt{n}$. Denote by n_i the number of vertices after the i-th round. This first phase stops when $\gamma\sqrt{n} > \frac{3}{\sqrt{2}}\sqrt{n_i}$. Due to $n_i \leq n/2^{i-1}$, it is easy to see that the index i^*, where this happens for the first time, can be bounded by a constant depending only on γ.

After the i^*-th round, we switch to $2/3$-separators guaranteed by the Planar Separator Theorem [AST], which will have size at most $\frac{3}{\sqrt{2}}\sqrt{n_i}$. During this second phase, $n_i \le (2/3)^{i-i^*} n_{i^*}$ holds and thus the number of vertices collected in S during the second phase can be bounded from above by

$$\sum_{i=i^*}^{\infty} \tfrac{3}{\sqrt{2}}\sqrt{n_i} \;\le\; \tfrac{3}{\sqrt{2}}\sqrt{n_{i^*}}\sum_{i=0}^{\infty}\left(\sqrt{\tfrac{2}{3}}\right)^i \;\le\; \alpha\sqrt{n_{i^*}}, \qquad \text{with } \alpha = \tfrac{9}{\sqrt{2}} + 3\sqrt{3}.$$

Summing up, we have $|S| \le i^*\gamma\sqrt{n}+\alpha\sqrt{n_{i^*}}$ in total. Now, computing an upper bound on i^* and on n_{i^*} will give the desired bound.

4 Concluding Remarks and Open Questions

In Theorem 1 and Theorem 2 we have established necessary conditions for trees and planar graphs to have large minimum bisection width, namely a small diameter and a large tree width, respectively. In both cases, these conditions are not sufficient, but it would be interesting to find additional conditions that give characterizations of graphs in certain classes with large minimum bisection.

References

[AKK] S. Arora, D. Karger, and M. Karpinski. Polynomial time approximation schemes for dense instances of NP-hard problems. *Journal of Computer and System Sciences*, 58(1):193–210, 1999.

[AST] N. Alon, P. Seymour, and R. Thomas. Planar separators. *SIAM Journal on Discrete Mathematics*, 7(2):184–193, 1994

[BK] P. Berman and M. Karpinski. Approximation hardness of bounded degree MIN-CSP and MIN-BISECTION. In *Automata, languages and programming*, vol. 2380 of *LNCS*, pages 623–632. Springer, Berlin, 2002.

[GJS] M. Garey, D. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237–267, 1976.

[JKLS] K. Jansen et al. Polynomial time approximation schemes for max-bisection on planar and geometric graphs. *SIAM Journal on Computing*, 35(1):110–119, 2005.

[Räc] H. Räcke. Optimal hierarchical decompositions for congestion minimization in networks. In *Proceedings of the 40th annual ACM symposium on Theory of computing*, STOC '08, pages 255–264, New York, NY, USA, 2008. ACM.

[RST] N. Robertson, P. Seymour, and R. Thomas. Quickly excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 62(2):323–348, 1994.