ALMOST SPANNING SUBGRAPHS OF RANDOM GRAPHS AFTER ADVERSARIAL EDGE REMOVAL

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Dedicated to Vojtěch Rödl on the occasion of his sixtieth birthday

ABSTRACT. Let $\Delta \geq 2$ be a fixed integer. We show that the random graph $\mathcal{G}_{n,p}$ with $p \gg (\log n/n)^{1/\Delta}$ is robust with respect to the containment of almost spanning bipartite graphs H with maximum degree Δ and sublinear bandwidth in the following sense: asymptotically almost surely, if an adversary deletes arbitrary edges from $\mathcal{G}_{n,p}$ in such a way that each vertex loses less than half of its neighbours, then the resulting graph still contains a copy of all such H.

1. Introduction and results

In this paper we study graphs that are robust in the following sense: even after adversarial removal of a specified proportion of their edges, they still contain copies of every graph from a certain class of graphs.

In order to make this precise, we use the notion of *resilience* (see [29]). Let $\mathcal{P}$ be a monotone increasing graph property and $G = (V, E)$ be a graph. The *global resilience* $R_g(G, \mathcal{P})$ of G with respect to $\mathcal{P}$ is the minimum $r \in \mathbb{R}$ such that deleting a suitable set of $r \cdot |E|$ edges from E creates a graph which is not in $\mathcal{P}$. The *local resilience* $R_\ell(G, \mathcal{P})$ of G with respect to $\mathcal{P}$ is the minimum $r \in \mathbb{R}$ such that deleting a suitable set of edges, respecting the restriction that at most $r \cdot \deg_G(v)$ edges incident to v should be removed for every vertex $v \in V$, creates a graph which is not in $\mathcal{P}$.

For example, using this terminology, the classical theorems of Turán [30] and Dirac [15] can be stated as follows: the global resilience of the complete graph K_n with respect to containing a clique on r vertices is $\frac{1}{r-1} - o(1)$, and the local resilience of K_n with respect to containing a Hamilton cycle is $\frac{1}{2} - o(1)$. In this paper we stay quite close to the scenario of these two examples insofar as we will also consider properties that deal with subgraph containment. However, we are interested in the resilience of graphs which are much sparser than the complete graph.

It turns out that the random graph $\mathcal{G}_{n,p}$ is well suited for this purpose ($\mathcal{G}_{n,p}$ is defined on vertex set $[n] = \{1, \ldots, n\}$ and edges exist independently of each other with probability p). Clearly, asymptotically almost surely (a.a.s.) the local resilience of $\mathcal{G}_{n,p}$ with respect to containing a Hamilton cycle (or in fact any connected graph on more than, say, $\frac{1}{2}n$ vertices) is at most $\frac{1}{2} + o(1)$, since for bigger values it is easy to disconnect the graph into components of size at most $\frac{1}{2}n$ by deleting edges respecting the corresponding resilience definition. Sudakov and Vu [29] showed that indeed a.a.s. the local resilience of $\mathcal{G}_{n,p}$ with respect to containing a Hamilton cycle is $\frac{1}{2} - o(1)$ if $p > \log^4 n/n$. A result of Dellamonica, Kohayakawa, Marciniszyn and Steger [12] implies that a.a.s. the local resilience of $\mathcal{G}_{n,p}$ with respect to containing cycles of length at least $(1 - \alpha)n$ is

Date: May 22, 2013.

The first and third author were partially supported by DFG grant TA 309/2-1. The first author was partially supported by an EUBRANEX grant of the EU programme EM ECW. The second author was partially supported by CNPq (Proc. 308509/2007-2, 485671/2007-7 and 486124/2007-0). The cooperation of the three authors was supported by a joint CAPES-DAAD project (415/ppp-probral/po/D08/11629, Proj. no. 333/09).

An extended abstract appeared in: V Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS), Gramado, Brazil, Electronic Notes in Discrete Mathematics, 2009. The authors are grateful to NUMEC/USP, Núcleo de Modelagem Estocástica e Complexidade of the University of São Paulo, and Project MaCLinC/USP, for supporting this research.

$\frac{1}{2} - o(1)$ for any $0 < \alpha < \frac{1}{2}$ and $p \gg 1/n$. We shall discuss the various lower bounds for the edge probability p occuring in these and later results at the end of Section 2.

Recently Balogh, Csaba, and Samotij [7] studied the local resilience of $\mathcal{G}_{n,p}$ with respect to containing all trees on $(1-\eta)n$ vertices with constant maximum degree Δ. They showed that there is a constant $c = c(\Delta, \eta)$ such that for $p \geq c/n$ this local resilience is also $\frac{1}{2} - o(1)$ a.a.s.

Now we extend the scope of investigations to the containment of a much larger class of subgraphs. A graph has *bandwidth* at most b if there exists a labelling of the vertices by numbers $1, \dots, n$, such that for every edge ij of the graph we have $|i - j| \leq b$. Let $\mathcal{H}(m, \Delta)$ denote the class of all graphs on m vertices with maximum degree at most Δ, and $\mathcal{H}_2^b(m, \Delta)$ denote the class of all *bipartite* graphs in $\mathcal{H}(m, \Delta)$ which have bandwidth at most b. Our result asserts that the local resilience of $\mathcal{G}_{n,p}$ with respect to containing all graphs H from $\mathcal{H}_2^{\beta n}((1-\eta)n, \Delta)$ is $\frac{1}{2} - o(1)$ for small β and η and for $p = p(n) = o(1)$ sufficiently large.

Theorem 1. *For each $\eta, \gamma > 0$ and $\Delta \geq 2$ there exist positive constants β and c such that the following holds for $p \geq c(\log n/n)^{1/\Delta}$. Asymptotically almost surely every spanning subgraph $G = (V, E)$ of $\mathcal{G}_{n,p}$ with $\deg_G(v) \geq (\frac{1}{2} + \gamma) \deg_{\mathcal{G}_{n,p}}(v)$ for all $v \in V$ contains a copy of every graph H in $\mathcal{H}_2^{\beta n}((1-\eta)n, \Delta)$.*

We note that several important classes of graphs have sublinear bandwidth, and hence Theorem 1 does apply to them: this is the case for, e.g., the class of all bounded degree planar graphs (see [10]).

As an application of this theorem we derive a result on rainbow H-copies with $H \in \mathcal{H}_2^{\beta n}((1-\eta)n, \Delta)$ for certain edge-colourings of K_n in Section 3. The proof of Theorem 1 is prepared in Sections 4–7 and presented in Section 8. First, however, we will compare our result to related results in the next section.

2. Background

As we saw at the end of the last section, we are looking for graphs that not only contain one specific subgraph but a large class of graphs. A graph G is called *universal* for a class of graphs $\mathcal{H}$ if G contains a copy of every graph from $\mathcal{H}$ as a subgraph. In this section, we first briefly sketch some results concerning universality in general and then come back to resilience with respect to universality.

Dellamonica, Kohayakawa, Rödl, and Ruciński [13] show that $\mathcal{G}_{n,p}$ is a.a.s. universal for $\mathcal{H}(n, \Delta)$ for some p in $\widetilde{\mathcal{O}}(n^{-1/2\Delta})$ (where $\widetilde{\mathcal{O}}$ hides polylogarithmic factors). It is also shown in [13] that the lower bound for the edge probability p can be improved if we restrict our attention to balanced bipartite graphs: Let $\mathcal{H}_2(m, m, \Delta)$ denote the class of bipartite graphs in $\mathcal{H}(2m, \Delta)$ with two colour classes of equal size. Then $\mathcal{G}_{2n,p}$ a.a.s. is universal for $\mathcal{H}_2(n, n, \Delta)$ for some p in $\widetilde{\mathcal{O}}(n^{-1/\Delta})$. The same lower bound for p also guarantees universality for *almost spanning* graphs of arbitrary chromatic number: Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi [4] prove that for every $\eta > 0$ and for some p in $\widetilde{\mathcal{O}}(n^{-1/\Delta})$, the random graph $\mathcal{G}_{n,p}$ a.a.s. is universal for $\mathcal{H}((1-\eta)n, \Delta)$. Recently, Dellamonica, Kohayakawa, Rödl, and Ruciński [14] generalised these results and obtained a corresponding lower bound for spanning graphs: They have shown that $\mathcal{G}_{n,p}$ is a.a.s. universal for $\mathcal{H}(n, \Delta)$ for some p in $\widetilde{\mathcal{O}}(n^{-1/2\Delta})$.

Alon and Capalbo [2, 3] gave explicit constructions of graphs with average degree $\widetilde{\mathcal{O}}(n^{-2/\Delta})n$ that are universal for $\mathcal{H}(n, \Delta)$. For results concerning universal graphs for trees see, e.g., [5].

Moving on to resilience, it is clear that an adversary can destroy any spanning subgraph by deleting the edges incident to a single vertex. Hence any graph must have trivial global resilience with respect to universality for spanning subgraphs.

However, if we focus on subgraphs of smaller order, then sparse random graphs have a global resilience arbitrarily close to 1: Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi [4] show that for every $\gamma > 0$ there is a constant $\eta > 0$ such that for some p in $\widetilde{\mathcal{O}}(n^{-1/2\Delta})$ the random graph $\mathcal{G}_{n,p}$ a.a.s. has global resilience $1 - \gamma$ with respect to universality for $\mathcal{H}_2(\eta n, \eta n, \Delta)$. In other words, $\mathcal{G}_{n,p}$ contains *many* copies of all graphs from $\mathcal{H}_2(\eta n, \eta n, \Delta)$ *everywhere*.

Finally, the concept of local resilience allows for non-trivial results concerning universality for almost spanning subgraphs. For example, a conjecture of Bollobás and Komlós proven in [11] asserts that the local resilience of the complete graph K_n with respect to universality for $\mathcal{H}_r^{\beta n}(n,\Delta)$ is $\frac{1}{r} - o(1)$. Here $\mathcal{H}_r^{\beta n}(n,\Delta)$ is the class of all r-colourable n-vertex graphs with maximum degree at most Δ and bandwidth at most βn, and one can show that the bandwidth constraint cannot be omitted.

Theorem 2 ([11])**.** *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds. If H is an r-chromatic graph on n vertices with $\Delta(H) \leq \Delta$, and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$, then G contains a copy of H.*

Our Theorem 1 replaces K_n by the much sparser graph $\mathcal{G}_{n,p}$, but it only treats the case $r = 2$ and almost spanning subgraphs.

Let us mention two more recent papers which continued this line of research. Huang, Lee and Sudakov considered almost spanning factors for constant probability p and showed that one cannot hope to obtain spanning subgraphs, as $\Omega(p^{-2})$ vertices may be forced to be left out (see [21]). Also, Balogh, Lee and Samotij considered the case of almost spanning triangle factors for $p \gg (\log n/n)^{1/2}$ (see [8]).

Before we conclude this section, let us briefly discuss the lower bounds for the edge probability p mentioned in the results above, summarized in Table 1. First, a straightforward counting argument shows that any graph that is universal for $\mathcal{H}(n,\Delta)$ must have at least $\Omega(n^{2-2/\Delta})$ edges. Moreover, it is easy to see that an edge probability $p = n^{\varepsilon - 2/\Delta}$ with $\varepsilon < \frac{1}{\Delta^2}$ is not sufficient to guarantee that $\mathcal{G}_{n,p}$ is universal even for the more restrictive class $\mathcal{H}_2(\eta n, \eta n, \Delta)$. Indeed, consider the graph $H \in \mathcal{H}_2(\eta n, \eta n, \Delta)$ consisting of $\eta n/\Delta$ copies of $K_{\Delta,\Delta}$. The expected number of copies of $K_{\Delta,\Delta}$ in $\mathcal{G}_{n,p}$ is at most

$$n^{2\Delta} p^{\Delta^2} = n^{2\Delta}(n^{-\frac{2}{\Delta}+\varepsilon})^{\Delta^2} = n^{2\Delta - 2\Delta + \varepsilon\Delta^2} \ll n,$$

and hence a.a.s. $\mathcal{G}_{n,p}$ does not contain a copy of H.

	Result	$\boldsymbol{p}$	**Reference**
Universality	$\mathcal{H}(n,\Delta) \subseteq \mathcal{G}_{n,p}$	$p = n^{-1/\Delta}$	[14]
Resilience	$R_g\big(\mathcal{G}_{n,p}, \mathcal{H}_2(\eta n, \eta n, \Delta)\big) \geq 1 - \gamma$	$p = n^{-1/2\Delta}$	[4]
	$R_\ell\big(\mathcal{G}_{n,p}, \mathcal{H}_2^{\beta n}((1-\eta)n, \Delta)\big) \geq \frac{1}{2} - \gamma$	$p = n^{-1/\Delta}$	Theorem 1

Table 1. Summary of (best) known universality and resilience results (logarithmic factors for p are omitted).

3. An application: rainbow copies of bipartite graphs

Let φ be an arbitrary colouring of the edges of the complete graph K_n. If φ uses no colour more than k times then we say that φ is *k-bounded*. Moreover, a copy of a graph H in K_n is a *rainbow* copy if φ uses no colour more than once on H. If there is a rainbow copy of H in K_n then φ is called *H-rainbow*.

Erdős, Nešetřil, and Rödl [16] asked for which $k = k(n)$ every k-bounded edge colouring of K_n has a rainbow Hamilton cycle. Frieze and Reed [17] showed that $k(n)$ can grow as fast as $\kappa n/\log n$ for some constant κ (for early progress on this problem see the references in [17]). Albert, Frieze, and Reed [1] improved this bound to $n/65$, which shows that k can grow linearly, as was previously conjectured by Hahn and Thomassen [19].

Here we consider the analogous question for H-rainbow colourings with $H \in \mathcal{H}_2^{\beta n}((1-\eta)n, \Delta)$. As a consequence of our main theorem, Theorem 1, we prove the following result.

Theorem 3. *For every $\eta > 0$ and $\Delta \geq 2$ there exist positive constants β and κ such that for n sufficiently large, for every graph $H \in \mathcal{H}_2^{\beta n}((1-\eta)n, \Delta)$ and $k \leq \kappa(n/\log n)^{1/\Delta}$, every k-bounded edge-colouring of K_n is H-rainbow.*

For the proof of this theorem we apply the strategy of [17] and do the following for a given k-bounded edge colouring φ of K_n. We first take a random subgraph $\Gamma = \mathcal{G}_{n,p}$ of K_n and then delete all edges in Γ whose colour appears more than once in Γ. Denote the resulting graph by $\Gamma(\varphi)$. Any subgraph of $\Gamma(\varphi)$ is trivially rainbow and hence it remains to show that there is a copy of H in $\Gamma(\varphi)$ in order to establish Theorem 3. In view of Theorem 1 it clearly suffices to prove the following lemma.

Lemma 4. *Let $p = p(n)$ and $k = k(n)$ be such that $p \geq 10^6 \log n/n$ and $pk \leq 10^{-3}$. For any k-bounded edge colouring φ of K_n, with probability $1 - o(1)$ all vertices v in $\Gamma = \mathcal{G}_{n,p}$ satisfy $\deg_{\Gamma(\varphi)}(v) \geq \frac{2}{3}\deg_\Gamma(v)$.*

Proof (sketch). Let v be an arbitrary vertex of Γ. We classify the 'deleted' edges incident to v, that is, those edges in $E\big(v, N_\Gamma(v) \setminus N_{\Gamma(\varphi)}(v)\big)$, into two sets: the set N_1 of those edges whose colour appears only once in $E(v, N_\Gamma(v))$ (but also somewhere else in Γ) and the set N_2 of those edges whose colour appears at least twice in $E(v, N_\Gamma(v))$. With probability $1 - o(1/n)$ we have that $\deg_\Gamma(v)$ lies in the interval $[(1-\frac{1}{20})np, (1+\frac{1}{20})np]$ by a Chernoff bound. Therefore, showing

(*i*) $\mathbb{P}(|N_1| \geq \frac{1}{10}np) = o(1/n)$ and
(*ii*) $\mathbb{P}(|N_2| \geq \frac{1}{10}np) = o(1/n)$

and applying the union bound proves the lemma.

For establishing (i) we expose the edges incident to v first, which enables us to determine $\deg_\Gamma(v)$. We have $\mathbb{P}\left(\deg_\Gamma(v) \geq \frac{21}{20}np\right) = o(1/n)$. Subsequently we expose the remaining edges. Recall that for any edge $vw \in N_1$ the colour $\varphi(vw)$ appears somewhere else in Γ, which happens with probability at most $p' := pk$. Since these events are independent for different colours, we have

$$\begin{aligned}\mathbb{P}(|N_1| \geq t) &\leq \mathbb{P}\left(\deg_\Gamma(v) \geq \tfrac{21}{20}np\right) + \mathbb{P}\left(|N_1| \geq t \,\middle|\, \deg_\Gamma(v) \leq \tfrac{21}{20}np\right) \\ &= o(1/n) + \mathbb{P}(X \geq t)\,,\end{aligned}$$

where X is a random variable with distribution $\mathrm{Bi}(n', p')$ where $n' = \deg_\Gamma(v) \leq \frac{21}{20}np$. Clearly $\mathbb{E}\,X \leq \frac{21}{20}np \cdot pk \leq \frac{1}{100}np$ and therefore (i) follows from an application of a Chernoff bound, since $np \geq 10^6 \log n$.

For establishing (ii) consider the random variable Y that counts edges in $E(v, N_\Gamma(v))$ whose colour appears only once in $E(v, N_\Gamma(v))$. Then $|N_2| = \deg_\Gamma(v) - Y$ and so it suffices to show that $\mathbb{P}(Y \leq \frac{19}{20}np) = o(1/n)$, using again that $\deg_\Gamma(v) > \frac{21}{20}np$ happens with probability $o(1/n)$. To see this, assume that $1, \dots, \ell$ are the colours that appear on the edges of K_n containing v, and let k_i be the number of such edges with colour $i \in [\ell]$. Then $Y = \sum_{i \in [\ell]} Y_i$ where Y_i is the indicator variable for the event that $E(v, N_\Gamma(v))$ contains exactly one edge of colour i. Observe that the Y_i are independent random variables and that $\mathbb{P}(Y_i = 1) = k_i p(1-p)^{k_i - 1}$. In addition $1 \geq (1-p)^{k_i-1} \geq (1-p)^k \geq \exp(-\frac{p}{1-p}k) \geq \frac{100}{101}$ and hence

$$\mathbb{E}\,Y = \sum_{i \in [\ell]} k_i p(1-p)^{k_i-1} \leq np \quad \text{and} \quad \mathbb{E}\,Y \geq \tfrac{100}{101}(n-1)p \geq \tfrac{99}{100}np\,.$$

We conclude $\mathbb{P}(Y \leq \frac{19}{20}np) = o(1/n)$ from $\mathbb{P}(Y \leq \mathbb{E}\,Y - t) \leq \exp(-\frac{1}{2}t^2/\mathbb{E}\,Y)$ (see [22, Theorem 2.10]) by setting $t := \frac{1}{100}np$ and using $np \geq 10^6 \log n$. □

As mentioned earlier, the bound on $k(n)$ established in [17] for rainbow Hamilton cycles is not best possible. As it turns out, the bound on k in Theorem 3 above can be improved as well. Indeed, such an improvement has recently been established in [9], where Lovász's local lemma is used. However, we observe that the method of proof above is more robust in the sense that one can, for instance, prove that suitably bounded colourings of *sparse random graphs* are H-rainbow—something that does not seem to be within reach of the method of proof in [9] (we omit the details).

4. Sparse regularity

In this section we will introduce one of the main tools for our proof, a sparse version of the regularity lemma developed by Rödl and one of the current authors (see [23, 25]). Before stating this lemma we introduce the necessary definitions.

Let $G = (V, E)$ be a graph, and suppose $p \in (0, 1]$ and $\varepsilon > 0$ are reals. For disjoint nonempty sets $U, W \subseteq V$ the *p-density* of the pair (U, W) is defined as $d_{G,p}(U, W) := e_G(U, W)/(p|U||W|)$. The pair (U, W) is *(ε, p)-regular* if $|d_{G,p}(U', W') - d_{G,p}(U, W)| \le \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \ge \varepsilon|U|$ and $|W'| \ge \varepsilon|W|$.

An *(ε, p)-regular partition* of $G = (V, E)$ is an *ε-equipartition* $V_0\dot\cup V_1\dot\cup \dots \dot\cup V_r$ of V, that is, with $|V_0| \le \varepsilon|V|$ and $|V_1| = \dots = |V_r|$, such that (V_i, V_j) is an (ε, p)-regular pair in G for all but at most $\varepsilon\binom{r}{2}$ pairs $ij \in \binom{[r]}{2}$. The partition classes V_i with $i \in [r]$ are called the *clusters* of the partition and V_0 is the *exceptional set*.

The sparse regularity lemma asserts the existence of (ε, p)-regular partitions for sparse graphs G without 'dense spots'. To quantify this latter property we need the following notion. Let $\eta > 0$ and $K > 1$ be real numbers. We say that $G = (V, E)$ is *(η, K)-bounded with respect to p* if for all disjoint sets $X, Y \subseteq V$ with $|X|, |Y| \ge \eta|V|$ we have $e_G(X, Y) \le Kp|X||Y|$.

Lemma 5 (sparse regularity lemma)**.** *For each $\varepsilon > 0$, $K > 1$, and $r_0 \ge 1$ there are constants r_1, ν, and n_0 such that for any $p \in (0, 1]$ the following holds. Any graph $G = (V, E)$ which has at least n_0 vertices and is (ν, K)-bounded with respect to p admits an (ε, p)-regular ε-equipartition with r clusters, for some $r_0 \le r \le r_1$.* □

As it turns out, we shall only make use of what one could call 'one-sided regularity'. We call a pair (U, W) *(ε, d, p)-dense* if $d_{G,p}(U', W') \ge d - \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \ge \varepsilon|U|$ and $|W'| \ge \varepsilon|W|$. Clearly, an (ε, p)-regular pair (U, W) is (ε, d, p)-dense for $d = d_{G,p}(U, W)$. Occasionally, in informal discussions, when the particular value of d or ε is not immediately relevant, we say that an (ε, p)-regular pair (U, W) is *(ε, p)-dense* or *p-dense*.

An ε-equipartition $V_0\dot\cup V_1\dot\cup \dots \dot\cup V_r$ of a graph $G = (V, E)$ is an *(ε, d, p)-dense partition* with *reduced graph* R if $V(R) = [r]$ and the pair (V_i, V_j) is (ε, d, p)-dense in G whenever $ij \in E(R)$. Note that, given an (ε, p)-regular partition as in Lemma 5 and a real number d, one has an (ε, d, p)-dense partition of G with the reduced graph R, with $ij \in E(R)$ if and only if only (V_i, V_j) is (ε, p)-regular and $d_{G,p}(V_i, V_j) \ge d$.

It follows directly from the definition that sub-pairs of p-dense pairs again form p-dense pairs.

Proposition 6. *Let (X, Y) be (ε, d, p)-dense and suppose $X' \subseteq X$ satisfies $|X'| \ge \mu|X|$. Then (X', Y) is $(\frac{\varepsilon}{\mu}, d, p)$-dense.* □

In addition, neighbourhoods of most vertices in a p-dense pair are not much smaller than expected. Again, this is a direct consequence of the definition of p-dense pairs.

Proposition 7. *Let (X, Y) be (ε, d, p)-dense. Then less than $\varepsilon|X|$ vertices $x \in X$ are such that $|N_Y(x)| < (d - \varepsilon)p|Y|$.* □

Some properties of the graph G translate to certain properties of the reduced graph R of the partition constructed by the sparse regularity lemma. For example the following well known consequence of Lemma 5 is a minimum degree version of the sparse regularity lemma. For a proof see Section B.1.

Lemma 8 (sparse regularity lemma, minimum degree version for $\mathcal{G}_{n,p}$)**.** *For all $\alpha \in [0, 1]$, $\varepsilon > 0$, and every integer r_0, there is an integer $r_1 \ge 1$ such that for all $d \in [0, 1]$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ if $\log^4 n/(pn) = o(1)$. Let $G = (V, E)$ be a spanning subgraph of Γ with $\deg_G(v) \ge \alpha \deg_\Gamma(v)$ for all $v \in V$. Then there is an (ε, d, p)-dense partition of G with reduced graph R of minimum degree $\delta(R) \ge (\alpha - d - \varepsilon)|V(R)|$ with $r_0 \le |V(R)| \le r_1$.*

We remark that we do observe "more" than a mere inheritance of properties here: the graph G we started with is *sparse*, but the reduced graph R we obtain in Lemma 8 is *dense*. This will enables us to apply results obtained for dense graphs to the reduced graph R, and hence use such dense results to draw conclusions about sparse graphs.

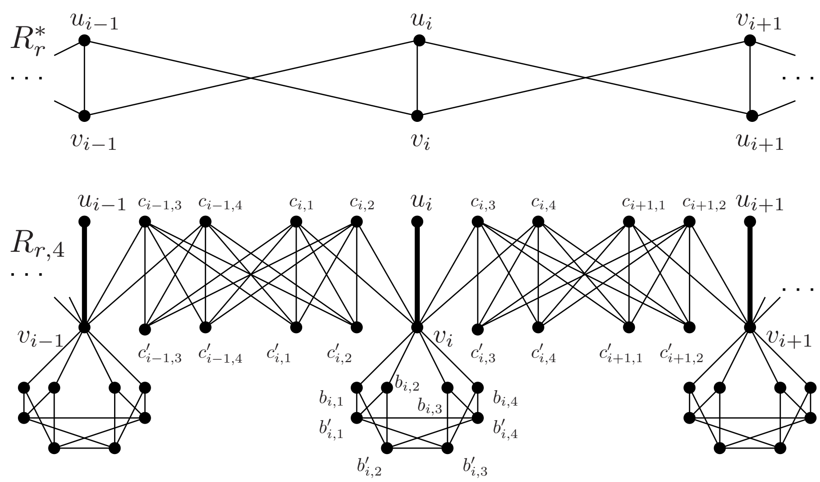

FIGURE 1. The ladder R_r^* and the spin graph $R_{r,t}$ for the special case $t = 2$.

5. Main Lemmas

In this section we will formulate the main lemmas and outline how they will be combined in Section 8 to give the proof of Theorem 1. For this we first need to define two (families of) special graphs.

For $r, t \in \mathbb{N}$, let $U = \{u_1, \dots, u_r\}$, $V = \{v_1, \dots, v_r\}$, $C = \{c_{i,j}, c'_{i,j} : i \in [r], j \in [2t]\}$, and $B = \{b_{i,j}, b'_{i,j} : i \in [r], j \in [2t]\}$. Let the *ladder* R_r^* be the graph with vertex set $U \dot\cup V$ and edge set $E(R_r^*) := \{u_i v_j : i, j \in [r], |i - j| \le 1\}$. Let the *spin graph* $R_{r,t}$ be the graph with vertex set $U \dot\cup V \dot\cup C \dot\cup B$ and the following edge set (see Figure 1):

$$E(R_{r,t}) := \bigcup_{\substack{i,i' \in [r], i' \neq 1 \\ j,j' \in [2t] \\ k,k' \in [t] \\ \ell,\ell' \in [t+1,2t]}} \Bigg(\Big\{ u_i v_i,\, b_{i,k} b'_{i,k'},\, b_{i,\ell} b'_{i,\ell'},\, c_{i,k} c'_{i,k'},\, c_{i,\ell} c'_{i,\ell'} \Big\} \cup \Big\{ b_{i,j} v_i,\, c_{i,j} v_i \Big\} \cup \Big\{ b'_{i,k} b'_{i,\ell},\, c_{i'-1,\ell} c'_{i',k},\, c'_{i'-1,\ell} c_{i',k} \Big\} \Bigg).$$

Now we can state our four main lemmas, two partition lemmas and two embedding lemmas. We start with the lemma for G, which constructs a partition of the host graph G. This lemma is a consequence of the sparse regularity lemma (Lemma 8) and asserts the existence of a p-dense partition of G such that its reduced graph contains a spin graph. We will indicate below why this is useful for the embedding of H. The lemma for G produces clusters of very different sizes: A set of larger clusters U_i and V_i which we call *big clusters* and which will accommodate most of the vertices of H later, and a set of smaller clusters $B_{i,j}$,$B'_{i,j}$, $C_{i,j}$, and $C'_{i,j}$. The $B_{i,j}$ and $B'_{i,j}$ are called *balancing clusters* and the $C_{i,j}$ and $C'_{i,j}$ *connecting clusters*. They will be used to host a small number of vertices of H. These vertices balance and connect the pieces of H that are embedded into the big clusters. The proof of Lemma 9 is given in Section 9.

Lemma 9 (Lemma for G)**.** *For all integers $t, r_0 > 0$ and reals $\eta_G, \gamma > 0$ there are positive reals η'_G and d such that for all $\varepsilon > 0$ there is r_1 such that the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $\log^4 n/(pn) = o(1)$. Let $G = (V, E)$ be a spanning subgraph of Γ with $\deg_G(v) \ge (\frac{1}{2} + \gamma) \deg_\Gamma(v)$ for all $v \in V$. Then there is $r_0 \le r \le r_1$, a subset V_0 of V with $|V_0| \le \varepsilon n$, and a mapping g from $V \setminus V_0$ to the spin graph $R_{r,t}$ such that for every $i \in [r], j \in [2t]$ we have*

(G1) $|U_i|, |V_i| \ge (1 - \eta_G)\frac{n}{2r}$ *for* $U_i := g^{-1}(u_i)$ *and* $V_i := g^{-1}(v_i)$,
(G2) $|C_{i,j}|, |C'_{i,j}|, |B_{i,j}|, |B'_{i,j}| \ge \eta'_G \frac{n}{2r}$
for $C_{i,j} := g^{-1}(c_{i,j})$, $C'_{i,j} := g^{-1}(c'_{i,j})$, $B_{i,j} := g^{-1}(b_{i,j})$, *and* $B'_{i,j} := g^{-1}(b'_{i,j})$,
(G3) *the pair* $\big(g^{-1}(x), g^{-1}(y)\big)$ *is* (ε, d, p)*-dense for all* $xy \in E(R_{r,t})$.

Since the dependencies of the constants appearing in this lemma are quite involved, we remark that their quantification is as follows:

$$\forall\, t,\, r_0,\, \eta_G,\, \gamma \quad \exists\, \eta'_G,\, d \quad \forall\, \varepsilon \quad \exists\, r_1\,.$$

Our second lemma provides a partition of H that fits the structure of the partition of G generated by Lemma 9. We will first state this lemma and then explain the different properties which it guarantees. A set S of vertices in a graph H is called *ℓ-independent* for an integer ℓ if each pair of distinct vertices in S has distance at least $\ell+1$ in H.

Lemma 10 (Lemma for H)**.** *For all integers Δ there is an integer $t>0$ such that for any $\eta_H>0$ and any integer $r\geq 1$ there is $\beta>0$ such that the following holds for all integers m and all bipartite graphs H on m vertices with $\Delta(H)\leq\Delta$ and $\mathrm{bw}(H)\leq\beta m$. There is a homomorphism h from H to the spin graph $R_{r,t}$ such that for every $i\in[r], j\in[2t]$*

- (H1) $|\widetilde{U}_i|,|\widetilde{V}_i|\leq(1+\eta_H)\frac{m}{2r}$ *for* $\widetilde{U}_i:=h^{-1}(u_i)$ *and* $\widetilde{V}_i:=h^{-1}(v_i)$,
- (H2) $|\widetilde{C}_{i,j}|,|\widetilde{C}'_{i,j}|,|\widetilde{B}_{i,j}|,|\widetilde{B}'_{i,j}|\leq\eta_H\frac{m}{2r}$ *for* $\widetilde{C}_{i,j}:=h^{-1}(c_{i,j})$, $\widetilde{C}'_{i,j}:=h^{-1}(c'_{i,j})$, $\widetilde{B}_{i,j}:=h^{-1}(b_{i,k})$, *and* $\widetilde{B}'_{i,j}:=h^{-1}(b'_{i,k})$,
- (H3) $\widetilde{C}_{i,j}$, $\widetilde{C}'_{i,j}$, $\widetilde{B}_{i,j}$, *and* $\widetilde{B}'_{i,j}$ *are* 3*-independent in* H,
- (H4) $\deg_{\widetilde{V}_i}(y)=\deg_{\widetilde{V}_i}(y')\leq\Delta-1$ *for all* $yy'\in\binom{\widetilde{C}_{i,j}}{2}\cup\binom{\widetilde{B}_{i,j}}{2}$, $\deg_{\widetilde{C}_i}(y)=\deg_{\widetilde{C}_i}(y')$ *for all* $y,y'\in\widetilde{C}'_{i,j}$, $\deg_{L(i,j)}(y)=\deg_{L(i,j)}(y')$ *for all* $y,y'\in\widetilde{B}'_{i,j}$,

where $\widetilde{C}_i:=\bigcup_{k\in[2t]}\widetilde{C}_{i,k}$ *and* $L(i,j):=\bigcup_{k\in[2t]}\widetilde{B}_{i,k}\cup\bigcup_{k<j}\widetilde{B}'_{i,k}$*. Further, let* $\widetilde{X}_i$ *with* $i\in[r]$ *be the set of vertices in* $\widetilde{V}_i$ *with neighbours outside* $\widetilde{U}_i$*. Then*

- (H5) $|\widetilde{X}_i|\leq\eta_H|\widetilde{V}_i|$.

The quantification of the constants appearing in this lemma is as follows:

$$\forall\,\Delta \quad \exists\, t \quad \forall\,\eta_H,\, r \quad \exists\,\beta\,.$$

This lemma asserts the existence of a homomorphism h from H to a spin graph $R_{r,t}$. Recall that $R_{r,t}$ is contained in the reduced graph of the p-dense partition provided by Lemma 9. As we will see, we can fix the parameters in this lemma such that, when we apply it together with Lemma 9, the homomorphism h has the following additional property. The number $\widetilde{L}$ of vertices that it maps to a vertex a of the spin graph is less than the number L contained in the corresponding cluster A provided by Lemma 9 (compare (G1) and (G2) with (H1) and (H2) and recall that m is slightly smaller than n). If A is a big cluster, then the numbers L and $\widetilde{L}$ differ only slightly (these vertices will be embedded using the constrained blow-up lemma), but for balancing and connecting clusters A the number $\widetilde{L}$ is much smaller than L (this is necessary for the embedding of these vertices using the connection lemma). With property (H5) Lemma 10 further guarantees that only few edges of H are not assigned either to two connecting or balancing clusters, or to two big clusters. This is helpful because it implies that we do not have to take care of "too many dependencies" between the applications of the blow-up lemma and the connection lemma. The remaining properties (H3)–(H4) of Lemma 10 are technical but required for the application of the connection lemma (see conditions (B) and (C) of Lemma 12).

The vertices in $\widetilde{C}_{i,j}$ and $\widetilde{C}'_{i,j}$ are also called *connecting vertices* of H, the vertices in $\widetilde{B}_{i,j}$ and $\widetilde{B}'_{i,j}$ *balancing vertices*.

We next describe the two embedding lemmas, the constrained blow-up lemma (Lemma 11) and the connection lemma (Lemma 12), which we would like to use on the partitions of G and H provided by Lemmas 9 and 10. The connecting lemma will be used to embed the connecting and balancing vertices into the connecting and balancing clusters *after* all the other vertices are embedded into the big clusters with the help of the constrained blow-up lemma.

The constrained blow-up lemma states that bipartite graphs H with bounded maximum degree can be embedded into a p-dense pair $G=(U,V)$ whose cluster sizes are just slightly bigger than the partition classes of H. This lemma further guarantees the following. If we specify a small family of small special sets in one of the partition classes of H and a small family of small forbidden sets in the corresponding cluster of G, then no special set is mapped to a forbidden set.

The existence of these forbidden sets is in fact a main difference to the classical blow-up lemma which is used in the dense setting, where a small family of special vertices of H can be guaranteed to be mapped to a *required set* of linear size in G. This is very useful in a dense graph, because its neighbourhoods (into which we would like to embed neighbours of already embedded vertices) are of linear size. In contrast, the property of having forbidden sets will be crucial for the sparse setting when we will apply this lemma together with the connection lemma in the proof of Theorem 1 in order to handle the "dependencies" between these applications. The proof of this lemma is given in Section 11 and relies on techniques developed in [4].

Lemma 11 (Constrained blow-up lemma). *For every integer $\Delta > 1$ and for all positive reals d, and η there exist positive constants ε and μ such that for all positive integers r_1 there is c such that for all integers $1 \le r \le r_1$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $p \ge c(\log n/n)^{1/\Delta}$. Let $G = (U,V) \subseteq \Gamma$ be an (ε, d, p)-dense pair with $|U|, |V| \ge n/r$ and let H be a bipartite graph on vertex classes $\widetilde{U} \dot\cup \widetilde{V}$ of sizes $|\widetilde{U}|, |\widetilde{V}| \le (1-\eta)n/r$ and with $\Delta(H) \le \Delta$. Moreover, suppose that there is a family $\mathcal{H} \subseteq \binom{\widetilde{V}}{\Delta}$ of* special Δ-sets *in $\widetilde{V}$ such that each $\widetilde{v} \in \widetilde{V}$ is contained in at most Δ special sets and a family $\mathcal{B} \subseteq \binom{V}{\Delta}$ of* forbidden Δ-sets *in V with $|\mathcal{B}| \le \mu |V|^\Delta$. Then there is an embedding of H into G such that no special set is mapped to a forbidden set.*

The quantification of the constants appearing in this lemma is as follows:

$$\forall \Delta, d, \eta \quad \exists \varepsilon, \mu \quad \forall r_1 \quad \exists c .$$

At first sight, the rôle of the integer r in Lemma 11 (and also in Lemma 12 below) seems a little obscure. The only reason for stating the lemma as above is that it is more readily applicable in this form, since we will need it for pairs of partition classes (U,V) whose size in relation to n will be determined by the regularity lemma.

Our last main lemma, the connection lemma (Lemma 12), embeds graphs H into graphs G forming a system of p-dense pairs. In contrast to the blow-up lemma, however, the graph H has to be much smaller than the graph G now (see condition (A)). In addition, each vertex $\widetilde{y}$ of H is equipped with a candidate set $C(\widetilde{y})$ in G from which the connection lemma will choose the image of $\widetilde{y}$ in the embedding. Lemma 12 requires that these candidate sets are big (condition (D)) and that pairs of candidate sets that correspond to an edge of H form p-dense pairs (condition (E)). The remaining conditions ((B) and (C)) are conditions on the neighbourhoods and degrees of the vertices in H (with respect to the given partition of H). For their statement we need the following additional definition.

For a graph H on vertex set $\widetilde{V} = \widetilde{V}_1 \dot\cup \dots \dot\cup \widetilde{V}_t$ and $y \in \widetilde{V}_i$ with $i \in [t]$ define the *left degree* of y with respect to the partition $\widetilde{V}_1 \dot\cup \dots \dot\cup \widetilde{V}_t$ to be $\mathrm{ldeg}(y; \widetilde{V}_1, \dots, \widetilde{V}_t) := \sum_{j=1}^{i-1} \deg_{\widetilde{V}_j}(y)$. When clear from the context we may also omit the partition and simply write $\mathrm{ldeg}(y)$. For two sets of vertices S, T we denote the *common neighbourhood* of (the vertices of) S in T by $N_T^\cap(S) := \bigcap_{s \in S} N_T(s)$.

Lemma 12 (Connection lemma). *For all integers $\Delta > 1$, $t > 0$ and reals $d > 0$ there are ε, $\xi > 0$ such that for all positive integers r_1 there is $c > 1$ such that for all integers $1 \le r \le r_1$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $p \ge c(\log n/n)^{1/\Delta}$. Let $G \subseteq \Gamma$ be any graph on vertex set $W = W_1 \dot\cup \dots \dot\cup W_t$ and let H be any graph on vertex set $\widetilde{W} = \widetilde{W}_1 \dot\cup \dots \dot\cup \widetilde{W}_t$. Suppose further that for each $i \in [t]$ each vertex $\widetilde{w} \in \widetilde{W}_i$ is equipped with an arbitrary set $X_{\widetilde{w}} \subseteq V(\Gamma) \setminus W$ with the property that the indexed set system $\big(X_{\widetilde{w}} \colon \widetilde{w} \in \widetilde{W}_i\big)$ consists of pairwise disjoint sets such that the following holds. We define the* external degree *of $\widetilde{w}$ to be $\mathrm{edeg}(\widetilde{w}) := |X_{\widetilde{w}}|$, its* candidate set *$C(\widetilde{w}) \subseteq W_i$ to be $C(\widetilde{w}) := N_{W_i}^\cap(X_{\widetilde{w}})$, and require that*

- (A) *$|W_i| \ge n/r$ and $|\widetilde{W}_i| \le \xi n/r$,*
- (B) *$\widetilde{W}_i$ is a 3-independent set in H,*
- (C) *$\mathrm{edeg}(\widetilde{w}) + \mathrm{ldeg}(\widetilde{w}) = \mathrm{edeg}(\widetilde{v}) + \mathrm{ldeg}(\widetilde{v})$ and $\deg_H(\widetilde{w}) + \mathrm{edeg}(\widetilde{w}) \le \Delta$ for all $\widetilde{w}, \widetilde{v} \in \widetilde{W}_i$,*
- (D) *$|C(\widetilde{w})| \ge ((d-\varepsilon)p)^{\mathrm{edeg}(\widetilde{w})} |W_i|$ for all $\widetilde{w} \in \widetilde{W}_i$, and*
- (E) *$(C(\widetilde{w}), C(\widetilde{v}))$ forms an (ε, d, p)-dense pair for all $\widetilde{w}\widetilde{v} \in E(H)$.*

Then there is an embedding of H into G such that every vertex $\widetilde{w} \in \widetilde{W}$ is mapped to a vertex in its candidate set $C(\widetilde{w})$.

The quantification of the constants appearing in this lemma is as follows:

$$\forall \Delta, t, d \quad \exists \varepsilon, \xi \quad \forall r_1 \quad \exists c.$$

The proof of this lemma is inherent in [26]. For the details in our setting see Section A.

6. Stars in random graphs

In this section we formulate two lemmas concerning properties of random graphs that will be useful when analysing neighbourhood properties of p-dense pairs in the following section. More precisely, we consider the following question here. Given a set of vertices X in a random graph $\Gamma = \mathcal{G}_{n,p}$ together with a family $\mathcal{F}$ of pairwise disjoint ℓ-sets in $V(\Gamma)$, we would like to determine how many pairs (x, F) with $x \in X$ and $F \in \mathcal{F}$ have the property that x lies in the common neighbourhood of the vertices in F.

Definition 13 (stars)**.** *Let $G = (V, E)$ be a graph, X be a subset of V and $\mathcal{F}$ be a family of pairwise disjoint ℓ-sets in $V \setminus X$ for some ℓ. Then the number of* stars *in G between X and $\mathcal{F}$ is*

$$\#\operatorname{stars}^G(X, \mathcal{F}) := \Big| \big\{ (x, F) \colon x \in X,\, F \in \mathcal{F},\, F \subseteq N_G(x) \big\} \Big|. \tag{1}$$

Observe that in a random graph $\Gamma = \mathcal{G}_{n,p}$ and for fixed sets X and $\mathcal{F}$ the random variable $\#\operatorname{stars}^\Gamma(X, \mathcal{F})$ has binomial distribution $\operatorname{Bi}(|X||\mathcal{F}|, p^\ell)$. This will be used in the proofs of the following lemmas. The first of these lemmas states that in $\mathcal{G}_{n,p}$ the number of stars between X and $\mathcal{F}$ does not exceed its expectation by more than seven times as long as X and $\mathcal{F}$ are not too small. This is a straightforward consequence of Chernoff's inequality.

Lemma 14 (star lemma for big sets)**.** *For every positive integer Δ and every positive real ν there is c such that if $p \geq c(\log n/n)^{1/\Delta}$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ on vertex set V. Let X be any subset of V and $\mathcal{F}$ be any family of pairwise disjoint Δ-sets in $V \setminus X$. If $\nu n \leq |X| \leq |\mathcal{F}| \leq n$, then*

$$\#\operatorname{stars}^\Gamma(X, \mathcal{F}) \leq 7p^\Delta |X||\mathcal{F}|.$$

Proof. Given Δ and ν let c be such that $7c^\Delta \nu^2 \geq 3\Delta$. From Chernoff's inequality (see [22, Chapter 2]) we know that $\mathbb{P}[Y \geq 7\,\mathbb{E}\,Y] \leq \exp(-7\,\mathbb{E}\,Y)$ for a binomially distributed random variable Y. We conclude that for fixed X and $\mathcal{F}$

$$\begin{aligned}\mathbb{P}\left[\#\operatorname{stars}^\Gamma(X, \mathcal{F}) > 7p^\Delta|X||\mathcal{F}|\right] &\leq \exp(-7p^\Delta|X||\mathcal{F}|)\\ &\leq \exp(-7c^\Delta(\log n/n)\nu^2 n^2) \leq \exp(-3\Delta n \log n)\end{aligned}$$

by the choice of c. Thus the probability that there are sets X and $\mathcal{F}$ violating the assertion of the lemma is at most

$$2^n n^{\Delta n} \exp(-3\Delta n \log n) \leq \exp(2\Delta n \log n - 3\Delta n \log n),$$

which tends to 0 as n tends to infinity. □

We will also need a variant of Lemma 14 for smaller sets X and families $\mathcal{F}$. As a trade-off, the bound on the number of stars provided by the next lemma will be somewhat worse. Lemma 15 appears almost in this form in [26]. The only (slight) modification that we need here is that X is allowed to be bigger than $\mathcal{F}$. However, the same proof as presented in [26] still works for this modified version. For the details see Section B.2.

Lemma 15 (star lemma for small sets)**.** *For all positive integers Δ and positive reals ξ there are positive constants ν and c such that if $p \geq c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ on vertex set V. Let X be any subset of V and $\mathcal{F}$ be any family of pairwise disjoint Δ-sets in $V \setminus X$. If $|X| \leq \nu n p^\Delta |\mathcal{F}|$ and $|X|, |\mathcal{F}| \leq \xi n$, then*

$$\#\operatorname{stars}^\Gamma(X, \mathcal{F}) \leq p^\Delta |X||\mathcal{F}| + 6\xi n p^\Delta |\mathcal{F}|. \tag{2}$$

7. Common neighbourhoods in p-dense pairs

As discussed in Section 4 it follows directly from the definition of p-denseness that sub-pairs of dense pairs form again dense pairs. In order to apply Lemma 11 and Lemma 12 together, we will need corresponding results on common neighbourhoods in systems of dense pairs (see Lemmas 17 and 20). For this it is necessary to first introduce some notation.

Let $G=(V,E)$ be a graph, $\ell, T>0$ be integers, p, ε, d be positive reals, and X, Y, $Z\subseteq V$ be disjoint vertex sets. Recall that for a set B of vertices from V and a vertex set $Y\subseteq V$ we call the set $N^\cap_Y(B)=\bigcap_{b\in B}N_Y(b)$ the common neighbourhood of (the vertices in) B in Y.

Definition 16 (Bad and good vertex sets). *Let G, ℓ, T, p, ε, d, X, Y, and Z be as above. We define the following family of ℓ-sets in X with small common neighbourhood in Y:*

$$\operatorname{bad}^{G,\ell}_{\varepsilon,d,p}(X,Y):=\left\{B\in\binom{X}{\ell}\colon\ |N^\cap_Y(B)|<(d-\varepsilon)^\ell p^\ell|Y|\right\}. \tag{3}$$

*If (X,Y) has p-density $d_{G,p}(X,Y)\geq d-\varepsilon$, then all ℓ-sets $T\in\binom{X}{\ell}$ that are not in $\operatorname{bad}^{G,\ell}_{\varepsilon,d,p}(X,Y)$ are called p-*good *in (X,Y).*

Let further

$$\operatorname{Bad}^{G,\ell}_{\varepsilon,d,p}(X,Y,Z)$$

be the family of ℓ-sets $B\in\binom{X}{\ell}$ that contain an ℓ'-set $B'\subseteq B$ with $\ell'>0$ such that either $|N^\cap_Y(B')|<(d-\varepsilon)^{\ell'}p^{\ell'}|Y|$ or $(N^\cap_Y(B'),Z)$ is not (ε,d,p)-dense in G.

The following lemma states that p-dense pairs in random graphs have the property that most ℓ-sets have big common neighbourhoods. Results of this type (with a slightly smaller exponent in the edge probability p) were established in [24]. The proof of Lemma 17 can be found in Section B.3.

Lemma 17 (common neighbourhood lemma). *For all integers $\Delta,\ell\geq 1$ and positive reals d, ε' and μ, there is $\varepsilon>0$ such that for all $\xi>0$ there is $c>1$ such that if $p\geq c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma=\mathcal{G}_{n,p}$. For $n_1\geq\xi p^{\Delta-1}n$, $n_2\geq\xi p^{\Delta-\ell}n$ let $G=(X\dot\cup Y,E)$ be any bipartite subgraph of Γ with $|X|=n_1$ and $|Y|=n_2$. If (X,Y) is an (ε,d,p)-dense pair, then $|\operatorname{bad}^{G,\ell}_{\varepsilon',d,p}(X,Y)|\leq\mu n_1^\ell$.*

Thus we know that typical vertex sets in dense pairs inside random graphs are p-good. In the next lemma we observe that families of such p-good vertex sets exhibit strong expansion properties.

Given Δ and p we say that a bipartite graph $G=(X\dot\cup Y,E)$ is (A,f)*-expanding*, if, for any family $\mathcal{F}\subseteq\binom{X}{\Delta}$ of pairwise disjoint p-good Δ-sets in (X,Y) with $|\mathcal{F}|\leq A$, we have $|N^\cap_Y(\mathcal{F})|\geq f|\mathcal{F}|$.

Lemma 18 (expansion lemma). *For all positive integers Δ and positive reals d and ε, there are positive ν and c such that if $p\geq c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma=\mathcal{G}_{n,p}$. Let $G=(X\dot\cup Y,E)$ be a bipartite subgraph of Γ. If (X,Y) is an (ε,d,p)-dense pair, then (X,Y) is $(1/p^\Delta,\nu np^\Delta)$-expanding.*

Proof. Given Δ, d, ε, set $\delta:=d-\varepsilon$, $\xi:=\delta^\Delta/7$ and let ν' and c be the constants from Lemma 15 for this Δ and ξ. Further, choose ν such that $\nu\leq\xi$ and $\nu\leq\nu'$. Let $\mathcal{F}\subseteq\binom{X}{\Delta}$ be a family of pairwise disjoint p-good Δ-sets with $|\mathcal{F}|\leq 1/p^\Delta$. Let $U=N^\cap_Y(\mathcal{F})$ be the common neighbourhood of $\mathcal{F}$ in Y. We wish to show that $|U|\geq(\nu np^\Delta)|\mathcal{F}|$. Suppose the contrary. Then $|U|<\nu' np^\Delta|\mathcal{F}|$, $|U|<\nu np^\Delta|\mathcal{F}|\leq\nu n\leq\xi n$ and $|\mathcal{F}|\leq 1/p^\Delta\leq c^\Delta n/\log n\leq\xi n$ for n sufficiently large and so we can apply Lemma 15 with parameters Δ and ξ to U and $\mathcal{F}$. Since every member of $\mathcal{F}$ is p-good in (X,Y), we thus have

$$\begin{aligned}\delta^\Delta p^\Delta n|\mathcal{F}|\leq\#\operatorname{stars}^G(U,\mathcal{F})\leq\#\operatorname{stars}^\Gamma(U,\mathcal{F})&\overset{(2)}{\leq}p^\Delta|U||\mathcal{F}|+6\xi np^\Delta|\mathcal{F}|\\&<p^\Delta(\nu np^\Delta)|\mathcal{F}||\mathcal{F}|+6\xi np^\Delta|\mathcal{F}|\leq\nu np^\Delta|\mathcal{F}|+6\xi np^\Delta|\mathcal{F}|\leq 7\xi np^\Delta|\mathcal{F}|,\end{aligned}$$

which yields that $\delta^\Delta<7\xi$, a contradiction. □

In the remainder of this section we are interested in the inheritance of p-denseness to sub-pairs (X',Y') of p-dense pairs (X,Y) in a graph $G=(V,E)$. It comes as a surprise that even for sets X' and Y' that are much smaller than the sets considered in the definition of p-denseness, such sub-pairs are typically dense. Phenomena of this type were observed in [24, 18].

Here, we will consider sub-pairs induced by neighbourhoods of vertices $v\in V$ (which may or may not be in $X\dot\cup Y$), i.e., sub-pairs (X',Y') where X' (or Y' or both) is the neighbourhood of v in Y (or in X). Further, we only consider the case when G is a subgraph of a random graph $\mathcal{G}_{n,p}$.

In [26] an inheritance result of this form was obtained for triples of dense pairs. More precisely, the following holds for subgraphs G of $\mathcal{G}_{n,p}$. For sufficiently large vertex set X, Y, and Z in G such that (X,Y) and (Y,Z) form p-dense pairs we have that most vertices $x\in X$ are such that $(N_Y(x),Y)$ forms again a p-dense pair (with slightly changed parameters). If, moreover, (X,Z) forms a p-dense pair, too, then $(N_Y(x),N_Z(x))$ is typically also a p-dense pair.

Lemma 19 (inheritance lemma for vertices [26])**.** *For all integers $\Delta>0$ and positive reals d_0, ε' and μ there is ε such that for all $\xi>0$ there is $c>1$ such that if $p>c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma=\mathcal{G}_{n,p}$. For n_1, $n_3\geq \xi p^{\Delta-1}n$ and $n_2\geq\xi p^{\Delta-2}n$ let $G=(X\dot\cup Y\dot\cup Z,E)$ be any tripartite subgraph of Γ with $|X|=n_1$, $|Y|=n_2$, and $|Z|=n_3$. If (X,Y) and (Y,Z) are (ε,d,p)-dense pairs in G with $d\geq d_0$, then there are at most μn_1 vertices $x\in X$ such that $(N(x)\cap Y,Z)$ is not an (ε',d,p)-dense pair in G.*

If, additionally, (X,Z) is (ε,d,p)-dense and n_1, n_2, $n_3\geq\xi p^{\Delta-2}n$, then there are at most μn_1 vertices $x\in X$ such that $(N(x)\cap Y,N(x)\cap Z)$ is not an (ε',d,p)-dense pair in G. □

In order to combine the constrained blow-up lemma (Lemma 11) and the connection lemma (Lemma 12) in the proof of Theorem 1 we will need a version of this result for ℓ-sets. Such a lemma, stating that common neighbourhoods of certain ℓ-sets form again p-dense pairs, can be obtained by an inductive argument from the first part of Lemma 19. For a proof see Section B.4.

Lemma 20 (inheritance lemma for ℓ-sets)**.** *For all integers $\Delta,\ell>0$ and positive reals d_0,ε', and μ there is ε such that for all $\xi>0$ there is $c>1$ such that if $p>c(\frac{\log n}{n})^{1/\Delta}$, then the following holds a.a.s. for $\Gamma=\mathcal{G}_{n,p}$. For $n_1,n_3\geq\xi p^{\Delta-1}n$ and $n_2\geq\xi p^{\Delta-\ell-1}n$ let $G=(X\dot\cup Y\dot\cup Z,E)$ be any tripartite subgraph of Γ with $|X|=n_1$, $|Y|=n_2$, and $|Z|=n_3$. Assume further that (X,Y) and (Y,Z) are (ε,d,p)-dense pairs with $d\geq d_0$. Then*

$$\big|\operatorname{Bad}^{G,\ell}_{\varepsilon',d,p}(X,Y,Z)\big|\leq\mu n_1^{\ell}.$$

8. Proof of Theorem 1

In this section we present a proof of Theorem 1 that combines our four main lemmas, namely the lemma for G (Lemma 9), the lemma for H (Lemma 10), the constrained blow-up lemma (Lemma 11), and the connection lemma (Lemma 12). This proof follows the outline given in Section 5. In addition we will apply the inheritance lemma for ℓ-sets (Lemma 20), which supplies an appropriate interface between the constrained blow-up lemma and the connection lemma.

Proof of Theorem 1. We first set up the constants. Given η, γ, and Δ let t be the constant promised by the lemma for H (Lemma 10) for input Δ. Set

$$\eta_G:=\eta/10,\qquad\text{and}\qquad r_0=1\,, \tag{4}$$

and apply the lemma for G (Lemma 9) with input t, r_0, η_G, and γ in order to obtain η'_G and d. Next, the connection lemma (Lemma 12) with input Δ, $2t$, and d provides us with $\varepsilon_{\rm CL}$, and $\xi_{\rm CL}$. We apply the constrained blow-up lemma (Lemma 11) with Δ, d, and $\eta/2$ in order to obtain $\varepsilon_{\rm BL}$ and $\mu_{\rm BL}$. With this we set

$$\eta_H:=\min\{\eta/10,\,\xi_{\rm CL}\eta'_G,\,1/(\Delta+1)\}. \tag{5}$$

Choose $\mu>0$ such that

$$100t^2\mu\leq\eta_{\rm BL}, \tag{6}$$

and apply Lemma 20 with Δ and $\ell=\Delta-1$, $d_0=d$, $\varepsilon'=\varepsilon_{\rm CL}$, and μ to obtain ε_{20}. Let

$$\xi_{20}:=\eta'_G/2r \tag{7}$$

and continue the application of Lemma 20 with ξ_{20} to obtain c_{20}. Now we can fix

$$\varepsilon := \min\{\varepsilon_{\mathrm{CL}}, \varepsilon_{\mathrm{BL}}, \varepsilon_{20}\} \tag{8}$$

and continue the application of Lemma 9 with input ε to get r_1. Let $\hat{r}_{\mathrm{BL}}$ and $\hat{r}_{\mathrm{CL}}$ be such that

$$\frac{2r_1}{1-\eta_G} \le \hat{r}_{\mathrm{BL}} \qquad \text{and} \qquad \frac{2r_1}{\eta_G} \le \hat{r}_{\mathrm{CL}} \tag{9}$$

and let c_{CL} and c_{BL} be the constants obtained from the continued application of Lemma 12 with r_1 replaced by $\hat{r}_{\mathrm{CL}}$ and Lemma 11 with r_1 replaced by $\hat{r}_{\mathrm{BL}}$, respectively.

We continue the application of Lemma 10 with input η_H. For each $r \in [r_1]$ Lemma 10 provides a value β_r , among all of which we choose the smallest one and set β to this value. Finally, we set $c := \max\{c_{\mathrm{BL}}, c_{\mathrm{CL}}, c_{20}\}$.

Consider a graph $\Gamma = \mathcal{G}_{n,p}$ with $p \ge c(\log n/n)^{1/\Delta}$. Then Γ a.a.s. satisfies the properties stated in Lemma 9, Lemma 11, Lemma 12, and Lemma 20, with the parameters previously specified. We assume in the following that this is the case and show that then also the following holds. For all subgraphs $G \subseteq \Gamma$ and all graphs H such that G and H have the properties required by Theorem 1 we have $H \subseteq G$. To summarise the definition of the constants above, we can now assume that Γ satisfies the conclusion of the following lemmas:

(L9) Lemma 9 for parameters t, $r_0 = 1$, η_G, γ, η'_G, d, ε, and r_1, i.e., if G is any spanning subgraph of Γ satisfying the requirements of Lemma 9, then we obtain a partition of G as specified in the lemma with these parameters,

(L11) Lemma 11 for parameters Δ, d, $\eta/2$, $\varepsilon_{\mathrm{BL}}$, μ_{BL}, and $\hat{r}_{\mathrm{BL}}$,

(L12) Lemma 12 for parameters Δ, $2t$, d, $\varepsilon_{\mathrm{CL}}$, ξ_{CL}, and $\hat{r}_{\mathrm{CL}}$,

(L20) Lemma 20 for parameters Δ, $\ell = \Delta - 1$, $d_0 = d$, $\varepsilon' = \varepsilon_{\mathrm{CL}}$, μ, ε_{20}, and ξ_{20}.

Now suppose we are given a graph $G = (V, E) \subseteq \Gamma$ with $\deg_G(v) \ge (\frac{1}{2} + \gamma)\deg_\Gamma(v)$ for all $v \in V$ and $|V| = n$, and a graph $H = (\widetilde{V}, \widetilde{E})$ with $|\widetilde{V}| = (1-\eta)n$. Before we show that H can be embedded into G we will use the lemma for G (Lemma 9) and the lemma for H (Lemma 10) to prepare G and H for this embedding.

First we use the fact that Γ has property (L9). Hence, for the graph G we obtain an r with $1 \le r \le r_1$ from Lemma 9, together with a set $V_0 \subseteq V$ with $|V_0| \le \varepsilon n$, and a mapping $g\colon V \setminus V_0 \to R_{r,t}$ such that (G1)–(G3) of Lemma 9 are fulfilled. For all $i \in [r], j \in [2t]$ let U_i, V_i, $C_{i,j}$, $C'_{i,j}$, $B_{i,j}$, and $B'_{i,j}$ be the sets defined in Lemma 9. Recall that these sets were called big clusters, connecting clusters, and balancing clusters. With this the graph G is prepared for the embedding. We now turn to the graph H.

We assume for simplicity that $2r/(1-\eta_G)$ and $r/(t\eta'_G)$ are integers and define

$$r_{\mathrm{BL}} := 2r/(1-\eta_G) \qquad \text{and} \qquad r_{\mathrm{CL}} := 2r/\eta'_G\,. \tag{10}$$

We apply Lemma 10 which we already provided with Δ and η_H. For input H this lemma provides a homomorphism h from H to $R_{r,t}$ such that (H1)–(H5) of Lemma 10 are fulfilled. For all $i \in [r], j \in [2t]$ let $\widetilde{U}_i$, $\widetilde{V}_i$, $\widetilde{C}_{i,j}$, $\widetilde{C}'_{i,j}$, $\widetilde{B}_{i,j}$, $\widetilde{B}'_{i,j}$, and $\widetilde{X}_i$ be the sets whose existence is guaranteed by Lemma 10. Further, set $C_i := C_{i,1}\dot\cup\ldots\dot\cup C_{i,2t}$, $\widetilde{C}_i := \widetilde{C}_{i,1}\dot\cup\ldots\dot\cup\widetilde{C}_{i,2t}$, that is, C_i consists of connecting clusters and $\widetilde{C}_i$ of connecting vertices. Define C'_i, $\widetilde{C}'_i$, B_i, $\widetilde{B}_i$, B'_i, and $\widetilde{B}'_i$ analogously (B_i consists of balancing clusters and $\widetilde{B}_i$ of balancing vertices).

Our next goal will be to appeal to property (L11) which asserts that we can apply the constrained blow-up lemma (Lemma 11) for each p-dense pair (U_i, V_i) with $i \in [r]$ individually and embed $H[\widetilde{U}_i\dot\cup\widetilde{V}_i]$ into this pair. For this we fix $i \in [r]$. We will first set up special Δ-sets $\mathcal{H}_i$ and forbidden Δ-sets $\mathcal{B}_i$ for the application of Lemma 11. The idea is as follows. With the help of Lemma 11 we will embed all vertices in $\widetilde{U}_i\dot\cup\widetilde{V}_i$. But all connecting and balancing vertices of H remain unembedded. They will be handled by the connection lemma, Lemma 12, later on. However, these two lemmas cannot operate independently. If, for example, a connecting vertex $\widetilde{y}$ has three neighbours in $\widetilde{V}_i$, then these neighbours will be already mapped to vertices v_1, v_2, v_3 in V_i (by the blow-up lemma) when we want to embed $\widetilde{y}$. Accordingly the image of $\widetilde{y}$ in the embedding is confined to the common neighbourhood of the vertices v_1, v_2, v_3 in G. In other words, this common neighbourhood will be the candidate set $C(\widetilde{y})$ in the application of Lemma 12. This lemma requires, however, that candidate sets are not too small (condition (D) of Lemma 12) and, in addition, that candidate sets of any two adjacent vertices induce p-dense pairs (condition (E)). Hence we need to be prepared for these requirements. This will be done via the special and forbidden sets. The family of special sets $\mathcal{H}_i$ will contain neighbourhoods in $\widetilde{V}_i$ of connecting or balancing vertices $\widetilde{y}$ of H (observe that such vertices do not have neighbours in $\widetilde{U}_i$, see Figure 1). The family of forbidden sets $\mathcal{B}_i$ will consist of sets in V_i which are "bad" for the embedding of these neighbourhoods in view of (D) and (E) of Lemma 12 (recall that Lemma 11 does not map special sets to forbidden sets). Accordingly, $\mathcal{B}_i$ contains Δ-sets that have small common neighbourhoods or do not induce p-dense pairs in one of the relevant balancing or connecting clusters. We will next give the details of this construction of $\mathcal{H}_i$ and $\mathcal{B}_i$.

We start with the special Δ-sets $\mathcal{H}_i$. As explained, we would like to include in the family $\mathcal{H}_i$ all neighbourhoods of vertices $\widetilde{w}$ of vertices outside $\widetilde{U}_i \dot\cup \widetilde{V}_i$. Such neighbourhoods clearly lie entirely in the set $\widetilde{X}_i$ provided by Lemma 10. However, they need not necessarily be Δ-sets (in fact, by (H4) of Lemma 10, they are of size at most $\Delta - 1$). Therefore we have to "pad" these neighbourhoods in order to obtain Δ-sets. This is done as follows. We start by picking an arbitrary set of $\Delta|\widetilde{X}_i|$ vertices (which will be used for the "padding") in $\widetilde{V}_i \setminus \widetilde{X}_i$. We add these vertices to $\widetilde{X}_i$ and call the resulting set $\widetilde{X}'_i$. This is possible because (H5) of Lemma 10 and (5) imply that $|\widetilde{X}'_i| \le (\Delta+1)|\widetilde{X}_i| \le (\Delta+1)\eta_H|\widetilde{V}_i| \le |\widetilde{V}_i|$.

Now let $\widetilde{Y}_i$ be the set of vertices in $\widetilde{B}_i \dot\cup \widetilde{C}_i$ with neighbours in $\widetilde{V}_i$. These are the vertices for whose neighbourhoods we will include Δ-sets in $\mathcal{H}_i$. It follows from the definition of $\widetilde{X}_i$ that $|\widetilde{Y}_i| \le \Delta|\widetilde{X}_i|$. Let $\widetilde{y} \in \widetilde{Y}_i \subseteq \widetilde{B}_i \cup \widetilde{C}_i$. By the definition of $\widetilde{X}_i$ we have $N_H(\widetilde{y}) \subseteq \widetilde{X}_i$. Next, we let

$$\widetilde{X}_{\widetilde{y}} \text{ be the set of neighbours of } \widetilde{y} \text{ in } \widetilde{V}_i\,. \tag{11}$$

As explained, $\widetilde{y}$ has strictly less than Δ neighbours in $\widetilde{V}_i$ and hence we choose additional vertices from $\widetilde{X}'_i \setminus \widetilde{X}_i$. In this way we obtain for each $\widetilde{y} \in \widetilde{Y}_i$ a Δ-set $N_{\widetilde{y}} \in \widetilde{X}'_i$ with

$$N_{\widetilde{X}_i}(\widetilde{y}) = N_{\widetilde{V}_i}(\widetilde{y}) = \widetilde{X}_{\widetilde{y}} \subseteq N_{\widetilde{y}}\,. \tag{12}$$

We make sure, in this process, that for any two different $\widetilde{y}$ and $\widetilde{y}'$ we never include the same additional vertex from $\widetilde{X}'_i \setminus \widetilde{X}_i$. This is possible because $|\widetilde{X}'_i \setminus \widetilde{X}_i| \ge \Delta|\widetilde{X}_i| \ge |\widetilde{Y}_i|$. We can thus guarantee that

$$\text{each vertex in } \widetilde{X}'_i \text{ is contained in at most } \Delta \text{ sets } N_{\widetilde{y}}. \tag{13}$$

The family of special Δ-sets for the application of Lemma 11 on (U_i, V_i) is then

$$\mathcal{H}_i := \{N_{\widetilde{y}} \colon \widetilde{y} \in \widetilde{Y}_i\}\,. \tag{14}$$

Note that this is indeed a family of Δ-sets encoding all neighbourhoods in $\widetilde{U}_i \dot\cup \widetilde{V}_i$ of vertices outside this set.

Now we turn to the family $\mathcal{B}_i$ of forbidden Δ-sets. Recall that this family should contain sets that are forbidden for the embedding of the special Δ-sets because their common neighbourhood in a (relevant) balancing or connecting cluster is small or does not induce a p-dense pair. More precisely, we are interested in Δ-sets S that have one of the following properties. Either S has a small common neighbourhood in some cluster from B_i or from C_i (observe that only balancing vertices from $\widetilde{B}_i$ and connecting vertices from $\widetilde{C}_i$ have neighbours in $\widetilde{V}_i$). Or the neighbourhood $N^{\cap}_D(S)$ of S in a cluster D from B_i or C_i, respectively, is such that $(N^{\cap}_D(S), D')$ is not p-dense for some cluster D' from $B'_i \cup B'_{i+1}$ or $C'_i \cup C'_{i+1}$ (observe that edges between balancing vertices run only between $\widetilde{B}_i$ and $\widetilde{B}'_i \cup \widetilde{B}'_{i+1}$ and edges between connecting vertices only between $\widetilde{C}_i$ and $\widetilde{C}'_i \cup \widetilde{C}'_{i+1}$).

For technical reasons, however, we need to digress from this strategy slightly: We want to bound the number of Δ-sets in $\mathcal{B}_i$ with the help of the inheritance lemma for ℓ-sets, Lemma 20, later. Notice that, thanks to the lower bound on n_2 in Lemma 20, this lemma cannot be applied (in a meaningful way) for Δ-sets. But it can be applied for $(\Delta-1)$-sets. Therefore, we will not consider Δ-sets directly but first construct an auxiliary family of $(\Delta-1)$-sets and then, again, "pad" these sets to obtain a family of Δ-sets. Observe that the strategy outlined while setting up the special sets $\mathcal{H}_i$ still works with these $(\Delta-1)$-sets: neighbourhoods of connecting or balancing vertices in $\widetilde{V}_i$ are of size at most $\Delta-1$ by (H4) of Lemma 10.

But now let us finally give the details. We first define the auxiliary family of $(\Delta-1)$-sets as follows:

$$\mathcal{B}'_i := \bigcup_{\substack{i'\in\{i,i+1\},j,j'\in[2t]\\ (c_{i,j},c'_{i',j'})\in R_{r,t}}} \mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i,C_{i,j},C'_{i',j'}) \quad\cup \bigcup_{\substack{j,j'\in[2t]\\ (b_{i,j},b'_{i,j'})\in R_{r,t}}} \mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i,B_{i,j},B_{i,j'}). \tag{15}$$

We will next bound the size of this family by appealing to property (L20), and hence Lemma 20, with the tripartite graphs $G[V_i,C_{i,j},C'_{i',j'}]$ and $G[V_i,B_{i,j},B'_{i,j'}]$ with indices as in the definition of $\mathcal{B}'_i$. For this we need to check the conditions appearing in this lemma. By the definition of $R_{r,t}$ and (G3) of Lemma 9 all pairs $(C_{i,j},C'_{i',j'})$ and $(B_{i,j},B'_{i,j'})$ appearing in the definition of $\mathcal{B}'_i$ as well as the pairs $(V_i,C_{i,j})$ and $(V_i,B_{i,j})$ with $j\in[2t]$ are (ε,d,p)-dense. For the vertex sets of these dense pairs we know $|V_i|,|C'_{i',j'}|,|B'_{i,j'}|\ge\eta'_G n/2r\ge\xi_{20}p^{\Delta-1}n$ and $|C_{i,j}|,|B_{i,j}|\ge\eta'_G n/2r=\xi_{20}n$ by (G1) and (G2) of Lemma 9 and (7). Thus, since $\varepsilon\le\varepsilon_{20}$, property (L20) implies that the family

$$\mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i,C_{i,j},C'_{i',j'}),\ \text{and}\ \mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i,B_{i,j},B'_{i,j'})$$

is of size $\mu|V_i|^{\Delta-1}$ at most. It follows from (15) that $|\mathcal{B}'_i|\le 8t^2\mu|V_i|^{\Delta-1}$ which is at most $\mu_{\mathrm{BL}}|V_i|^{\Delta-1}$ by (6). The family of forbidden Δ-sets is then defined by

$$\mathcal{B}_i := \mathcal{B}'_i\times V_i \quad\text{and we have}\quad |\mathcal{B}_i|\le\mu_{\mathrm{BL}}|V_i|^{\Delta}\,. \tag{16}$$

Having defined the special and forbidden Δ-sets we are now ready to appeal to (L11) and use the constrained blow-up lemma (Lemma 11) with parameters Δ, d, $\eta/2$, $\varepsilon_{\mathrm{BL}}$, μ_{BL}, $\hat{r}_{\mathrm{BL}}$, and r_{BL} separately for each pair of graphs $G_i:=(U_i,V_i)$ and $H_i:=H[\widetilde{U}_i\dot\cup\widetilde{V}_i]$. Let us quickly check that the constant r_{BL} and the graphs G_i and H_i satisfy the required conditions. Observe first, that $1\le r_{\mathrm{BL}}=2r/(1-\eta_G)\le 2r_1/(1-\eta_G)\le\hat{r}_{\mathrm{BL}}$ by (10) and (9). Moreover (U_i,V_i) is an $(\varepsilon_{\mathrm{BL}},d,p)$-dense pair by (G3) of Lemma 9 and (8). (G1) implies

$$|U_i|\ge(1-\eta_G)\frac{n}{2r}\overset{(10)}{=}\frac{n}{r_{\mathrm{BL}}}$$

and similarly $|V_i|\ge n/r_{\mathrm{BL}}$. By (H1) of Lemma 10 we have

$$\begin{aligned}|\widetilde{U}_i|&\le(1+\eta_H)\frac{m}{2r}\le(1+\eta_H)(1-\eta)\frac{n}{2r}\le(1+\eta_H-\eta)\frac{n}{2r}\overset{(4),(5)}{\le}(1-\tfrac12\eta-\eta_G)\frac{n}{2r}\\&\le(1-\tfrac12\eta)(1-\eta_G)\frac{n}{2r}\overset{(10)}{=}(1-\tfrac12\eta)\frac{n}{r_{\mathrm{BL}}}\end{aligned}$$

and similarly $|\widetilde{V}_i|\le(1-\frac{\eta}{2})n/r_{\mathrm{BL}}$. For the application of Lemma 11, let the families of special and forbidden Δ-sets be defined in (14) and (16), respectively. Observe that (13) and (16) guarantee that the required conditions (of Lemma 11) are satisfied. Consequently there is an embedding of H_i into G_i for each $i\in[r]$ such that no special Δ-set is mapped to a forbidden Δ-set. Denote the united embedding resulting from these r applications of the constrained blow-up lemma by $f_{\mathrm{BL}}:\bigcup_{i\in[r]}\widetilde{U}_i\cup\widetilde{V}_i\to\bigcup_{i\in[r]}U_i\cup V_i$.

It remains to verify that f_{BL} can be extended to an embedding of all vertices of H into G. We still need to take care of the balancing and connecting vertices. For this purpose we will, again, fix $i\in[r]$ and use property (L12) which states that the conclusion of the connection lemma (Lemma 12) holds for parameters Δ, $2t$, d, $\varepsilon_{\mathrm{CL}}$, ξ_{CL}, and $\hat{r}_{\mathrm{CL}}$. We will apply this lemma with input r_{CL} to the graphs $G'_i:=G[W_i]$ and $H'_i:=H[\widetilde{W}_i]$ where W_i and $\widetilde{W}_i$ and their partitions for the application of the connection lemma are as follows (see Figure 2). Let $W_i:=W_{i,1}\dot\cup\ldots\dot\cup W_{i,8t}$ where for all $j\in[t],k\in[2t]$ we set

$$\begin{aligned}W_{i,j}&:=C_{i,t+j}\,, & W_{i,t+j}&:=C_{i+1,j}\,, & W_{i,2t+j}&:=C'_{i,t+j}\,,\\ W_{i,3t+j}&:=C'_{i+1,j}\,, & W_{i,4t+k}&:=B_{i,k}\,, & W_{i,6t+k}&:=B'_{i,k}\,.\end{aligned}$$

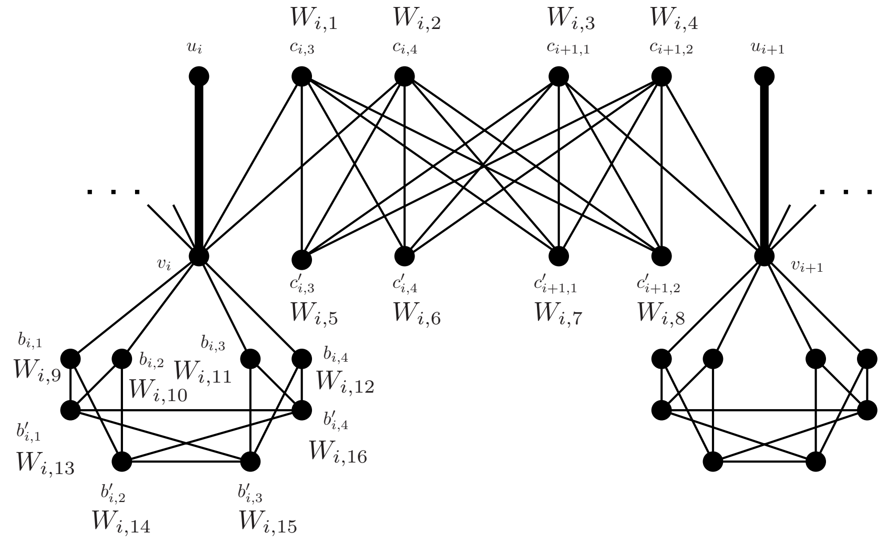

FIGURE 2. The partition $W_i = W_{i,1} \dot\cup \ldots \dot\cup W_{i,8t}$ of $G'_i = G[W_i]$ for the special case $t = 2$.

(*This means that we propose the clusters in the following order to the connection lemma. The connecting clusters without primes come first, then the connecting clusters with primes, then the balancing clusters without primes, and finally the balancing clusters with primes.*)

The partition $\widetilde{W}_i := \widetilde{W}_{i,1} \dot\cup \ldots \dot\cup \widetilde{W}_{i,8t}$ of the vertex set $\widetilde{W}_i$ of H'_i is defined accordingly, i.e., for all $j \in [t], k \in [2t]$ we set

$$\widetilde{W}_{i,j} := \widetilde{C}_{i,t+j}\,, \qquad \widetilde{W}_{i,t+j} := \widetilde{C}_{i+1,j}\,, \qquad \widetilde{W}_{i,2t+j} := \widetilde{C}'_{i,t+j}\,,$$
$$\widetilde{W}_{i,3t+j} := \widetilde{C}'_{i+1,j}\,, \qquad \widetilde{W}_{i,4t+k} := \widetilde{B}_{i,k}\,, \qquad \widetilde{W}_{i,6t+k} := \widetilde{B}'_{i,k}\,.$$

To check whether we can apply the connecting lemma observe first that

$$1 \le 2r/\eta_G \le 2r_1/\eta_G \le \hat{r}_{\mathrm{CL}}$$

by (9). For $\widetilde{y} \in \widetilde{W}_{i,j}$ with $j \in [8t]$ recall from (11) (using that each vertex in H has neighbours in at most one set $\widetilde{V}_{i'}$, see Figure 1) that

$$\widetilde{X}_{\widetilde{y}} \text{ is the set of neighbours of } \widetilde{y} \text{ in } \widetilde{V}_i \cup \widetilde{V}_{i+1} \text{ and set } X_{\widetilde{y}} := f_{\mathrm{BL}}(\widetilde{X}_{\widetilde{y}}). \tag{17}$$

Then the indexed set system $\left(\widetilde{X}_{\widetilde{y}} \colon \widetilde{y} \in \widetilde{W}_{i,j}\right)$ consists of pairwise disjoint sets because $\widetilde{W}_{i,j}$ is 3-independent in H by (H3) of Lemma 10. Thus also $\left(X_{\widetilde{y}} : \widetilde{y} \in \widetilde{W}_{i,j}\right)$ consists of pairwise disjoint sets, as required by Lemma 12. Now let the external degree and the candidate set of $\widetilde{y} \in \widetilde{W}_{i,j}$ be defined as in Lemma 12, i.e.,

$$\mathrm{edeg}(\widetilde{y}) := |X_{\widetilde{y}}| \qquad \text{and} \qquad C(\widetilde{y}) := N^{\cap}_{W_{i,j}}(X_{\widetilde{y}})\,. \tag{18}$$

Observe that this implies $C(\widetilde{y}) = W_{i,j}$ if $\widetilde{X}_{\widetilde{y}} = \emptyset$ and hence $X_{\widetilde{y}} = \emptyset$. Now we will check that conditions (A)–(E) of Lemma 12 are satisfied. From (G2) of Lemma 9 and (H2) of Lemma 10 it follows that

$$|W_{i,j}| \overset{\text{(G2)}}{\ge} \eta'_G \frac{n}{2r} \overset{(10)}{=} \frac{n}{r_{\mathrm{CL}}} \qquad \text{and}$$
$$|\widetilde{W}_{i,j}| \overset{\text{(H2)}}{\le} \eta_H \frac{m}{2r} \le \eta_H \frac{n}{2r} \overset{(10)}{=} \frac{\eta_H}{\eta'_G}\frac{n}{r_{\mathrm{CL}}} \overset{(5)}{\le} \xi_{\mathrm{CL}} \frac{n}{r_{\mathrm{CL}}}$$

and thus we have condition (A). By (H3) of Lemma 10 we also get condition (B) of Lemma 12. Further, it follows from (H4) of Lemma 10 that $\mathrm{edeg}(\widetilde{y}) = \mathrm{edeg}(\widetilde{y}')$ and $\mathrm{ldeg}(\widetilde{y}) = \mathrm{ldeg}(\widetilde{y}')$ for all

$\widetilde{y}, \widetilde{y}' \in \widetilde{W}_{i,j}$ with $j \in [8t]$. In addition $\Delta(H) \leq \Delta$ and hence

$$\begin{aligned}\deg_{H'_i}(\widetilde{y}) + \mathrm{edeg}(\widetilde{y}) &\overset{(18)}{=} |N_{\widetilde{W}_i}(\widetilde{y})| + |X_{\widetilde{y}}| \\ &\overset{(17)}{=} |N_{\widetilde{W}_i}(\widetilde{y})| + |N_{\widetilde{V}_i \cup \widetilde{V}_{i+1}}(\widetilde{y})| \leq \deg_H(\widetilde{y}) \leq \Delta\end{aligned}$$

and thus condition (C) of Lemma 12 is satisfied. To check conditions (D) and (E) of Lemma 12 observe that for all $\widetilde{y} \in \widetilde{C}'_{i',j}$ with $i' \in \{i, i+1\}$ and $j \in [2t]$ we have $C(\widetilde{y}) = C'_{i',j}$ as $\widetilde{y}$ has no neighbours in $\widetilde{V}_i$ or $\widetilde{V}_{i+1}$ and hence the external $\mathrm{edeg}(\widetilde{y}) = 0$ (see (17) and (18)). Thus (D) is satisfied for $\widetilde{y} \in \widetilde{C}'_{i',j}$, and similarly for $\widetilde{y} \in \widetilde{B'}_{i',j}$. For all $\widetilde{y} \in \widetilde{C}_{i,j}$ with $t < j \leq 2t$ on the other hand we have $\widetilde{X}_{\widetilde{y}} \subseteq N_{\widetilde{y}} \in \binom{\widetilde{V}_i}{\Delta}$ by (11). Recall that $N_{\widetilde{y}}$ was a special Δ-set in the application of the restricted blow-up lemma on $G_i = (U_i, V_i)$ and $H_i = H[\widetilde{U_i} \dot\cup \widetilde{V_i}]$ owing to (14). Therefore $N_{\widetilde{y}}$ is not mapped to a forbidden Δ-set in $\mathcal{B}_i \subseteq \binom{V_i}{\Delta}$ by f_{BL} and thus, by (15), to no Δ-set in $\mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i, C_{i,j}, C'_{i',j'}) \times V_i$ with $i' \in \{i, i+1\}, j, j' \in [2t]$ and $(c_{i,j}, c'_{i',j'}) \in R_{r,t}$. We infer that the set $f_{\mathrm{BL}}(\widetilde{X}_{\widetilde{y}}) = X_{\widetilde{y}} \in \binom{V_{i(\widetilde{y})}}{\mathrm{edeg}(\widetilde{y})}$ satisfies $|N^{\cap}_{C_{i,j}}(X_{\widetilde{y}})| \geq (d - \varepsilon_{\mathrm{CL}})^{\mathrm{edeg}(\widetilde{y})} p^{\mathrm{edeg}(\widetilde{y})} |C_{i,j}|$ and is such that

$$(N^{\cap}_{C_{i,j}}(X_{\widetilde{y}}), C'_{i',j'}) \text{ is } (\varepsilon_{\mathrm{CL}}, d, p)\text{-dense} \\ \text{for all } i' \in \{i, i+1\}, j, j' \in [2t] \text{ with } (c_{i,j}, c'_{i',j'}) \in R_{r,t}. \tag{19}$$

Since we chose $C(\widetilde{y}) = N^{\cap}(X_{\widetilde{y}}) \cap C_{i,j}$ in (18) we get condition (D) of Lemma 12 also for $\widetilde{y} \in \widetilde{C}_{i,j}$ with $t < j \leq 2t$. For $\widetilde{y} \in \widetilde{C}_{i+1,j}$ with $j \in [t]$ the same argument applies with $\widetilde{X}_{\widetilde{y}} \subseteq N_{\widetilde{y}} \in \binom{\widetilde{V}_{i+1}}{\Delta}$, and for $\widetilde{y} \in \widetilde{B}_{i,j}$ with $j \in [2t]$ the same argument applies with $\widetilde{X}_{\widetilde{y}} \subseteq N_{\widetilde{y}} \in \binom{\widetilde{V}_i}{\Delta}$.

Now it will be easy to see that we get (E) of Lemma 12. Indeed, recall again that $C(\widetilde{y}) = C'_{i',j'}$ for all $\widetilde{y} \in \widetilde{C}'_{i',j'}$ and $C(\widetilde{y}) = B'_{i',j'}$ for all $\widetilde{y} \in \widetilde{B}'_{i',j'}$ with $i' \in \{i, i+1\}$ and $j \in [2t]$. In addition, the mapping h constructed by Lemma 10 is a homomorphism from H to $R_{r,t}$. Hence (19) and property (G3) of Lemma 9 assert that condition (E) of Lemma 12 is satisfied for all edges $\widetilde{y}\widetilde{y}'$ of $H'_i = H[\widetilde{W}_i]$ with at least one end, say $\widetilde{y}$, in a cluster $\widetilde{C}'_{i',j'}$ or $\widetilde{B}'_{i',j'}$. This is true because then $C(\widetilde{y}) = W_{i,k}$ where $\widetilde{W}_{i,k}$ is the cluster containing $\widetilde{y}$, and $C(\widetilde{y}') = N^{\cap}(X_{\widetilde{y}'}) \cap W_{i,k'}$ where $\widetilde{W}_{i,k'}$ is the cluster containing $\widetilde{y}'$. Moreover, since h is a homomorphism all edges $\widetilde{y}\widetilde{y}'$ in $H'_i = H[\widetilde{W}_i]$ have at least one end in a cluster $\widetilde{C}'_{i',j'}$ or $\widetilde{B}'_{i',j'}$.

So conditions (A)–(E) are satisfied and we can apply Lemma 12 to get embeddings of $H'_i = H[\widetilde{W}_i]$ into $G'_i = G[W_i]$ for all $i \in [r]$ that map vertices $\widetilde{y} \in \widetilde{W}_i$ (i.e. connecting and balancing vertices) to vertices $y \in W_i$ in their candidate sets $C(\widetilde{y})$. Let f_{CL} be the united embedding resulting from these r applications of the connection lemma and denote the embedding that unites f_{BL} and f_{CL} by f.

To finish the proof we verify that f is an embedding of H into G. Let $\widetilde{x}\widetilde{y}$ be an edge of H. By definition of the spin graph $R_{r,t}$ and since the mapping h constructed by Lemma 10 is a homomorphism from H to $R_{r,t}$ we only need to distinguish the following cases for $i \in [r]$ and $j, j' \in [2t]$ (see also Figure 1):

case 1: If $\widetilde{x} \in \widetilde{V}_i$ and $\widetilde{y} \in \widetilde{U}_i$, then $f(\widetilde{x}) = f_{\mathrm{BL}}(\widetilde{x})$ and $f(\widetilde{y}) = f_{\mathrm{BL}}(\widetilde{y})$ and thus the constrained blow-up lemma guarantees that $f(\widetilde{x})f(\widetilde{y})$ is an edge of G_i.

case 2: If $\widetilde{x} \in \widetilde{W}_i$ and $\widetilde{y} \in \widetilde{W}_i$, then $f(\widetilde{x}) = f_{\mathrm{CL}}(\widetilde{x})$ and $f(\widetilde{y}) = f_{\mathrm{CL}}(\widetilde{y})$ and thus the connection lemma guarantees that $f(\widetilde{x})f(\widetilde{y})$ is an edge of G'_i.

case 3: If $\widetilde{x} \in \widetilde{V}_i$ and $\widetilde{y} \in \widetilde{W}_i$, then either $\widetilde{y} \in \widetilde{C}_{i,j}$ or $\widetilde{y} \in \widetilde{B}_{i,j}$ for some j. Moreover, $f(\widetilde{x}) = f_{\mathrm{BL}}(\widetilde{x})$ and therefore by (18) the candidate set $C(\widetilde{y})$ of $\widetilde{y}$ satisfies $C(\widetilde{y}) \subseteq N_{C_{i,j}}(f(\widetilde{x}))$ or $C(\widetilde{y}) \subseteq N_{B_{i,j}}(f(\widetilde{x}))$, respectively. As $f(\widetilde{y}) = f_{\mathrm{CL}}(\widetilde{y}) \in C(\widetilde{y})$ we also get that $f(\widetilde{x})f(\widetilde{y})$ is an edge of G in this case.

It follows that f maps all edges of H to edges of G, which finishes the proof of the theorem. □

9. A p-dense partition of G

For the proof of the Lemma for G we shall apply the minimum degree version of the sparse regularity lemma (Lemma 8). Observe that this lemma guarantees that the reduced graph of the regular partition we obtain is dense. Thus we can apply Theorem 2 to this reduced graph. In the proof of Lemma 9 we use this theorem to find a copy of the ladder R^*_r in the reduced graph (the graphs R^*_r and $R_{r,t}$ are defined in Section 5 on page 6, see also Figure 1). Then we further partition the clusters in this ladder to obtain a regular partition whose reduced graph contains a spin graph $R_{r,t}$. Recall that this partition will consist of a series of so-called *big clusters* which we denote by U_i and V_i, and a series of smaller clusters called *balancing clusters* $B_{i,j}$, $B'_{i,j}$ and *connecting clusters* $C_{i,j}$, $C'_{i,j}$ with $i \in [r]$, $j \in [2t]$. We will now give the details.

Proof of Lemma 9. Given t, r_0, η_G, and γ choose η'_G such that

$$\frac{\eta_G}{5} + \left(\frac{4}{\gamma} + 2\right) t \cdot \eta'_G \le \frac{\eta_G}{2} \tag{20}$$

and set $d := \gamma/4$. Apply Theorem 2 with input $r_{\text{BK}} := 2$, $\Delta = 3$ and $\gamma/2$ to obtain the constants β and $k_{\text{BK}} := n_0$. For input ε set

$$r'_0 := \max\{2r_0 + 1, k_{\text{BK}}, 3/\beta, 6/\gamma, 2/\varepsilon, 10/\eta_G\} \tag{21}$$

and choose ε' such that

$$\varepsilon'/\eta'_G \le \varepsilon/2, \qquad \text{and} \qquad \varepsilon' \le \min\{\gamma/4, \eta_G/10\}. \tag{22}$$

Lemma 8 applied with $\alpha := \frac{1}{2} + \gamma$, ε', r'_0 then gives us the missing constant r_1.

Assume that Γ is a typical graph from $\mathcal{G}_{n,p}$ with $\log^4 n/(pn) = o(1)$, in the sense that it satisfies the conclusion of Lemma 8, and let $G = (V, E) \subseteq \Gamma$ satisfy $\deg_G(v) \ge (\frac{1}{2} + \gamma)\deg_\Gamma(v)$ for all $v \in V$. Lemma 8 applied with $\alpha = \frac{1}{2} + \gamma$, ε', r'_0, and d to G gives us an (ε', d, p)-dense partition $V = V'_0 \dot\cup V'_1 \dot\cup \ldots \dot\cup V'_{r'}$ of G with reduced graph R' with $|V(R')| = r'$ such that $2r_0+1 \le r'_0 \le r' \le r_1$ and with minimum degree at least $(\frac{1}{2} + \gamma - d - \varepsilon')r' \ge (\frac{1}{2} + \frac{\gamma}{2})r'$ by (22). If r' is odd, then set $V_0 := V'_0 \dot\cup V'_{r'}$ and $r := (r'-1)/2$, otherwise set $V_0 := V'_0$ and $r := r'/2$. Clearly $r_0 \le r \le r_1$, the graph $R := R'[2r]$ still has minimum degree at least $(\frac{1}{2} + \frac{\gamma}{3})2r$ and $|V_0| \le \varepsilon' n + (n/r'_0) \le (\eta_G/5)n$ by the choice of r'_0 and ε'. It follows from Theorem 2 applied with $\Delta = 3$ and $\gamma/2$ that R contains a copy of the ladder R^*_r on $2r$ vertices (R^*_r has bandwidth $2 \le \beta \cdot 2r$ by the choice of r'_0 in (21)). Hence we can rename the vertices of the graph $R = R'[2r]$ with $u_1, v_1, \ldots, u_r, v_r$ according to the spanning copy of R^*_r. This naturally defines an equipartite mapping f from $V \setminus V_0$ to the vertices of the ladder R^*_r, where f maps all vertices in some cluster V_i with $i \in [2r]$ to a vertex $u_{i'}$ or $v_{i'}$ of R^*_r for some index $i' \in [r]$. We will show that subdividing the clusters $f^{-1}(x)$ for all $x \in V(R^*_r)$ will give the desired mapping g.

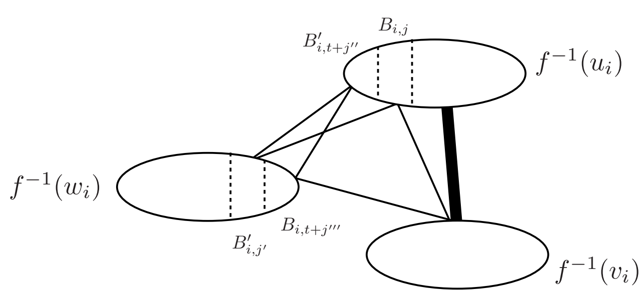

Figure 3. Cutting off a set of balancing clusters from $f^{-1}(u_i)$ and $f^{-1}(w_i)$. These clusters build p-dense pairs (thanks to the triangle $u_iv_iw_i$ in R) in the form of a C_5.

We will now construct the balancing clusters $B_{i,j}$ and $B'_{i,j}$ with $i \in [r]$, $j \in [2t]$ and afterwards turn to the connecting clusters $C_{i,j}$ and $C'_{i,j}$ and big clusters U_i and V_i with $i \in [r]$, $j \in [2t]$.

Notice that $\delta(R) \geq (\frac{1}{2}+\frac{\gamma}{3})2r$ implies that every edge u_iv_i of $R^*_r \subseteq R$ is contained in more than γr triangles in R. Therefore, we can choose vertices w_i of R for all $i \in [r]$ such that $u_iv_iw_i$ forms a triangle in R and no vertex of R serves as w_i more than $2/\gamma$ times. We continue by choosing in cluster $f^{-1}(u_i)$ arbitrary disjoint vertex sets $B_{i,1}, \dots, B_{i,t}, B'_{i,t+1}, \dots, B'_{i,2t}$, of size $\eta'_G n/(2r)$ each, for all $i \in [r]$. We will show below that $f^{-1}(u_i)$ is large enough so that these sets can be chosen. We then remove all vertices in these sets from $f^{-1}(u_i)$. Similarly, we choose in cluster $f^{-1}(w_i)$ arbitrary disjoint vertex sets $B_{i,t+1}, \dots, B_{i,2t}, B'_{i,1}, \dots, B'_{i,t}$, of size $\eta'_G n/(2r)$ each, for all $i \in [r]$. We also remove these sets from $f^{-1}(w_i)$. Observe that this construction asserts the following property. For all $i \in [r]$ and $j, j', j'', j''' \in [t]$ each of the pairs $(f^{-1}(v_i), B_{i,j})$, $(B_{i,j}, B'_{i,j'})$, $(B'_{i,j'}, B'_{i,t+j''})$, $(B'_{i,t+j''}, B_{i,t+j'''})$, and $(B_{i,t+j'''}, f^{-1}(v_i))$ is a sub-pair of a p-dense pair corresponding to an edge of $R[\{u_i, v_i, w_i\}]$ (see Figure 3). Accordingly this is a sequence of p-dense pairs in the form of a C_5, as needed for the balancing clusters in view of condition (G3) (see also Figure 1). Hence we call the sets $B_{i,j}$ and $B'_{i,j}$ with $i \in [r]$, $j \in [2t]$ balancing clusters from now on and claim that they have the required properties. This claim will be verified below.

We now turn to the construction of the connecting clusters and big clusters. Recall that we already removed balancing clusters from all clusters $f^{-1}(u_i)$ and possibly from some clusters $f^{-1}(v_i)$ (because v_i might have served as $w_{i'}$) with $i \in [r]$. For each $i \in [r]$ we arbitrarily partition the remaining vertices of cluster $f^{-1}(u_i)$ into sets $C_{i,1} \dot\cup \dots \dot\cup C_{i,2t} \dot\cup U_i$ and the remaining vertices of cluster $f^{-1}(v_i)$ into sets $C'_{i,1} \dot\cup \dots \dot\cup C'_{i,2t} \dot\cup V_i$ such that $|C_{i,j}|, |C'_{i,j}| = \eta'_G n/(2r)$ for all $i \in [r], j \in [2t]$. This gives us the connecting and the big clusters and we claim that also these clusters have the required properties. Observe, again, that for all $i \in [r]$, $i' \in \{i-1, i, i+1\} \setminus \{0\}$, $j, j' \in [2t]$ each of the pairs (U_i, V_i), $(C_{i',j}, V_i)$, and $(C_{i,j}, C'_{i,j'})$ is a sub-pair of a p-dense pair corresponding to an edge of R^*_r (see Figure 4).

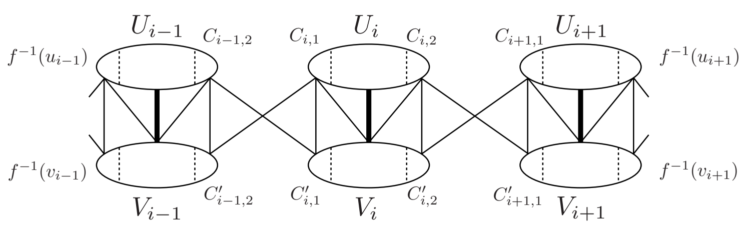

FIGURE 4. Partitioning the remaining vertices of cluster $f^{-1}(u_i)$ and $f^{-1}(v_i)$ into sets $C_{i,1} \dot\cup C_{i,2} \dot\cup U_i$ and $C'_{i,1} \dot\cup C'_{i,2} \dot\cup V_i$ (for the special case $t = 1$). These clusters form p-dense pairs (thanks to the ladder R^*_r in R) as indicated by the edges.

We will now show that the balancing clusters, connecting clusters and big clusters satisfy conditions (G1)–(G3). Note that condition (G2) concerning the sizes of the connecting and balancing clusters is satisfied by construction. To determine the sizes of the big clusters observe that from each cluster V'_j with $j \in [2r]$ vertices for at most $2t \cdot 2/\gamma$ balancing clusters were removed. In addition, at most $2t$ connecting clusters were split off from V'_j. Since $|V \setminus V_0| \geq (1 - \eta_G/5)n$ we get

$$|V_i|, |U_i| \geq \left(1 - \frac{\eta_G}{5}\right)\frac{n}{2r} - \left(\frac{4}{\gamma} + 2\right) t \cdot \eta'_G \frac{n}{2r} \geq (1 - \eta_G)\frac{n}{2r}$$

by (20). This is condition (G1). It remains to verify condition (G3). It can easily be checked that for all $xy \in E(R_{r,t})$ the corresponding pair $(g^{-1}(x), g^{-1}(y))$ is a sub-pair of some cluster pair $(f^{-1}(x'), f^{-1}(y'))$ with $x'y' \in E(R)$ by construction. In addition, all big, connecting, and balancing clusters are of size at least $\eta'_G n/(2r)$. Hence we have $|g^{-1}(x)| \geq \eta'_G |f^{-1}(x')|$ and $|g^{-1}(y)| \geq \eta'_G |f^{-1}(y')|$. We conclude from Proposition 6 that $(g^{-1}(x), g^{-1}(y))$ is (ε, d, p)-dense since $\varepsilon'/\eta'_G \leq \varepsilon$ by (22). This finishes the verification of (G3). □

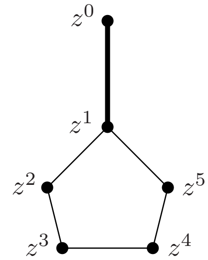

FIGURE 5. The graph $\bar{R}$ in Proposition 22.

10. A PARTITION OF H

Hajnal and Szemerédi determined the minimum degree that forces a certain number of vertex disjoint K_r copies in G. In addition their result guarantees that the remaining vertices can be covered by copies of K_{r-1}. Another way to express this, which actually resembles the original formulation, is obtained by considering the complement $\bar{G}$ of G and its maximum degree. Then, so the theorem asserts, the graph $\bar{G}$ contains a certain number of vertex disjoint independent sets of almost equal sizes. In other words, $\bar{G}$ admits a vertex colouring such that the sizes of the colour classes differ by at most 1. Such a colouring is also called *equitable colouring*.

Theorem 21 (Hajnal & Szemerédi [20])**.** *Let $\bar{G}$ be a graph on n vertices with maximum degree $\Delta(\bar{G}) \le \Delta$. Then there is an equitable vertex colouring of G with $\Delta + 1$ colours.* □

In the proof of Lemma 10 that we shall present in this section we will use this theorem in order to guarantee property (H3). This will be the very last step in the proof, however. First, we need to take care of the remaining properties.

Before we start, let us agree on some terminology that will turn out to be useful in the proof of Lemma 10. When defining a homomorphism h from a graph H to a graph R, we write $h(S) := z$ for a set S of vertices in H and a vertex z in R to say that all vertices from S are mapped to z. Recall that we have a bandwidth hypothesis on H. Consider an ordering of the vertices of H achieving its bandwidth. Then we can deal with the vertices of H in this order. In particular, we can refer to vertices as the *first* or *last* vertices in some set, meaning that they are the vertices with the smallest or largest label from this set.

We start with the following proposition.

Proposition 22. *Let $\bar{R}$ be the following graph with six vertices and six edges:*

$$\bar{R} := \left(\{z^0, z^1, \dots, z^5\}, \{z^0z^1, z^1z^2, z^2z^3, z^3z^4, z^4z^5, z^5z^1\}\right),$$

see Figure 5 for a picture of $\bar{R}$. For every real $\bar{\eta} > 0$ there exists a real $\bar{\beta} > 0$ such that the following holds: Consider an arbitrary bipartite graph $\bar{H}$ with $\bar{m}$ vertices, colour classes Z^0 and Z^1, and $\mathrm{bw}(\bar{H}) \le \bar{\beta}\bar{m}$ and denote by T the union of the first $\bar{\beta}\bar{m}$ vertices and the last $\bar{\beta}\bar{m}$ vertices of H. Then there exists a homomorphism $\bar{h}\colon V(\bar{H}) \to V(\bar{R})$ from $\bar{H}$ to $\bar{R}$ such that for all $j \in \{0,1\}$ and all $k \in [2,5]$

$$\frac{\bar{m}}{2} - 5\bar{\eta}\bar{m} \le |\bar{h}^{-1}(z^j)| \le \frac{\bar{m}}{2} + \bar{\eta}\bar{m}\,, \tag{23}$$

$$|\bar{h}^{-1}(z^k)| \le \bar{\eta}\bar{m}\,, \tag{24}$$

$$h(T \cap Z^j) = z^j. \tag{25}$$

Roughly speaking, Proposition 22 shows that we can find a homomorphism from a bipartite graph $\bar{H}$ to a graph $\bar{R}$ which consists of an edge z^0z^1 which has an attached 5-cycle in such a way that most of the vertices of $\bar{H}$ are mapped about evenly to the vertices z^0 and z^1. If we knew that the colour classes of $\bar{H}$ were of almost equal size, then this would be a trivial task, but since this is not guaranteed, we will have to make use of the additional vertices $z^2, \dots, z^5$.

Proof of Proposition 22. Given $\bar{\eta}$, choose an integer $\ell \geq 6$ and a real $\bar{\beta} > 0$ such that

$$\frac{5}{\ell} < \bar{\eta} \quad \text{and} \quad \bar{\beta} := \frac{1}{\ell^2}. \tag{26}$$

For the sake of a simpler exposition we assume that $\bar{m}/\ell$ and $\bar{\beta}\bar{m}$ are integers. Now consider a graph $\bar{H}$ as given in the statement of the proposition. Partition $V(\bar{H})$ along the ordering induced by the bandwidth labelling into sets $\bar{W}_1, \dots, \bar{W}_\ell$ of sizes $|\bar{W}_i| = \bar{m}/\ell$ for $i \in [\ell]$. For each $\bar{W}_i$, consider its last $5\bar{\beta}\bar{m}$ vertices and partition them into sets $X_{i,1}, \dots, X_{i,5}$ of size $|X_{i,k}| = \bar{\beta}\bar{m}$. For $i \in [\ell]$ let

$$W_i := \bar{W}_i \setminus (X_{i,1} \cup \dots \cup X_{i,5}), \quad W := \bigcup_{i=1}^{\ell} W_i,$$

and note that

$$L := |W_i| = \frac{\bar{m}}{\ell} - 5\bar{\beta}\bar{m} \stackrel{(26)}{=} \left(\frac{1}{\ell} - \frac{5}{\ell^2}\right)\bar{m} \geq \frac{1}{\ell^2}\bar{m} \stackrel{(26)}{=} \bar{\beta}\bar{m}.$$

For $i \in [\ell]$, $j \in \{0,1\}$, and $1 \leq k \leq 5$ let

$$W_i^j := W_i \cap Z^j, \quad X_{i,k}^j := X_{i,k} \cap Z^j.$$

Thanks to the fact that $\mathrm{bw}(\bar{H}) \leq \bar{\beta}\bar{m}$, we know that there are no edges between W_i and $W_{i'}$ for $i \neq i' \in [\ell]$. In a first round, for each $i \in [\ell]$ we will either map all vertices from W_i^j to z^j for both $j \in \{0,1\}$ (call such a mapping a *normal* embedding of W_i) or we map all vertices from W_i^j to z^{1-j} for both $j \in \{0,1\}$ (call this an *inverted* embedding). We will do this in such a way that the difference between the number of vertices that get sent to z^0 and the number of those that get sent to z^1 is as small as possible. Since $|W_i| \leq L$ the difference is therefore at most L. If, in addition, we guarantee that both W_1 and W_ℓ receive a normal embedding, it is at most $2L$. So, to summarize and to describe the mapping more precisely: there exist integers $\varphi_i \in \{0,1\}$ for all $i \in [\ell]$ such that $\varphi_1 = 0 = \varphi_\ell$ and the function $h : W \to \{z^0, z^1\}$ defined by

$$h(W_i^j) := \begin{cases} z^j & \text{if } \varphi_i = 0, \\ z^{1-j} & \text{if } \varphi_i = 1, \end{cases}$$

is a homomorphism from $\bar{H}[W]$ to $\bar{R}[\{z^0, z^1\}]$, satisfying that for both $j \in \{0,1\}$

$$\begin{aligned} |h^{-1}(z^j)| &\leq \frac{\ell L}{2} + 2L = \left(\frac{\ell}{2} + 2\right)\frac{\bar{m}}{\ell} - \left(\frac{\ell}{2} + 2\right)5\bar{\beta}\bar{m} \\ &\stackrel{(26)}{=} \frac{\bar{m}}{2} + \bar{m}\left(\frac{2}{\ell} - \frac{5}{2}\frac{1}{\ell} - \frac{10}{\ell^2}\right) \leq \frac{\bar{m}}{2}\,. \end{aligned} \tag{27}$$

In the second round we extend this homomorphism to the vertices in the classes $X_{i,k}$. Recall that these vertices are by definition situated after those in W_i and before those in W_{i+1}. The idea for the extension is simple. If W_i and W_{i+1} have been embedded in the same way by h (either both normal or both inverted), then we map all the vertices from all $X_{i,k}$ to z^0 and z^1 accordingly. If they have been embedded in different ways (one normal and one inverted), then we walk around the 5-cycle $z^1, \dots, z^5, z^1$ to switch colour classes.

Here is the precise definition. Consider an arbitrary $i \in [\ell]$. Since $h(W_i^0)$ and $h(W_i^1)$ are already defined, choose (and fix) $j \in \{0,1\}$ in such a way that $h(W_i^j) = z^1$. Note that this implies that $h(W_i^{1-j}) = z^0$. Now define $h_i : \bigcup_{k=0}^{5} X_{i,k} \to \bigcup_{k=1}^{5}\{z^k\}$ as follows:

Suppose first that $\varphi_i = \varphi_{i+1}$. Observe that in this case we must also have $h(W_{i+1}^j) = z^1$ and $h(W_{i+1}^{1-j}) = z^0$. So we can happily define for all $k \in [5]$

$$h_i(X_{i,k}^j) = z^1 \quad \text{and} \quad h_i(X_{i,k}^{1-j}) = z^0.$$

Now suppose that $\varphi_i \neq \varphi_{i+1}$. Since we are still assuming that j is such that $h(W_i^j) = z^1$ and thus $h(W_i^{1-j}) = z^0$, the fact that $\varphi_i \neq \varphi_{i+1}$ implies that $h(W_{i+1}^j) = z^0$ and $h(W_{i+1}^{1-j}) = z^1$. In this case we define h_i as follows:

$h(W_i^{1-j})$ $= z^0$	$h_i(X_{i,1}^{1-j})$ $:= z^2$	$h_i(X_{i,2}^{1-j})$ $:= z^2$	$h_i(X_{i,3}^{1-j})$ $:= z^4$	$h_i(X_{i,4}^{1-j})$ $:= z^4$	$h_i(X_{i,5}^{1-j})$ $:= z^1$	$h(W_{i+1}^{1-j})$ $= z^1$
$h(W_i^{j})$ $= z^1$	$h_i(X_{i,1}^{j})$ $:= z^1$	$h_i(X_{i,2}^{j})$ $:= z^3$	$h_i(X_{i,3}^{j})$ $:= z^3$	$h_i(X_{i,4}^{j})$ $:= z^5$	$h_i(X_{i,5}^{j})$ $:= z^5$	$h(W_{i+1}^{j})$ $= z^0$

Finally, we set $\bar{h} : V(\bar{H}) \to V(\bar{R})$ by letting $\bar{h}(x) := h(x)$ if $x \in W_i$ for some $i \in [\ell]$ and $\bar{h}(x) := h_i(x)$ if $x \in X_{i,k}$ for some $i \in [\ell]$ and $k \in [5]$.

In order to verify that this is a homomorphism from $\bar{H}$ to the sets $\bar{R}$, we first let

$$X_{i,0}^0 := W_i^0, X_{i,0}^1 := W_i^1, X_{i,6}^0 := W_{i+1}^0, X_{i,6}^1 := W_{i+1}^1.$$

Using this notation, it is clear that any edge xx' in $\bar{H}[W_i \cup \bigcup_{k=1}^5 X_{i,k} \cup W_{i+1}]$ with $x \in Z^j$ and $x' \in Z^{1-j}$ is of the form

$$xx' \in (X_{i,k}^j \times X_{i,k}^{1-j}) \cup (X_{i,k}^j \times X_{i,k+1}^{1-j}) \cup (X_{i,k+1}^j \times X_{i,k}^{1-j})$$

for some $k \in [0,6]$. It is therefore easy to check in the above table that $\bar{h}$ maps xx' to an edge of R.

We conclude the proof by showing that the cardinalities of the preimages of the vertices in R match the required sizes. In the second round we mapped a total of

$$\ell \cdot 5\bar{\beta}\bar{m} \overset{(26)}{=} \frac{5}{\ell}\bar{m} \overset{(26)}{\leq} \bar{\eta}\bar{m}$$

additional vertices from $\bar{H}$ to the vertices of $\bar{R}$, which guarantees that

$$|\bar{h}^{-1}(z^j)| \overset{(27)}{\leq} \frac{\bar{m}}{2} + \bar{\eta}\bar{m} \text{ for all } j \in \{0,1\}, \quad |\bar{h}^{-1}(z^k)| \leq \bar{\eta}\bar{m} \text{ for all } k \in [2,5].$$

Finally, the lower bound in (23) immediately follows from the upper bounds:

$$|\bar{h}^{-1}(z^j)| \geq \bar{m} - |\bar{h}^{-1}(z^{1-j})| - \sum_{k=2}^{5} |\bar{h}^{-1}(z^k)| \geq \frac{\bar{m}}{2} - 5\bar{\eta}\bar{m}.$$

□

We remark that Proposition 22 (and thus Lemma 10) would remain true if we replaced the 5-cycle in $\bar{R}$ by a 3-cycle. However, we need the properties of the 5-cycle in the proof of the main theorem. Now we will prove Lemma 10.

Proof of Lemma 10. Given the integer Δ, set $t := (\Delta+1)^3(\Delta^3+1)$. Given a real $0 < \eta_H < 1$ and integers m and r, set $\bar{\eta} := \eta_H/20 < 1/20$ and apply Proposition 22 to obtain a real $\bar{\beta} > 0$. Choose $\beta > 0$ sufficiently small so that all the inequalities

$$\frac{1}{r} - 4\beta \geq \beta/\bar{\beta}, \quad 4\beta r \leq \frac{\eta_H}{20r}, \quad 16\Delta\beta r \leq \eta_H \left(\frac{1}{r} - 4\beta\right)\left(\frac{1}{2} - 5\bar{\eta}\right) \tag{28}$$

hold. Again, we assume that m/r and βm are integers.

Next we consider the spin graph $R_{r,t}$ with $t = 1$, i.e., let $R := R_{r,1}$. For the sake of simpler reference, we will change the names of its vertices as follows: For all $i \in [r]$ we set (see Figure 6)

$$z_i^0 := u_i, z_i^1 := v_i, z_i^2 := b_{i,1}, z_i^3 := b'_{i,1}, z_i^4 := b'_{i,2}, z_i^5 := b_{i,2},$$

$$q_i^2 := c_{i,1}, q_i^3 := c'_{i,1}, q_i^4 := c_{i,2}, q_i^5 := c'_{i,2}.$$

Note that for every $i \in [r]$ the graph $R[\{z_i^0, \ldots, z_i^5\}]$ is isomorphic to the graph $\bar{R}$ defined in Proposition 22.

Partition $V(H)$ along the ordering (induced by the bandwidth labelling) into sets $\bar{S}_1, \ldots, \bar{S}_r$ of sizes $|\bar{S}_i| = m/r$ for $i \in [r]$.

Define sets $T_{i,k}$ for $i \in [r]$ and $k \in [0,5]$ with $|T_{i,k}| = \beta m$ such that $T_{i,0} \cup \cdots \cup T_{i,4}$ contain the last $5\beta m$ vertices of $\bar{S}_i$ and $T_{i,5}$ the first βm vertices of $\bar{S}_{i+1}$ (according to the ordering). Set

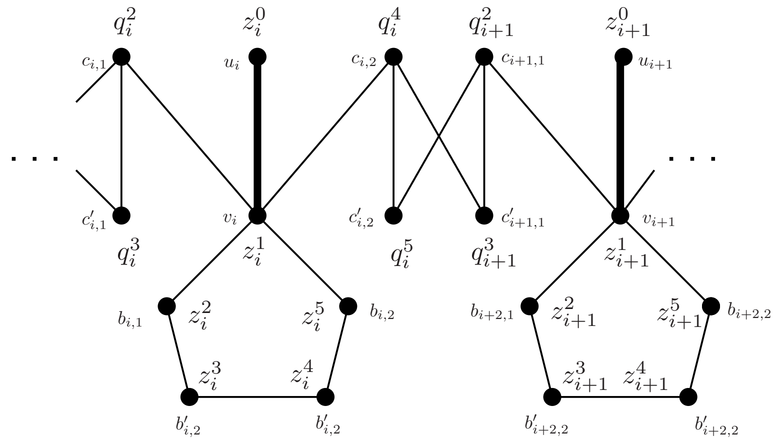

FIGURE 6. The subgraph $R[\{z_i^0,\dots,z_i^5,q_i^4,q_i^5,q_{i+1}^2,q_{i+1}^3,z_{i+1}^0,\dots,z_{i+1}^5\}]$ of $R_{r,1}$ in the proof of Lemma 10.

$S_i := \bar{S}_i \setminus (T_{i,1}\cup\dots\cup T_{i,4})$ and observe that this implies that $T_{i,0}$ is the set of the last βm vertices of S_i and $T_{i,5}$ is the set of the first βm vertices in $\bar{S}_{i+1}$. Set

$$\bar{m} := |S_i| = (m/r) - 4\beta m = \left(\frac{1}{r} - 4\beta\right) m \overset{(28)}{\geq} \beta m/\bar{\beta}, \quad \text{thus} \quad \bar{\beta}\bar{m} \geq \beta m. \tag{29}$$

Denote by Z^0 and Z^1 the two colour classes of the bipartite graph H. For $i\in[\ell]$, $k\in[0,5]$ and $j\in[0,1]$ let

$$S_i^j := S_i \cap Z^j, \quad T_{i,k}^j := T_{i,k}\cap Z^j.$$

Now for each $i\in[r]$ apply Proposition 22 to $\bar{H}_i := H[S_i]$ and $\bar{R}_i := R[\{z_i^0,\dots,z_i^5\}]$. Observe that

$$\operatorname{bw}(\bar{H}_i) \leq \operatorname{bw}(H) \leq \beta m \overset{(29)}{\leq} \bar{\beta}\bar{m},$$

so we obtain a homomorphism $\bar{h}_i : S_i \to \{z_i^0,\dots,z_i^5\}$ of $\bar{H}_i$ to $\bar{R}_i$. Combining these yields a homomorphism

$$\bar{h} : \bigcup_{i=1}^{r} S_i \to \bigcup_{i=1}^{r}\{z_i^0,\dots,z_i^5\},$$

$$\text{from} \quad H[\bigcup_{i=1}^{r} S_i] \quad \text{to} \quad R[\bigcup_{i=1}^{r}\{z_i^0,\dots,z_i^5\}]$$

with the property that for every $i\in[r]$, $j\in[0,1]$ and $k\in[2,5]$

$$\frac{\bar{m}}{2} - 5\bar{\eta}\bar{m} \overset{(23)}{\leq} |\bar{h}^{-1}(z_i^j)| \overset{(23)}{\leq} \frac{\bar{m}}{2} + \bar{\eta}\bar{m} \leq \left(1+\frac{\eta_H}{10}\right)\frac{m}{2r} \quad \text{and}$$

$$|\bar{h}^{-1}(z_i^k)| \overset{(24)}{\leq} \bar{\eta}\bar{m} \leq \frac{\eta_H}{10}\frac{m}{2r}.$$

Thanks to (29), we know that $\bar{\beta}\bar{m} \geq \beta m$, and therefore applying the information from (25) in Proposition 22 yields that for all $i\in[r]$ and $j\in[0,1]$

$$\bar{h}(T_{i,0}^j) = z_i^j \quad \text{and} \quad \bar{h}(T_{i,5}^j) = z_{i+1}^j.$$

In the second round, our task is to extend this homomorphism to the vertices in $\bar{S}_i \setminus S_i$ by defining a function

$$h_i : T_{i,1}\cup\dots\cup T_{i,4} \to \{z_i^1, q_i^4, q_i^5, q_{i+1}^2, q_{i+1}^3, z_{i+1}^1\}$$

for each $i\in[r]$ as follows:

$\bar{h}(T^0_{i,0}) = z^0_i$	$h_i(T^0_{i,1}) := q^4_i$	$h_i(T^0_{i,2}) := q^4_i$	$h_i(T^0_{i,3}) := q^2_{i+1}$	$h_i(T^0_{i,4}) := q^2_{i+1}$	$\bar{h}(T^0_{i,5}) = z^0_{i+1}$
$\bar{h}(T^1_{i,0}) = z^1_i$	$h_i(T^1_{i,1}) := z^1_i$	$h_i(T^1_{i,2}) := q^5_i$	$h_i(T^1_{i,3}) := q^3_{i+1}$	$h_i(T^1_{i,4}) := z^1_{i+1}$	$\bar{h}(T^1_{i,5}) = z^1_{i+1}$

Now set $h(x) := \bar{h}(x)$ if $x \in S_i$ for some $i \in [r]$ and $h(x) := h_i(x)$ if $x \in T_{i,k}$ for some $i \in [r]$ and $k \in [4]$.

Let us verify that h is a homomorphism from H to R. For edges xx' with both endpoints inside a set S_i we do not need to check anything because here $h(x) = \bar{h}(x)$ and $h(x') = \bar{h}(x')$ and we know from Proposition 22 that $\bar{h}$ is a homomorphism. Due to the bandwidth condition $\mathrm{bw}(H) \leq \beta m$, any other edge xx' with $x \in Z^0$ and $x' \in Z^1$ is of the form

$$xx' \in (T^0_{i,k} \times T^1_{i,k}) \cup (T^0_{i,k} \times T^1_{i,k+1}) \cup (T^0_{i,k+1} \times T^1_{i,k})$$

for some $i \in [\ell]$ and $0 \leq k, k+1 \leq 5$. It is therefore easy to check in the above table that h maps xx' to an edge of R.

What can we say about the cardinalities of the preimages? In the second round we have mapped $4\beta mr$ additional vertices from H to vertices in R, hence for any vertex z in R with $z \notin \{z^0_i, z^1_i\}$, $i \in [\ell]$ we have

$$|h_i^{-1}(z)| \leq 4\beta mr \overset{(28)}{\leq} \frac{\eta_H}{10}\frac{m}{2r}, \tag{30}$$

and therefore the required upper bounds immediately follow from (10).

At this point we have found a homomorphism h from H to $R = R_{r,1}$ of which we know that it satisfies properties (H1) and (H2).

So far we have been working with the graph $R = R_{r,1}$, and therefore we know which vertices have been mapped to $u_i = z^0_i$ and $v_i = z^1_i$:

$$\widetilde{U_i} := h^{-1}(u_i) = h^{-1}(z^0_i) \quad \text{and} \quad \widetilde{V_i} := h^{-1}(v_i) = h^{-1}(z^1_i).$$

Moreover for $i \in [r]$ and $k \in [2,5]$ set

$$Z^k_i := h^{-1}(z^k_i) \quad \text{and} \quad Q^k_i := h^{-1}(q^k_i).$$

Let us deal with property (H5) next. By definition, a vertex in $\widetilde{X}_i \subseteq \widetilde{V}_i$ must have at least one neighbour in Q^2_i or Q^4_i or Z^2_i or Z^5_i. We know from (30) that the two latter sets contain at most $4\beta mr$ vertices each, and each of their vertices has at most Δ neighbours. Thus

$$\begin{aligned} |\widetilde{X}_i| \quad &\leq \quad \Delta \cdot 16\beta mr \overset{(28)}{\leq} \eta_H \left(\frac{1}{r} - 4\beta\right)\left(\frac{1}{2} - 5\bar{\eta}\right) m \overset{(29)}{=} \eta_H \bar{m}\left(\frac{1}{2} - 5\bar{\eta}\right) \\ &\overset{(10)}{\leq} \quad \eta_H |\bar{h}^{-1}(z^1_i)| \leq \eta_H |h^{-1}(z^1_i)| = \eta_H |\widetilde{V}_i|, \end{aligned}$$

which shows that (H5) is also satisfied.

Next we would like to split up the sets Z^k_i and Q^k_i for $i \in [r]$ and $k \in [2,5]$ into smaller sets in order to meet the additional requirements (H3) and (H4). This means that we need to partition them further into sets of vertices which have no path of length 1, 2, or 3 between them and which have the same degree into certain sets.

To achieve this, first denote by H^3 the 3rd power of H. Then an upper bound on the maximum degree of H^3 is obviously given by

$$\Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)(\Delta - 1) \leq \Delta^3.$$

Hence H^3 has a vertex colouring $c : V(H) \to \mathbb{N}$ with at most $\Delta^3 + 1$ colours. Notice that a set of vertices that receives the same colour by c forms a 3-independent set in H. To formalize this argument, we define a 'fingerprint' function

$$f : \bigcup_{i=1}^{r} \bigcup_{k=2}^{5} (Z^k_i \cup Q^k_i) \to [0, \Delta] \times [0, \Delta] \times [0, \Delta] \times [\Delta^3 + 1]$$

as follows:

$$f(y) := \begin{cases} \Big(\deg_{\widetilde{V}_i}(y), \deg_{Q_i^2\cup Q_i^4}(y), \deg_{Z_i^2}(y), c(y)\Big) & \text{if } y \in \Big(\bigcup_{k=2}^5 (Q_i^k \cup Z_i^k)\Big) \setminus Z_i^4, \\ \Big(\deg_{\widetilde{V}_i}(y), \deg_{Q_i^2\cup Q_i^4}(y), \deg_{Z_i^3\cup Z_i^5}(y), c(y)\Big) & \text{if } y \in Z_i^4, \end{cases}$$

for some $i \in [r]$.

Recall that we defined $t := (\Delta+1)^3(\Delta^3+1)$, so let us identify the codomain of f with the set $[t]$. Now for $i \in [r]$ and $j \in [t]$ we set

$$\widetilde{B}_{i,j} := Z_i^2 \cap f^{-1}(j), \qquad \widetilde{B}_{i,t+j} := Z_i^5 \cap f^{-1}(j),$$
$$\widetilde{B}'_{i,j} := Z_i^3 \cap f^{-1}(j), \qquad \widetilde{B}'_{i,t+j} := Z_i^4 \cap f^{-1}(j),$$
$$\widetilde{C}_{i,j} := Q_i^2 \cap f^{-1}(j), \qquad \widetilde{C}_{i,t+j} := Q_i^4 \cap f^{-1}(j),$$
$$\widetilde{C}'_{i,j} := Q_i^3 \cap f^{-1}(j), \qquad \widetilde{C}'_{i,t+j} := Q_i^5 \cap f^{-1}(j).$$

Observe, for example, that for $y \in \widetilde{B}_{i,j}$ the third component of $f(y)$ is exactly equal to $\deg_{L(i,j)}(y)$. Now, for any

$$yy' \in \binom{\widetilde{C}_{i,j}}{2} \cup \binom{\widetilde{B}_{i,j}}{2} \cup \binom{\widetilde{C}'_{i,j}}{2} \cup \binom{\widetilde{B}'_{i,j}}{2},$$

we have $f(y) = j = f(y')$ and hence any of the parameters required in (H3) and (H4) have the same value for y and y'.

The only thing missing before the proof of Lemma 10 is complete is that we need to guarantee that every $y \in Z_i^2 \cup Z_i^5 \cup Q_i^2 \cup Q_i^4$ has at most $\Delta - 1$ neighbours in $\widetilde{V}_i$, as required in the first line of (H4). If a vertex y does not satisfy this, it must have *all* its Δ neighbours in $\widetilde{V}_i$. Since by definition of $\widetilde{V}_i$ these neighbours have been mapped to z_i^1, we can map y to z_i^0 (instead of mapping it to z_i^2, z_i^5, q_i^2 or q_i^4).

Even if, in this way, all of the vertices in $Z_i^2 \cup Z_i^5 \cup Q_i^2 \cup Q_i^4$ would have to be mapped to z_i^0, (30) assures us that these are at most $4\frac{\eta_H}{10}\frac{m}{2r}$ vertices. Since by (10) at most $(1+\frac{\eta_H}{10})\frac{m}{2r}$ have already been mapped to z_i^0 in the first round and by (30) at most $\frac{\eta_H}{10}\frac{m}{2r}$ in the second round, this does not violate the upper bound in (H1). □

11. The constrained blow-up lemma

As explained earlier, the proof of the constrained blow-up lemma uses techniques developed in [4, 28] adapted to our setting. In fact, the proof we present here follows the embedding strategy used in the proof of [4, Theorem 1.5]. This strategy is roughly as follows. Assume we want to embed the bipartite graph H on vertex set $\widetilde{U}\dot\cup\widetilde{V}$ into the host graph G on vertex set $U\dot\cup V$. Then we consider injective mappings $f\colon \widetilde{V} \to V$, and try to find one that can be extended to $\widetilde{U}$ such that the resulting mapping is an embedding of H into G. For determining whether a particular mapping f can be extended in this way we shall construct an auxiliary bipartite graph B_f, the so-called candidate graph (see Definition 23), which contains a matching covering one of its partition classes if and only if f can be extended. Accordingly, our goal will be to check whether B_f contains such a matching M which we will do by appealing to Hall's condition. On page 26 we will explain the details of this part of the proof, determine necessary conditions for the application of Hall's theorem, and collect them in form of a matching lemma (Lemma 30). It will then remain to show that there is a mapping f such that B_f satisfies the conditions of this matching lemma. This will require most of the work. The idea here is as follows.

We will show that mappings f usually have the necessary properties as long as they do not map neighbourhoods $N_H(\widetilde{u}) \subseteq \widetilde{V}$ of vertices in $\widetilde{u} \in \widetilde{U}$ to certain "bad" spots in V. The existence of (many) mappings that avoid these "bad" spots is verified with the help of a hypergraph packing lemma (Lemma 28). This lemma states that half of all possible mappings f avoid *almost all* "bad" spots and can easily be turned into mappings f' avoiding *all* "bad" spots with the help of so-called switchings.

11.1. **Candidate graphs.** If we have injective mappings $f\colon \widetilde{V} \to V$ as described in the previous paragraph we would like to decide whether f can be extended to an embedding of H into G. Observe that in such an embedding each vertex $\widetilde{u} \in \widetilde{U}$ has to be embedded to a vertex $u \in U$ such that the following holds. The neighbourhood $N_H(\widetilde{u})$ has its image $f(N_H(\widetilde{u}))$ in the set $N_G(u)$. Determining which vertices u are "candidates" for the embedding of $\widetilde{u}$ in this sense gives rise to the following bipartite graph.

Definition 23 (candidate graph)**.** *Let H and G be bipartite graphs on vertex sets $\widetilde{U}\dot\cup\widetilde{V}$ and $U\dot\cup V$, respectively. For an injective function $f\colon \widetilde{V} \to V$ we say that a vertex $u \in U$ is an* f-candidate *for $\widetilde{u} \in \widetilde{U}$ if and only if $f(N_H(\widetilde{u})) \subseteq N_G(u)$.*

The candidate graph $B_f(H,G) := (\widetilde{U}\dot\cup U, E_f)$ *for f is the bipartite graph with edge set*

$$E_f := \left\{\widetilde{u}u \in \widetilde{U} \times U\colon \ u \text{ is an } f\text{-candidate for } \widetilde{u}\right\}.$$

Now it is easy to see that the mapping f described above can be extended to an embedding of H into G if and only if the corresponding candidate graph has a matching covering $\widetilde{U}$. Clearly, if the candidate graph $B_f(H,G)$ of f has vertices $\widetilde{u} \in \widetilde{U}$ of degree 0, then $B_f(H,G)$ has no such matching and hence f cannot be extended. More generally we would like to avoid that $\deg_{B_f(H,G)}(\widetilde{u})$ is too small. Notice that this means precisely that f should not map $N_H(\widetilde{u})$ to a set $B \subseteq V$ that has a small common neighbourhood in G. These sets B are the "bad" spots (see the beginning of this section) that should be avoided by f.

We explained above that, in order to avoid "bad" spots, we will have to change certain mappings f slightly. The exact definition of this operation is as follows.

Definition 24 (switching)**.** *Let $f, f'\colon X \to Y$ be injective functions. We say that f' is obtained from f by a* switching *if there are $u, v \in X$ with $f'(u) = f(v)$ and $f'(v) = f(u)$ and $f(w) = f'(w)$ for all $w \notin \{u, v\}$. The* switching distance $\mathrm{d_{sw}}(f, f')$ *of f and f' is at most s if the mapping f' can be obtained from f by a sequence of at most s switchings.*

These switchings will alter the candidate graph corresponding to the injective function slightly (but not much, see Lemma 26). In order to quantify this, we further define the neighbourhood distance between two bipartite graphs B and B' which determines the number of vertices (in one partition class) whose neighbourhoods differ in B and B'.

Definition 25 (neighbourhood distance)**.** *Let $B = (U\dot\cup\widetilde{U}, E)$, $B' = (U\dot\cup\widetilde{U}, E')$ be bipartite graphs. We define the* neighbourhood distance *of B and B' with respect to $\widetilde{U}$ as*

$$\mathrm{d}_{N(\widetilde{U})}(B, B') := \left|\{\widetilde{u} \in \widetilde{U}\colon \ N_B(\widetilde{u}) \neq N_{B'}(\widetilde{u})\}\right|.$$

The next simple lemma now examines the effect of switchings on the neighbourhood distance of candidate graphs and shows that functions with small switching distance correspond to candidate graphs with small neighbourhood distance.

Lemma 26 (switching lemma)**.** *Let H and G be bipartite graphs on vertex sets $\widetilde{U}\dot\cup\widetilde{V}$ and $U\dot\cup V$, respectively, such that $\deg_H(\widetilde{v}) \leq \Delta$ for all $\widetilde{v} \in \widetilde{V}$ and let $f, f'\colon \widetilde{V} \to V$ be injective functions with switching distance $\mathrm{d_{sw}}(f, f') \leq s$. Then the neighbourhood distance of the candidate graphs $B_f(H,G)$ and $B_{f'}(H,G)$ satisfies*

$$\mathrm{d}_{N(\widetilde{U})}\Big(B_f(H,G), B_{f'}(H,G)\Big) \leq 2s\Delta\,.$$

Proof. We proceed by induction on s. For $s = 0$ the lemma is trivially true. Thus, consider $s > 0$ and let g be a function with $\mathrm{d_{sw}}(f, g) \leq s-1$ and $\mathrm{d_{sw}}(g, f') = 1$. Define

$$N(f, f') := \left\{\widetilde{u} \in \widetilde{U}\colon N_{B_f(H,G)}(\widetilde{u}) \neq N_{B_{f'}(H,G)}(\widetilde{u})\right\}.$$

Clearly, $|N(f, f')| = \mathrm{d}_{N(\widetilde{U})}(B_f(H,G), B_{f'}(H,G))$ and $N(f, f') \subseteq N(f, g) \cup N(g, f')$. By induction hypothesis we have $|N(f, g)| \leq 2(s-1)\Delta$. The remaining switching from g to f' interchanges only the images of two vertices from $\widetilde{V}$, say $\widetilde{v}_1$ and $\widetilde{v}_2$. It follows that

$$N(g, f') = \left\{\widetilde{u} \in N_H(\widetilde{v}_1) \cup N_H(\widetilde{v}_2)\colon N_{B_g(H,G)}(\widetilde{u}) \neq N_{B_{f'}(H,G)}(\widetilde{u})\right\},$$

which implies $|N(g,f')| \le 2\Delta$ and therefore we get $|N(f,f')| \le 2s\Delta$. □

11.2. **A hypergraph packing lemma.** The main ingredient to the proof of the constrained blow-up lemma is the following hypergraph packing result (Lemma 28). To understand what this lemma says and how we will apply it, recall that we would like to embed the vertex set $\widetilde{U}$ of H into the vertex set U of G such that subsets of $\widetilde{U}$ that form neighbourhoods in the graph H avoiding certain "bad" spots in U. If H is a Δ-regular graph, then these neighbourhoods form Δ-sets. In this case, as we will see, also the "bad" spots form Δ-sets. Accordingly, we have to solve the problem of packing the neighbourhood Δ-sets $\mathcal{N}$ and the "bad" Δ-sets $\mathcal{B}$, which is a hypergraph packing problem. Lemma 28 below states that this is possible under certain conditions. One of these conditions is that the "bad" sets should not "cluster" too much (although there might be many of them). The following definition makes this precise.

Definition 27 (corrupted sets)**.** *For $\Delta \in \mathbb{N}$ and a set V let $\mathcal{B} \subseteq \binom{V}{\Delta}$ be a collection of Δ-sets in V and let x be a positive real. We say that all $B \in \mathcal{B}$ are x-*corrupted *by $\mathcal{B}$. Recursively, for $i \in [\Delta-1]$ an i-set $B \in \binom{V}{i}$ in V is called x-corrupted by $\mathcal{B}$ if it is contained in more than x of the $(i+1)$-sets that are x-corrupted by $\mathcal{B}$.*

Observe that, if a vertex $v \in V$ is not x-corrupted by $\mathcal{B}$, then it is also not x'-corrupted by $\mathcal{B}$ for any $x' > x$.

The hypergraph packing lemma now implies that $\mathcal{N}$ and $\mathcal{B}$ can be packed if $\mathcal{B}$ contains no corrupted sets. In fact this lemma states that *half of all* possible ways to map the vertices of $\mathcal{N}$ to $\mathcal{B}$ can be turned into such a packing by performing a sequence of few switchings.

Lemma 28 (hypergraph packing lemma [28])**.** *For all integers $\Delta \ge 2$ and $\ell \ge 1$ and every positive σ there are positive constants η_{28}, and n_{28} such that the following holds. Let $\mathcal{B}$ be a Δ-uniform hypergraph on $n' \ge n_{28}$ vertices such that no vertex of $\mathcal{B}$ is $\eta_{28}n'$-corrupted by $\mathcal{B}$. Let $\mathcal{N}$ be a Δ-uniform hypergraph on $n \le n'$ vertices such that no vertex of $\mathcal{N}$ is contained in more than ℓ edges of $\mathcal{N}$.*

Then for at least half of all injective functions $f\colon V(\mathcal{N}) \to V(\mathcal{B})$ there are packings f' of $\mathcal{N}$ and $\mathcal{B}$ with switching distance $\mathrm{d_{sw}}(f,f') \le \sigma n$. □

When applying this lemma we further make use of the following lemma which helps us to bound corruption.

Lemma 29 (corruption lemma)**.** *Let $n,\Delta > 0$ be integers and μ and η be positive reals. Let V be a set of size n and $\mathcal{B} \subseteq \binom{V}{\Delta}$ be a family of Δ-sets of size at most μn^Δ. Then at most $(\Delta!/\eta^{\Delta-1})\mu n$ vertices are ηn-corrupted by $\mathcal{B}$.*

Proof. For $i \in [\Delta]$ let $\mathcal{B}_i$ be the family of all those i-sets $B' \in \binom{V}{i}$ that are ηn-corrupted by $\mathcal{B}$. We will prove by induction on i (starting at $i = \Delta$) that

$$|\mathcal{B}_i| \le \frac{\Delta!/i!}{\eta^{\Delta-i}}\mu n^i. \tag{31}$$

For $i = 1$ this establishes the lemma. For $i = \Delta$ the assertion is true by assumption. Now assume that (31) is true for $i > 1$. By definition every $B' \in \mathcal{B}_{i-1}$ is contained in more than ηn sets $B \in \mathcal{B}_i$. On the other hand, clearly every $B \in \mathcal{B}_i$ contains at most i sets from $\mathcal{B}_{i-1}$. Double counting thus gives

$$\eta n\,|\mathcal{B}_{i-1}| \le \left|\left\{(B',B) : B' \in \mathcal{B}_{i-1}, B \in \mathcal{B}_i, B' \subseteq B\right\}\right| \le i\,|\mathcal{B}_i| \overset{(31)}{\le} i\frac{\Delta!/i!}{\eta^{\Delta-i}}\mu n^i,$$

which implies (31) for i replaced by $i-1$. □

11.3. **A matching lemma.** We indicated earlier that we are interested in determining whether a candidate graph has a matching covering one of its partition classes. In order to do so we will make use of the following matching lemma which is an easy consequence of Hall's theorem. This lemma takes two graphs B and B' as input that have small neighbourhood distance. In our application these two graphs will be candidate graphs that correspond to two injective mappings f and f' with small switching distance (such as promised by the hypergraph packing lemma, Lemma 28). Recall that Lemma 26 guarantees that mappings with small switching distance correspond to candidate graphs with small neighbourhood distance.

The matching lemma asserts that B' has the desired matching if certain vertex degree and neighbourhood conditions are satisfied. These conditions are somewhat technical. They are tailored exactly to match the conditions that we establish for candidate graphs in the proof of the constrained blow-up lemma (see Claims 33–35).

Lemma 30 (matching lemma)**.** *Let $B = (\widetilde{U}\dot{\cup}U, E)$ and $B' = (\widetilde{U}\dot{\cup}U, E')$ be bipartite graphs with $|U| \geq |\widetilde{U}|$ and $\mathrm{d}_{N(\widetilde{U})}(B, B') \leq s$. If there are positive integers x and n_1, n_2, n_3 such that*

(i) $\deg_{B'}(\widetilde{u}) \geq n_1$ *for all* $\widetilde{u} \in \widetilde{U}$,
(ii) $|N_{B'}(\widetilde{S})| \geq x|\widetilde{S}|$ *for all* $\widetilde{S} \subseteq \widetilde{U}$ *with* $|\widetilde{S}| \leq n_2$
(iii) $e_{B'}(\widetilde{S}, S) \leq \frac{n_1}{n_3}|\widetilde{S}||S|$ *for all* $\widetilde{S} \subseteq \widetilde{U}$, $S \subseteq U$ *with* $xn_2 \leq |S| < |\widetilde{S}| < n_3$,
(iv) $|N_B(S) \cap \widetilde{S}| > s$ *for all* $\widetilde{S} \subseteq \widetilde{U}$, $S \subseteq U$ *with* $|\widetilde{S}| \geq n_3$ *and* $|S| > |U| - |\widetilde{S}|$,

then B' has a matching covering $\widetilde{U}$.

Proof. We will check Hall's condition in B' for all sets $\widetilde{S} \subseteq \widetilde{U}$. We clearly have $|N_{B'}(\widetilde{S})| \geq |\widetilde{S}|$ for $|\widetilde{S}| \leq xn_2$ by (ii) (if $|\widetilde{S}| > n_2$, then consider a subset of $\widetilde{S}$ of size n_2).

Next, consider the case $xn_2 < |\widetilde{S}| < n_3$. Set $S := N_{B'}(\widetilde{S})$ and assume, for a contradiction, that $|S| < |\widetilde{S}|$. Since $|S| < |\widetilde{S}| < n_3$ we have $|S|/n_3 < 1$. Therefore, applying (i), we can conclude that

$$e_{B'}(\widetilde{S}, S) = \sum_{\widetilde{u} \in \widetilde{S}} |N_{B'}(\widetilde{u})| \geq n_1|\widetilde{S}| > \frac{n_1}{n_3}|\widetilde{S}||S|,$$

which is a contradiction to (ii). Thus $|N_{B'}(\widetilde{S})| \geq |\widetilde{S}|$.

Finally, for sets $\widetilde{S}$ of size at least n_3 set $S := U \setminus N_{B'}(\widetilde{S})$ and assume, again for a contradiction, that $|N_{B'}(\widetilde{S})| < |\widetilde{S}|$. This implies $|S| > |U| - |\widetilde{S}|$. Accordingly we can apply (iv) to $\widetilde{S}$ and S and infer that $|N_B(S) \cap \widetilde{S}| > s$. Since $\mathrm{d}_{N(\widetilde{U})}(B, B') \leq s$, at most s vertices from $\widetilde{U}$ have different neighbourhoods in B and B' and so

$$\begin{aligned}\left|N_{B'}(S) \cap \widetilde{S}\right| &= \left|\left\{\widetilde{u} \in \widetilde{S} : N_{B'}(\widetilde{u}) \cap S \neq \emptyset\right\}\right| \\ &\geq \left|\left\{\widetilde{u} \in \widetilde{S} : N_{B}(\widetilde{u}) \cap S \neq \emptyset\right\}\right| - s = \left|N_B(S) \cap \widetilde{S}\right| - s > 0,\end{aligned}$$

which is a contradiction as $S = U \setminus N_{B'}(\widetilde{S})$. □

11.4. **Proof of Lemma 11.** Now we are almost ready to present the proof of the constrained blow-up lemma (Lemma 11). We just need one further technical lemma as preparation. This lemma considers a family of pairwise disjoint Δ-sets $\mathcal{S}$ in a set S and states that a random injective function from S to a set T usually has the following property. The images $f(\mathcal{S})$ of sets in $\mathcal{S}$ "almost" avoid a small family of "bad" sets $\mathcal{T}$ in T.

Lemma 31. *For all positive integers Δ and positive reals β and μ_S there is $\mu_T > 0$ such that the following holds. Let S and T be disjoint sets, $\mathcal{S} \subseteq \binom{S}{\Delta}$ be a family of pairwise disjoint Δ-sets in S with $|\mathcal{S}| \leq \frac{1}{\Delta}(1-\mu_S)|T|$, and $\mathcal{T} \subseteq \binom{T}{\Delta}$ be a family of Δ-sets in T with $|\mathcal{T}| \leq \mu_T|T|^\Delta$.*

Then a random injective function $f : S \to T$ satisfies $|f(\mathcal{S}) \setminus \mathcal{T}| > (1-\beta)|\mathcal{S}|$ with probability at least $1 - \beta^{|\mathcal{S}|}$.

Proof. Given Δ, β, and μ_S choose

$$\mu_T := \sqrt[\beta]{\beta}\left(\frac{e}{\beta}\left(\frac{\Delta}{\mu_S}\right)^\Delta\right)^{-1}. \tag{32}$$

Let S, T, $\mathcal{S}$, and $\mathcal{T}$ be as required and let f be a random injective function from S to T. We consider f as a consecutive random selection (without replacement) of images for the elements of S where the images of the elements of the (disjoint) sets in $\mathcal{S}$ are chosen first. Let S_i be the i-th such set in $\mathcal{S}$. Then the probability that f maps S_i to a set in $\mathcal{T}$, which we denote by p_i, is at most

$$p_i \leq \frac{|\mathcal{T}|}{\binom{|T|-(i-1)\Delta}{\Delta}} \leq \frac{\mu_T|T|^\Delta}{\binom{\mu_S|T|}{\Delta}} \leq \mu_T \frac{|T|^\Delta}{\left(\frac{\mu_S|T|}{\Delta}\right)^\Delta} = \mu_T \left(\frac{\Delta}{\mu_S}\right)^\Delta =: p\,,$$

where the second inequality follows from $(i-1)\Delta \leq |\bigcup \mathcal{S}| \leq (1-\mu_S)|T|$. Let Z be a random variable with distribution $\mathrm{Bi}(|\mathcal{S}|, p)$. It follows that $\mathbb{P}[|f(\mathcal{S}) \cap \mathcal{T}| \geq z] \leq \mathbb{P}[Z \geq z]$. Since

$$\mathbb{P}[Z \geq z] \leq \binom{|\mathcal{S}|}{z} p^z < \left(\frac{e|\mathcal{S}|p}{z}\right)^z ,$$

we infer that

$$\mathbb{P}\Big[|f(\mathcal{S}) \cap \mathcal{T}| \geq \beta|\mathcal{S}|\Big] < \left(\frac{ep}{\beta}\right)^{\beta|\mathcal{S}|} = \left(\frac{e\mu_T}{\beta}\left(\frac{\Delta}{\mu_S}\right)^\Delta\right)^{\beta|\mathcal{S}|} \overset{(32)}{=} \beta^{|\mathcal{S}|},$$

which proves the lemma since $|f(\mathcal{S}) \cap \mathcal{T}| \geq \beta|\mathcal{S}|$ holds iff $|f(\mathcal{S}) \setminus \mathcal{T}| \leq (1-\beta)|\mathcal{S}|$. □

Now we can finally give the proof of Lemma 11.

Proof of Lemma 11. We first define a sequence of constants. Given Δ, d, and η fix $\Delta' := \Delta^2 + 1$. Choose β and σ such that

$$\beta^{\frac{1}{7}(\frac{d}{2})^\Delta} \leq \frac{1}{5} \qquad \text{and} \qquad \frac{(1-\beta)d^\Delta}{100^\Delta} \geq 2\sigma \tag{33}$$

Apply the hypergraph packing lemma, Lemma 28, with input Δ, $\ell = 2\Delta + 1$, and σ to obtain constants η_{28}, and n_{28}. Next, choose η'_{28}, μ_{BL}, and μ_S such that

$$\frac{\eta'_{28}}{1-\eta} \leq \eta_{28}\,, \qquad \frac{\Delta! \cdot 2\mu_{\mathrm{BL}}}{(\eta'_{28})^{\Delta-1}} \leq \eta\,, \qquad \frac{1}{\Delta'} \leq \frac{1}{\Delta}(1-\mu_S)\,. \tag{34}$$

Lemma 31 with input Δ, β, μ_S provides us with a constant μ_T. We apply Lemma 17 two times, once with input $\Delta = \ell$, d, $\varepsilon' := \frac{1}{2}d$, and $\mu = \mu_{\mathrm{BL}}/\Delta'$ and once with input $\Delta = \ell$, d, $\varepsilon' := \frac{1}{2}d$, and $\mu = \mu_T$ and get constants ε_{17} and $\widetilde{\varepsilon}_{17}$, respectively. Now we can fix the promised constant ε such that

$$\varepsilon \leq \min\left\{\frac{\varepsilon_{17}}{\Delta'}\,,\ \frac{d}{2\Delta}\right\}, \qquad \text{and} \qquad \frac{\varepsilon\Delta'}{\eta(1-\eta)} < \min\{d, \widetilde{\varepsilon}_{17}\}. \tag{35}$$

As last input let r_1 be given and set

$$\xi_{17} := \eta(1-\eta)/(r_1\Delta'). \tag{36}$$

Next let c_{17} be the maximum of the two constants obtained from the two applications of Lemma 17, that we started above, with the additional parameter ξ_{17}. Further, let ν and c_{18} be the constants from Lemma 18 for input Δ, d, and ε, and let c_{14} be the constant from Lemma 14 for input Δ and ν. Finally, we choose $c = \max\{c_{17}, c_{18}, c_{14}\}$. With this we defined all necessary constants.

Now assume we are given any $1 \leq r \leq r_1$, and a random graph $\Gamma = \mathcal{G}_{n,p}$ with $p \geq c(\log n/n)^{1/\Delta}$, where, without loss of generality, n is such that

$$(1-\eta')\tfrac{n}{r} \geq n_{28}. \tag{37}$$

Then, with high probability, the graph Γ satisfies the assertion of the different lemmas concerning random graphs, that we started to apply in the definition of the constants. More precisely, by the choice of the constants above,

(P1) Γ satisfies the assertion of Lemma 14 for parameters Δ and ν, i.e., for any set X and any family $\mathcal{F}$ with the conditions required in this lemma, the conclusion of the lemma holds.
(P2) Similarly Γ satisfies the assertion of Lemma 17 for parameters $\Delta = \ell$, d, $\varepsilon' = \frac{1}{2}d$, $\mu = \mu_{\mathrm{BL}}/\Delta'$, ε_{17}, and ξ_{17}. The same holds for parameters $\Delta = \ell$, d, $\varepsilon' = \frac{1}{2}d$, $\mu = \mu_T$, $\widetilde{\varepsilon}_{17}$, and ξ_{17}.
(P3) Γ satisfies the assertion of Lemma 18 for parameters Δ, d, ε, and ν.

In the following we will assume that Γ has these properties and show that it then also satisfies the conclusion of the constrained blow-up lemma, Lemma 11.

Let $G \subseteq \Gamma$ and H be two bipartite graphs on vertex sets $U\dot\cup V$ and $\widetilde{U}\dot\cup\widetilde{V}$, respectively, that fulfil the requirements of Lemma 11. Moreover, let $\mathcal{H} \subseteq \binom{\widetilde{V}}{\Delta}$ be the family of special Δ-sets, and $\mathcal{B} \subseteq \binom{V}{\Delta}$ be the family of forbidden Δ-sets. It is not difficult to see that, by possibly adding some edges to H, we can assume that the following holds.

(Ũ) All vertices in $\widetilde{U}$ have degree *exactly* Δ.
(Ṽ) All vertices in $\widetilde{V}$ have degree maximal $\Delta + 1$.

Our next step will be to split the partition class U of G and the corresponding partition class $\widetilde{U}$ of H into Δ' parts of equal size. From the partition of H we require that no two vertices in one part have a common neighbour. This will guarantee that the neighbourhoods of two different vertices from one part form disjoint vertex sets (which we need because we would like to apply Lemma 18 later, in the proof of Claim 33, and Lemma 18 asserts certain properties for families of disjoint vertex sets).

Let us now explain precisely how we split U and $\widetilde{U}$. We assume for simplicity that $|\widetilde{U}|$ and $|U|$ are divisible by Δ' and partition the sets U arbitrarily into Δ' parts $U = U_1\dot\cup \dots U_{\Delta'}$ of equal size, i.e., sets of size at least $n/(r\Delta')$. Similarly let $\widetilde{U} = \widetilde{U}_1\dot\cup \dots \dot\cup\widetilde{U}_{\Delta'}$ be a partition of $\widetilde{U}$ into sets of equal size such that each $\widetilde{U}_j$ is 2-independent in H. Such a partition exists by the Theorem of Hajnal and Szemerédi (Theorem 21) applied to $H^2[\widetilde{U}]$ because the maximum degree of H^2 is less than $\Delta' = \Delta^2 + 1$.

In Claim 32 below we will assert that there is an embedding f' of $\widetilde{V}$ into V that can be extended to *each* of the $\widetilde{U}_j$ separately such that we obtain an embedding of H into G. To this end we will consider the candidate graphs $B_{f'}(H_j, G_j)$ defined by f' (see Definition 23) and show, that there is an f' such that each $B_{f'}(H_j, G_j)$ has a matching covering $\widetilde{U}_j$. This, as discussed earlier, will ensure the existence of the desired embedding. For preparing this argument, we first need to exclude some vertices of V which are not suitable for such an embedding. For identifying these vertices, we define the following family of Δ-sets which contains $\mathcal{B}$ and all sets in V that have a small common neighbourhood in some $\widetilde{U}_j$.

Define $\mathcal{B}' := \mathcal{B} \cup \bigcup_{j\in\Delta'} \mathcal{B}_j$ where

$$\mathcal{B}_j := \left\{ B \in \binom{V}{\Delta} \colon \; \left|N_G^{\cap}(B) \cap U_j\right| < (\tfrac{1}{2}d)^{\Delta} p^{\Delta} |U_j| \right\} \overset{(3)}{=} \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V, U_j). \tag{38}$$

We claim that we obtain a set $\mathcal{B}'$ that is not much larger than $\mathcal{B}$. Indeed, by Proposition 6 the pair

$$(V, U_j) \text{ is } (\varepsilon\Delta', d, p)\text{-dense for all } j \in [\Delta'], \tag{39}$$

and $\varepsilon\Delta' \le \varepsilon_{17}$ by (35). Moreover we have $|U_j| \ge n/(r\Delta') \ge n/(r_1\Delta') \ge \xi_{17} n$ by (36). We can thus use the fact that our random graph Γ satisfies property (P2) (with $\mu = \mu_{\mathrm{BL}}/\Delta'$) on the bipartite subgraph $G[V\dot\cup U_j]$ and conclude that $|\mathcal{B}_j| \le \mu_{\mathrm{BL}}|V|^{\Delta}/\Delta'$. Since $|\mathcal{B}| \le \mu_{\mathrm{BL}}|V|^{\Delta}$ by assumption we infer

$$|\mathcal{B}'| \le \mu_{\mathrm{BL}}|V|^{\Delta} + \Delta' \cdot \mu_{\mathrm{BL}}|V|^{\Delta}/\Delta' = 2\mu_{\mathrm{BL}}|V|^{\Delta}.$$

Set

$$V' := V \setminus V'' \quad \text{with} \quad V'' := \Big\{ v \in V \colon \; v \text{ is } \eta'_{28}|V|\text{-corrupted by } \mathcal{B}' \Big\} \tag{40}$$

and delete all sets from $\mathcal{B}'$ that contain vertices from V''. This determines the set V'' of vertices that we exclude from V for the embedding. We will next show that we did not exclude too many vertices in this process. For this we use the corruption lemma, Lemma 29. Indeed, Lemma 29 applied with n replaced by $|V|$, with Δ, $\mu = 2\mu_{\mathrm{BL}}$, and η'_{28} to V and $\mathcal{B}'$ implies that

$$|V''| \le \frac{\Delta!}{(\eta'_{28})^{\Delta-1}} 2\mu_{\mathrm{BL}}|V| \overset{(34)}{\le} \eta|V| \quad \text{and thus} \quad n' := |V'| \ge (1-\eta)|V|. \tag{41}$$

Let

$$H_j := H\big[\widetilde{U}_j\dot\cup\widetilde{V}\big] \qquad \text{and} \qquad G_j := G\big[U_j\dot\cup V'\big].$$

Now we are ready to state the claim announced above, which asserts that there is an embedding f' of the vertices in $\widetilde{V}$ to the vertices in V' such that the corresponding candidate graphs $B_{f'}(H_j, G_j)$ have matchings covering $\widetilde{U}_j$. As we will shall show, this claim implies the assertion of the constrained blow-up lemma. Its proof, which we will provide thereafter, requires the matching lemma (Lemma 30), and the hypergraph packing lemma (Lemma 28).

Claim 32. *There is an injection $f' : \widetilde{V} \to V'$ with $f'(T) \notin \mathcal{B}$ for all $T \in \mathcal{H}$ such that for* all *$j \in [\Delta']$ the candidate graph $B_{f'}(H_j, G_j)$ has a matching covering $\widetilde{U}_j$.*

Let us show that proving this claim suffices to establish the constrained blow-up lemma. Indeed, let $f' : \widetilde{V} \to V'$ be such an injection and denote by $M_j : \widetilde{U}_j \to U_j$ the corresponding matching in $B_{f'}(H_j, G_j)$ for $j \in [\Delta]$. We claim that the function $g \colon \widetilde{U} \dot\cup \widetilde{V} \to U \dot\cup V$, defined by

$$g(\widetilde{w}) = \begin{cases} M_j(\widetilde{w}) & \widetilde{w} \in \widetilde{U}_j, \\ f'(\widetilde{w}) & \widetilde{w} \in \widetilde{V}, \end{cases}$$

is an embedding of H into G. To see this, notice first that g is injective since f' is an injection and all M_j are matchings. Furthermore, consider an edge $\widetilde{u}\widetilde{v}$ of H with $\widetilde{u} \in \widetilde{U}_j$ for some $j \in [\Delta']$ and $\widetilde{v} \in \widetilde{V}$ and let

$$u := g(\widetilde{u}) = M_j(\widetilde{u}) \qquad \text{and} \qquad v := g(\widetilde{v}) = f'(\widetilde{v}).$$

It follows from the definition of M_j that $\widetilde{u}u$ is an edge of the candidate graph $B_{f'}(H_j, G_j)$. Hence, by the definition of $B_{f'}(H_j, G_j)$, u is an f'-candidate for $\widetilde{u}$, i.e.,

$$f'\left(N_{H_j}(\widetilde{u})\right) \subseteq N_{G_j}(u).$$

Since $v = f'(\widetilde{v}) \in f'\left(N_{H_j}(\widetilde{u})\right)$ this implies that uv is an edge of G. Because f' also satisfies $f'(T) \notin \mathcal{B}$ for all $T \in \mathcal{H}$ the embedding g also meets the remaining requirement of the constrained blow-up lemma that no special Δ-set is mapped to a forbidden Δ-set. □

For completing the proof of Lemma 11, we still need to prove Claim 32 which we shall be occupied with for the remainder of this section. We will assume throughout that we have the same setup as in the preceding proof. In particular all constants, sets, and graphs are defined as there.

For proving Claim 32 we will use the matching lemma (Lemma 30) on candidate graphs $B = B_f(H_j, G_j)$ and $B' = B_{f'}(H_j, G_j)$ for injections $f, f' \colon \widetilde{V} \to V'$. As we will see, the following three claims imply that there are suitable f and f' such that the conditions of this lemma are satisfied. More precisely, Claim 33 will take care of conditions (i) and (ii) in this lemma, Claim 34 of condition (iii), and Claim 35 of condition (iv). Before proving these claims we will show that they imply Claim 32.

The first claim states that *many* injective mappings $f \colon \widetilde{V} \to V'$ can be turned into injective mappings f' (with the help of a few switchings) such that the candidate graphs $B_{f'}(H_j, G_j)$ for f' satisfy certain degree and expansion properties.

Claim 33. *For* at least half *of all injections $f : \widetilde{V} \to V'$ there is an injection $f' : \widetilde{V} \to V'$ with $\mathrm{d_{sw}}(f, f') \leq \sigma n / r$ such that the following is satisfied for all $j \in [\Delta']$. For all $\widetilde{u} \in \widetilde{U}_j$ and all $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| \leq p^{-\Delta}$ we have*

$$\deg_{B_{f'}(H_j,G_j)}(\widetilde{u}) \geq (\tfrac{d}{2})^{\Delta} p^{\Delta} |U_j| \qquad \textit{and} \qquad |N_{B_{f'}(H_j,G_j)}(\widetilde{S})| \geq \nu n p^{\Delta} |\widetilde{S}|. \tag{42}$$

Further, no special Δ-set from $\mathcal{H}$ is mapped to a forbidden Δ-set from $\mathcal{B}$ by f'.

The second claim asserts that *all* injective mappings f' are such that the candidate graphs $B_{f'}(H_j, G_j)$ do not contain sets of certain sizes with too many edges between them.

Claim 34. All *injections $f' : \widetilde{V} \to V'$ satisfy the following for all $j \in [\Delta']$ and all $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$. If $\nu n \leq |S| < |\widetilde{S}| < \frac{1}{7}(\frac{d}{2})^{\Delta} |U_j|$, then*

$$e_{B_{f'}(H_j,G_j)}(\widetilde{S}, S) \leq 7p^{\Delta} |\widetilde{S}||S|.$$

The last of the three claims states that for *random* injective mappings f the graphs $B_{f'}(H_j, G_j)$ have edges between any pair of large enough sets $S \subseteq U_j$ and $\widetilde{S} \subseteq \widetilde{U}_j$.

Claim 35. *A* random *injection* $f : \widetilde{V} \to V'$ *a.a.s. satisfies the following. For all* $j \in [\Delta']$ *and all* $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ *with* $|\widetilde{S}| \geq \frac{1}{7}(\frac{d}{2})^\Delta |U_j|$ *and* $|S| > |U_j| - |\widetilde{S}|$ *we have*

$$\left| N_{B_f(H_j,G_j)}(S) \cap \widetilde{S} \right| > 2\sigma n/r.$$

Proof of Claim 32. Our aim is to apply the matching lemma (Lemma 30) to the candidate graphs $B_f(H_j, G_j)$ and $B_{f'}(H_j, G_j)$ for all $j \in [\Delta']$ with carefully chosen injections f and f'.

Let $f : \widetilde{V} \to V'$ be an injection satisfying the assertions of Claim 33 and Claim 35 and let f' be the injection promised by Claim 33 for this f. Such an f exists as at least half of all injections from $\widetilde{V}$ to V' satisfy the assertion of Claim 33 and almost all of those satisfy the assertion of Claim 35. We will now show that for all $j \in [\Delta']$ the conditions of Lemma 30 are satisfied for input

$$B = B_f(H_j, G_j), \qquad B' = B_{f'}(H_j, G_j), \qquad s = 2\sigma n/r\,,$$
$$x = \nu n p^\Delta, \qquad n_1 = (\tfrac{d}{2})^\Delta p^\Delta |U_j|, \qquad n_2 = p^{-\Delta}, \qquad n_3 = \tfrac{1}{7}(\tfrac{d}{2})^\Delta |U_j|,$$

Claim 33 asserts that $\mathrm{d}_{\mathrm{sw}}(f, f') \leq \sigma n/r$. Since $\widetilde{U}_j$ is 2-independent in H we have $\deg_{H_j}(\widetilde{v}) \leq 1$ for all $\widetilde{v} \in \widetilde{V}$. Thus the switching lemma, Lemma 26, applied to H_j and G_j and with s replaced by $\sigma n/r$ implies

$$\mathrm{d}_{N(\widetilde{U}_j)}(B, B') = \mathrm{d}_{N(\widetilde{U}_j)}\Big(B_f(H_j, G_j), B_{f'}(H_j, G_j)\Big) \leq 2\sigma n/r = s.$$

Moreover, by Claim 33, for all $\widetilde{u} \in \widetilde{U}_j$ we have

$$\deg_{B'}(\widetilde{u}) = \deg_{B_{f'}(H_j,G_j)}(\widetilde{u}) \geq (\tfrac{d}{2})^\Delta p^\Delta |U_j| = n_1$$

and thus condition (i) of Lemma 30 holds true. Further, we conclude from Claim 33 that $|N_{B'}(\widetilde{S})| \geq x|\widetilde{S}|$ for all $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| < p^{-\Delta} = n_2$. This gives condition (ii) of Lemma 30. In addition, Claim 34 states that for all $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with $xn_2 = \nu n \leq |S| < |\widetilde{S}| < \frac{1}{7}(\frac{d}{2})^\Delta |U_j| = n_3$ we have

$$e_{B'}(\widetilde{S}, S) = e_{B_{f'}(H_j,G_j)}(\widetilde{S}, S) \leq 7p^\Delta |\widetilde{S}||S| = \frac{n_1}{n_3}|\widetilde{S}||S|$$

and accordingly also condition (iii) of Lemma 30 is satisfied. To see (iv), observe that the choice of f and Claim 35 assert

$$\left| N_B(S) \cap \widetilde{S} \right| = \left| N_{B_f(H_j,G_j)}(S) \cap \widetilde{S} \right| > 2\sigma n/r = s$$

for all $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| \geq \frac{1}{7}(\frac{d}{2})^\Delta |U_j| = n_3$ and $|S| > |U| - |\widetilde{S}|$. Therefore, all conditions of Lemma 30 are satisfied and we infer that for *all* $j \in [\Delta']$ the candidate graph $B_{f'}(H_j, G_j)$ with f' as chosen above has a matching covering $\widetilde{U}$. Moreover, by Claim 33, f' maps no special Δ-set to a forbidden Δ-set. This establishes Claim 32. □

It remains to show Claims 33–35. We start with Claim 33. For the proof of this claim we apply the hypergraph packing lemma (Lemma 28).

Proof of Claim 33. Notice that ($\tilde{\mathrm{U}}$) on page 29 implies that $N_H(\widetilde{u})$ contains exactly Δ elements for each $\widetilde{u} \in \widetilde{U}$. Hence we may define the following family of Δ-sets. Let

$$\mathcal{N} := \Big\{ N_H(\widetilde{u}) \colon\; \widetilde{u} \in \widetilde{U} \Big\} \cup \mathcal{H} \subseteq \binom{\widetilde{V}}{\Delta}.$$

We want to apply the hypergraph packing lemma (Lemma 28) with Δ, with ℓ replaced by $2\Delta + 1$, and with σ to the hypergraphs with vertex sets $\widetilde{V}$ and V' and edge sets $\mathcal{N}$ and $\mathcal{B}'$, respectively (see (38) on page 29). We will first check that the necessary conditions are satisfied.

Observe that

$$|V'| \overset{(41)}{\geq} (1-\eta')|V| \geq (1-\eta')n/r \overset{(37)}{\geq} n_{28}, \qquad \text{and} \qquad |\widetilde{V}| \leq |V'|\,.$$

Furthermore, a vertex $\widetilde{v} \in \widetilde{V}$ is neither contained in more than Δ sets from $\mathcal{H}$ nor is $\widetilde{v}$ in $N_H(\widetilde{u})$ for more than $\Delta+1$ vertices $\widetilde{u} \in \widetilde{U}$ (by ($\tilde{\mathrm{V}}$) on page 29). Therefore the condition Lemma 28 imposes on $\mathcal{N}$ is satisfied with ℓ replaced by $2\Delta+1$. Moreover, according to (40) no vertex in V' is $\eta'_{28}|V|$-corrupted by $\mathcal{B}'$. Since

$$\eta'_{28}|V| \overset{(41)}{\leq} \eta'_{28}(1-\eta)^{-1}n' \overset{(34)}{\leq} \eta_{28}n',$$

this (together with the observation in Definition 27) implies that no vertex in V' is $\eta_{28}n'$-corrupted by $\mathcal{B}'$ and therefore all prerequisites of Lemma 28 are satisfied.

It follows that the conclusion of Lemma 28 holds for at least half of all injective functions $f\colon \widetilde{V} \to V'$, namely that there are packings f' of (the hypergraphs with edges) $\mathcal{N}$ and $\mathcal{B}$ with switching distance $\mathrm{d_{sw}}(f,f') \leq \sigma|\widetilde{V}| \leq \sigma n/r$. Clearly, such a packing f' does not send any special Δ-set from $\mathcal{H}$ to any forbidden Δ-set from $\mathcal{B}$. Our next goal is to show that f' satisfies the first part of (42) for all $j \in [\Delta']$ and $\widetilde{u} \in \widetilde{U}_j$. For this purpose, fix j and $\widetilde{u}$. The definition of the candidate graph $B_{f'}(H_j,G_j)$, Definition 23, implies

$$\begin{aligned}\deg_{B_{f'}(H_j,G_j)}(\widetilde{u}) &= \left|\left\{u \in U_j\colon\ f'\left(N_{H_j}(\widetilde{u})\right) \subseteq N_{G_j}(u)\right\}\right| \\ &= \left|\left\{u \in U_j\colon\ u \in N^{\cap}_{G_j}\Big(f'\left(N_{H_j}(\widetilde{u})\right)\Big)\right\}\right| \\ &= \left|N^{\cap}_{G_j}\Big(f'\left(N_{H_j}(\widetilde{u})\right)\Big)\right| \geq (\tfrac{1}{2}d)^{\Delta}p^{\Delta}|U_j|\,.\end{aligned}$$

where the first inequality follows from the fact that $N_{H_j}(\widetilde{u}) \in \mathcal{N}$ and thus, as f' is a packing of $\mathcal{N}$ and $\mathcal{B}'$, we have $f'(N_{H_j}(\widetilde{u})) \notin \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V,U_j) \subseteq \mathcal{B}'$ (see the definition of $\mathcal{B}'$ in (38)). This in turn means that all Δ-sets $f'(N_{H_j}(\widetilde{u}))$ with $\widetilde{u} \in \widetilde{U}_j$ are p-good (see Definition 16) in (V,U_j), because (V,U_j) has p-density at least $d-\varepsilon\Delta' \geq \frac{d}{2}$ by (39) and (35). With this information at hand we can proceed to prove the second part of (42). Let $\widetilde{S} \subseteq \widetilde{U}_j$ with $\widetilde{S} < 1/p^{\Delta}$ and consider the family $\mathcal{F} \subseteq \binom{V}{\Delta}$ with

$$\mathcal{F} := \{f'(N_H(\widetilde{u}))\colon\ \widetilde{u} \in \widetilde{S}\}.$$

Because U_j is 2-independent in H the sets $N_H(\widetilde{u})$ with $\widetilde{u} \in \widetilde{S}$ form a family of disjoint Δ-sets in $\widetilde{V}$. It follows that also the sets $f'(N_H(\widetilde{u}))$ with $\widetilde{u} \in \widetilde{S}$ form a family of disjoint Δ-sets in V. By (P3) on page 28 the conclusion of Lemma 18 holds for Γ. We conclude that the pair (V,U_j) is $(1/p^{\Delta}, \nu np^{\Delta})$-expanding. Since $|\mathcal{F}| = |\widetilde{S}| < 1/p^{\Delta}$ by assumption and all members of $\mathcal{F}$ are p-good in (V,U_j) this implies that $|N^{\cap}_{U_j}(\mathcal{F})| \geq \nu np^{\Delta}|\mathcal{F}|$. On the other hand $N^{\cap}_{U_j}(\mathcal{F}) = N_{B_{f'}(H_j,G_j)}(\widetilde{S})$ by the definition of $B_{f'}(H_j,G_j)$ and $\mathcal{F}$ and thus we get the second part of (42). □

Recall that property (P1) states that Γ satisfies the conclusion of Lemma 14 for certain parameters. We will use this fact to prove Claim 34.

Proof of Claim 34. Fix $f'\colon \widetilde{V} \to V'$, $j \in [\Delta']$, $S \subseteq U_j$, and $\widetilde{S} \subseteq \widetilde{U}_j$ with $\nu n \leq |S| < |\widetilde{S}| < \frac{1}{7}(\frac{d}{2})^{\Delta}|U_j|$. For the candidate graphs $B_{f'}(H_j,G_j)$ of f' we have

$$\begin{aligned}e_{B_{f'}(H_j,G_j)}(\widetilde{S},S) &= \left|\left\{\widetilde{u}u \in \widetilde{S} \times S\colon\ f'\big(N_H(\widetilde{u})\big) \subseteq N_G(u)\right\}\right| \\ &\overset{(1)}{=} \#\,\mathrm{stars}^G\left(S, \left\{f'\big(N_H(\widetilde{u})\big)\colon\ \widetilde{u} \in \widetilde{S}\right\}\right) \\ &\leq \#\,\mathrm{stars}^{\Gamma}\left(S, \left\{f'\big(N_H(\widetilde{u})\big)\colon\ \widetilde{u} \in \widetilde{S}\right\}\right) = \#\,\mathrm{stars}^{\Gamma}(S,\mathcal{F}'),\end{aligned}$$

where $\mathcal{F}' := \{f'(N_H(\widetilde{u}))\colon \widetilde{u} \in \widetilde{S}\}$. As before the sets $f'(N_H(\widetilde{u}))$ with $\widetilde{u} \in \widetilde{S}$ form a family of $|\widetilde{S}|$ disjoint Δ-sets in V'. Since $\nu n \le |S| < |\widetilde{S}| = |\mathcal{F}'| \le n$ we can appeal to property (P1) (and hence Lemma 14) with the set $X := S$ and the family $\mathcal{F}'$ and infer that

$$e_{B_{f'}(H_j,G_j)}(\widetilde{S}, S) \le \#\,\mathrm{stars}^{\Gamma}(S, \mathcal{F}') \le 7p^{\Delta}|\mathcal{F}'||S| = 7p^{\Delta}|\widetilde{S}||S|$$

as required. □

Finally, we prove Claim 35. For this proof we will use the fact that Δ-sets in p-dense graphs have big common neighbourhoods (the conclusion of Lemma 17 holds by property (P2)) together with Lemma 31.

Proof of Claim 35. Let f be an injective function from $\widetilde{V}$ to V'. First, consider a fixed $j \in [\Delta']$ and fixed sets $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| \ge \frac{1}{7}(\frac{d}{2})^{\Delta}|U_j|$ and $|S| > |U_j| - |\widetilde{S}|$. Define

$$\mathcal{S} := \{N_{H_j}(\widetilde{u})\colon \widetilde{u} \in \widetilde{S}\} \qquad \text{and} \qquad \mathcal{T} := \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V', S).$$

and observe that

$$\begin{aligned}\left|N_{B_f(H_j,G_j)}(S) \cap \widetilde{S}\right| &= \left|\left\{\widetilde{u} \in \widetilde{S}\colon \exists u \in S \text{ with } f\left(N_{H_j}(\widetilde{u})\right) \subseteq N_{G_j}(u)\right\}\right| \\ &= \left|\left\{\widetilde{u} \in \widetilde{S}\colon N^{\cap}_{G_j}\Big(f\left(N_{H_j}(\widetilde{u})\right)\Big) \cap S \ne \emptyset\right\}\right| \\ &\ge \left|\left\{\widetilde{u} \in \widetilde{S}\colon f\left(N_{H_j}(\widetilde{u})\right) \notin \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V', S)\right\}\right| = |f(\mathcal{S}) \setminus \mathcal{T}|\end{aligned}$$

since all Δ-sets $B \notin \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V', S)$ satisfy $|N^{\cap}_{G_j}(B) \cap S| \ge (\frac{d}{2})^{\Delta}p^{\Delta}|S| > 0$. Thus, for proving the claim, it suffices to show that a random injection $f : \widetilde{V} \to V'$ violates $|f(\mathcal{S}) \setminus \mathcal{T}| > 2\sigma n/r$ with probability at most $5^{-|U_j|}$ because this implies that f violates the conclusion of Claim 35 for *some* $j \in [\Delta']$, and *some* $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with probability at most $\mathcal{O}(2^{|U_j|}2^{|\widetilde{U}_j|} \cdot 5^{-|U_j|}) = o(1)$. For this purpose, we will use the fact that the pair (V', S) is p-dense. Indeed, observe that

$$|S| > |U_j| - |\widetilde{S}| > |U_j| - |\widetilde{U}_j| = \frac{|U| - |\widetilde{U}|}{\Delta'} \ge \frac{\eta|U|}{\Delta'}$$

by the assumptions of the constrained blow-up lemma, Lemma 11. As $|V'| \ge (1-\eta)|V|$ by (41) we can apply Proposition 6 twice to infer from the (ε, d, p)-density of (V, U) that (V', S) is $(\widetilde{\varepsilon}, d, p)$-dense with $\widetilde{\varepsilon} := \varepsilon\Delta'/(\eta(1-\eta))$. Furthermore $\widetilde{\varepsilon} \le \widetilde{\varepsilon}_{17}$ by (35) and

$$|V'| \overset{(41)}{\ge} (1-\eta)\frac{n}{r} \overset{(36)}{\ge} \xi_{17}n, \qquad \text{and} \qquad |S| > \frac{\eta|U|}{\Delta'} \ge \frac{\eta n}{r\Delta'} \overset{(36)}{\ge} \xi_{17}n.$$

Hence we conclude from (P2) on page 28 (with $\mu = \mu_T$) that $|\mathcal{T}| = |\,\mathrm{bad}^{G,\Delta}_{d/2,d,p}(V', S)| \le \mu_T|V'|^{\Delta}$. In addition

$$\tfrac{1}{7}\left(\tfrac{d}{2}\right)^{\Delta}|U_j| \le |\widetilde{S}| = |\mathcal{S}| \le |\widetilde{U}_j| \le (1-\eta)\frac{n}{\Delta'} \overset{(41)}{\le} \frac{|V'|}{\Delta'} \overset{(34)}{\le} \frac{1}{\Delta}(1-\mu_S)|V'|. \tag{43}$$

Thus, we can apply Lemma 31 with Δ, β, and μ_S to $S = \widetilde{V}$, $T = V'$, and to $\mathcal{S}$ and $\mathcal{T}$ and conclude that f violates

$$|f(\mathcal{S}) \setminus \mathcal{T}| > (1-\beta)|\mathcal{S}| \overset{(43)}{\ge} (1-\beta)\tfrac{1}{7}\left(\tfrac{d}{2}\right)^{\Delta}|U_j| \ge \frac{(1-\beta)d^{\Delta}n}{7 \cdot 2^{\Delta}r\Delta'} \ge \frac{(1-\beta)d^{\Delta}n}{100^{\Delta}r} \overset{(33)}{\ge} 2\sigma\frac{n}{r}$$

with probability at most

$$\beta^{|\mathcal{S}|} \le \beta^{\frac{1}{7}(\frac{d}{2})^{\Delta}|U_j|} \le 5^{-|U_j|}$$

where the first inequality follows from (43) and the second from (33). □

Acknowledgement. We thank two anonymous referees for their comments.

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Appendix A. The connection lemma

The proof of Lemma 12 which we present in this appendix is inherent in the proof of [26, Lemma 18]. The only difference is that we have a somewhat more special set-up here (given by the pre-defined partitions and candidate sets). This set-up however is chosen exactly in such a way that this proof continues to work if we adapt the parameters involved accordingly.

Proof of Lemma 12. For the proof of Lemma 12 we use an inductive argument and embed a partition class of H into the corresponding partition class of G one at a time. Before describing this strategy we will define two graph properties $\mathrm{D}_p(d_0, \varepsilon', \mu, \varepsilon, \xi)$ and $\mathrm{STAR}_p(k, \xi, \nu)$, which a random graph $\Gamma = \mathcal{G}_{n,p}$ enjoys a.a.s. for suitable sets of parameters. Then we will set up these parameters accordingly and define all other constants involved in the proof.

For a fixed n-vertex graph Γ, fixed positive reals d_0, ε', μ, ε, ξ, and ν, a fixed integer k, and a function $p = p(n)$ we define the following properties of Γ.

$\mathrm{D}_p(d_0, \varepsilon', \mu, \varepsilon, \xi)$: *We say that* Γ *has property* $\mathrm{D}_p(d_0, \varepsilon', \mu, \varepsilon, \xi)$ *if it satisfies the property stated in Lemma 19 with these parameters and with* Δ*, i.e., whenever* $G = (X\dot\cup Y\dot\cup Z, E)$ *is a tripartite subgraph of* Γ *with the required properties, then it satisfies the conclusion of this lemma.*

$\mathrm{STAR}_p(k, \xi, \nu)$: *Similarly* Γ *has property* $\mathrm{STAR}_p(k, \xi, \nu)$ *if* Γ *has the property stated in Lemma 15 with* Δ *replaced by* k*, with parameters* ξ*,* ν*, and for* $p = p(n)$*.*

Now we set up the constants. Let Δ, t and d be given and assume without loss of generality that $d \le \frac{1}{4}$. First we set

$$\mu = \frac{1}{4\Delta^2} \tag{44}$$

and we fix ε_i for $i = t, t-1, \dots, 0$ by setting

$$\begin{aligned} \varepsilon_t &= \frac{d}{12\Delta t}\,, \qquad d_0 := d\,, \quad \text{and} \\ \varepsilon_{i-1} &= \min\left\{\varepsilon(\Delta, d_0, \varepsilon' = \varepsilon_i, \mu),\, \varepsilon_i\right\} \text{ for } i = t, \dots, 1\,, \end{aligned} \tag{45}$$

where $\varepsilon(\Delta - 1, d_0, \varepsilon' = \varepsilon_i, \mu)$ is given by Lemma 19. We choose $\varepsilon := \varepsilon_0$ and $\xi := (d/100)^{\Delta}$ and receive r_1 as input. For each $k \in [\Delta]$ and each $r' \in [r_1]$ Lemma 15 with Δ replaced by k and with ξ replaced by ξ/r' provides positive constants $\nu(k, r')$ and $c(k, r')$. Let ν be the minimum among the $\nu(k, r')$ and let c_{15} be the maximum among the $c(k, r')$ as we let both k and r' vary. Similarly Lemma 19 with input $\Delta - 1$, d_0, $\varepsilon' = \varepsilon_i$, μ, and ξ replaced by ξ/r' provides constants $c'(i, r')$ for $i \in [0, t]$ and $r' \in [r_1]$. We let c_{19} be the maximum among these $c'(i, r')$. Then we fix $c := \max\{c_{15}, c_{19}\}$, and receive $r \in [r_1]$ as input. Finally, we set

$$\xi_{15} := \xi_{19} := \xi/r = (d/100)^{\Delta}(1/r)\,. \tag{46}$$

This finishes the definition of the constants.

Let $p = p(n) \ge c(\log n/n)^{1/\Delta}$ and let Γ be a graph from $\mathcal{G}_{n,p}$. By Lemma 15, Lemma 19, and the choice of constants the graph Γ a.a.s. satisfies properties $\mathrm{D}_p(d, \varepsilon_i, \mu, \varepsilon_{i-1}, \xi_{19})$ for all $i \in [t]$, and properties $\mathrm{STAR}_p(k, \xi_{15}, \nu)$ for all $k \in [\Delta]$. In the remainder of this proof we assume that Γ has these properties and show that then Γ also satisfies the conclusion of Lemma 12.

Let $G \subseteq \Gamma$ and H be arbitrary graphs satisfying the requirements stated in the lemma on vertex sets $W = W_1\dot\cup \dots \dot\cup W_t$ and $\widetilde{W} = \widetilde{W}_1\dot\cup \dots \dot\cup \widetilde{W}_t$, respectively. Let $h\colon \widetilde{W} \to [t]$ be the "partition function" for the vertex partition of H, i.e.,

$$h(\widetilde{w}) = j \quad \text{if and only if} \quad \widetilde{w} \in \widetilde{W}_j.$$

For an integer $i \le h(\widetilde{w})$ we denote by

$$\mathrm{ldeg}^i(\widetilde{w}) := \left|N_H(\widetilde{w}) \cap \{\widetilde{x} \in \widetilde{W}\colon\ h(\widetilde{x}) \le i\}\right|$$

the left degree of $\widetilde{w}$ with respect to $\widetilde{W}_1\dot\cup \dots \dot\cup \widetilde{W}_i$. Clearly $\mathrm{ldeg}^{h(\widetilde{w})}(\widetilde{w}) = \mathrm{ldeg}(\widetilde{w})$. Before we continue, recall that each vertex $\widetilde{w} \in \widetilde{W}_i$ is equipped with a set $X_{\widetilde{w}} \subseteq V(\Gamma) \setminus W$ and that we defined an external degree $\mathrm{edeg}(\widetilde{w}) = |X_{\widetilde{w}}|$ of $\widetilde{w}$ as well as a candidate set $C(\widetilde{w}) = N^{\cap}_{W_i}(X_{\widetilde{w}}) \subseteq W_i$. In the course of our embedding procedure, that we will describe below, we shall shrink this candidate set but keep certain invariants as we explain next.

We proceed inductively and embed the vertex class $\widetilde{W}_i$ into W_i one at a time, for $i = 1, \dots, t$. To this end, we verify the following statement $(\mathcal{S}_i)$ for $i = 0, \dots, t$.

$(\mathcal{S}_i)$ There exists a partial embedding φ_i of $H[\bigcup_{j=1}^{i} \widetilde{W}_j]$ into $G[\bigcup_{j=1}^{i} W_j]$ such that for every $\widetilde{z} \in \bigcup_{j=i+1}^{t} \widetilde{W}_j$ there exists a candidate set $C_i(\widetilde{z}) \subseteq C(\widetilde{z})$ given by

(a) $C_i(\widetilde{z}) = \bigcap\{N_G(\varphi_i(\widetilde{x}))\colon\ \widetilde{x} \in N_H(\widetilde{z}) \text{ and } h(\widetilde{x}) \le i\} \cap C(\widetilde{z})$,

and satisfying

(b) $|C_i(\widetilde{z})| \ge (dp/2)^{\mathrm{ldeg}^i(\widetilde{z})}|C(\widetilde{z})|$, and

(c) for every edge $\{\widetilde{z}, \widetilde{z}'\} \in E(H)$ with $h(\widetilde{z}), h(\widetilde{z}') > i$ the pair $\big(C_i(\widetilde{z}), C_i(\widetilde{z}')\big)$ is (ε_i, d, p)-dense in G.

Statement $(\mathcal{S}_i)$ ensures the existence of a partial embedding of the first i classes $\widetilde{W}_1, \dots, \widetilde{W}_i$ of H into G such that for every unembedded vertex $\widetilde{z}$ there exists a candidate set $C_i(\widetilde{z}) \subseteq C(\widetilde{z})$ that is not too small (see part (b)). Moreover, if we embed $\widetilde{z}$ into its candidate set, then its image will be adjacent to all vertices $\varphi_i(\widetilde{x})$ with $\widetilde{x} \in (\widetilde{W}_1 \cup \dots \cup \widetilde{W}_i) \cap N_H(\widetilde{z})$ (see part (a)). The last property, part (c), says that for edges of H such that none of the endvertices are embedded already the respective candidate sets induce (ε, d', p)-dense pairs for some positive d'. This property will be crucial for the inductive proof.

Remark. In what follows we shall use the following convention. Since the embedding of H into G will be divided into t rounds, we shall find it convenient to distinguish among the vertices of H. We shall use $\widetilde{x}$ for vertices that have already been embedded, $\widetilde{y}$ for vertices that will be embedded in the current round, while $\widetilde{z}$ will denote vertices that we shall embed at a later step.

Before we verify $(\mathcal{S}_i)$ for $i = 0, \dots, t$ by induction on i we note that $(\mathcal{S}_t)$ implies that H can be embedded into G in such a way that every vertex $\widetilde{w} \in \widetilde{W}$ is mapped to a vertex in its candidate set $C(\widetilde{w})$. Consequently, verifying $(\mathcal{S}_t)$ concludes the proof of Lemma 12.

Basis of the induction: $i = 0$. We first verify $(\mathcal{S}_0)$. In this case φ_0 is the empty mapping and for every $\widetilde{z} \in \widetilde{W}$ we have, according to (a), $C_0(\widetilde{z}) = C(\widetilde{z})$, as there is no vertex $\widetilde{x} \in N_H(\widetilde{z})$ with $h(\widetilde{x}) \le 0$. Property (b) holds because $C_0(\widetilde{z}) = C(\widetilde{z})$ and $\mathrm{ldeg}^0(\widetilde{z}) = 0$ for every $\widetilde{z} \in \widetilde{W}$. Finally, property (c) follows from the property that $(C(\widetilde{z}), C(\widetilde{z}'))$ is (ε_0, d, p)-dense by (E) of Lemma 12.

Induction step: $i \to i+1$. For the inductive step, we suppose that $i < t$ and assume that statement $(\mathcal{S}_i)$ holds; we have to construct φ_{i+1} with the required properties. Our strategy is as follows. In the first step, we find for every $\widetilde{y} \in \widetilde{W}_{i+1}$ an appropriate subset $C'(\widetilde{y}) \subseteq C_i(\widetilde{y})$ of its candidate set such that if $\varphi_{i+1}(\widetilde{y})$ is chosen from $C'(\widetilde{y})$, then the new candidate set $C_{i+1}(\widetilde{z}) := C_i(\widetilde{z}) \cap N_G(\varphi_{i+1}(\widetilde{y}))$ of every "right-neighbour" $\widetilde{z}$ of $\widetilde{y}$ will not shrink too much and property (c) will continue to hold.

Note, however, that in general $|C'(\widetilde{y})| \le |C_i(\widetilde{y})| = o(n) \ll |\widetilde{W}_{i+1}|$ (if $\mathrm{ldeg}^i(\widetilde{y}) \ge 1$) and, hence, we cannot "blindly" select $\varphi_{i+1}(\widetilde{y})$ from $C'(\widetilde{y})$. Instead, in the second step, we shall verify Hall's condition to find a system of distinct representatives for the indexed set system $\big(C'(\widetilde{y})\colon\ \widetilde{y} \in \widetilde{W}_{i+1}\big)$ and we let $\varphi_{i+1}(\widetilde{y})$ be the representative of $C'(\widetilde{y})$. (A similar idea was used in [6, 27].) We now give the details of those two steps.

First step: For the first step, fix $\widetilde{y} \in \widetilde{W}_{i+1}$ and set

$$N_H^{i+1}(\widetilde{y}) := \{\widetilde{z} \in N_H(\widetilde{y})\colon\ h(\widetilde{z}) > i+1\}\,.$$

A vertex $v \in C_i(\widetilde{y})$ will be "bad" (i.e., we shall not select v for $C'(\widetilde{y})$) if there exists a vertex $\widetilde{z} \in N_H^{i+1}(\widetilde{y})$ for which $N_G(v) \cap C_i(\widetilde{z})$, in view of (b) and (c) of $(\mathcal{S}_{i+1})$, cannot play the rôle of $C_{i+1}(\widetilde{z})$.

We first prepare for (b) of $(\mathcal{S}_{i+1})$. Fix a vertex $\widetilde{z} \in N_H^{i+1}(\widetilde{y})$. Since $(C_i(\widetilde{y}), C_i(\widetilde{z}))$ is an (ε_i, d, p)-dense pair by (c) of $(\mathcal{S}_i)$, Proposition 7 implies that there exist at most $\varepsilon_i|C_i(\widetilde{y})| \le \varepsilon_t|C_i(\widetilde{y})|$ vertices v in $C_i(\widetilde{y})$ such that

$$|N_G(v) \cap C_i(\widetilde{z})| < \big(d - \varepsilon_t\big)p|C_i(\widetilde{z})|\,.$$

Repeating the above for all $\widetilde{z} \in N_H^{i+1}(\widetilde{y})$, we infer from (a) and (b) of $(\mathcal{S}_i)$, that there are at most $\Delta\varepsilon_t|C_i(\widetilde{y})|$ vertices $v \in C_i(\widetilde{y})$ such that the following fails to be true for some $\widetilde{z} \in N_H^{i+1}(\widetilde{y})$:

$$|N_G(v) \cap C_i(\widetilde{z})| \ge \big(d - \varepsilon_t\big)p|C_i(\widetilde{z})| \stackrel{\text{(a),(b)}}{\ge} (d-\varepsilon_t)\,p\left(\frac{dp}{2}\right)^{\mathrm{ldeg}^i(\widetilde{z})}|C(\widetilde{z})| \stackrel{(45)}{\ge} \left(\frac{dp}{2}\right)^{\mathrm{ldeg}^{i+1}(\widetilde{z})}|C(\widetilde{z})|\,. \quad (47)$$

For property (c) of $(\mathcal{S}_{i+1})$, we fix an edge $e=\{\widetilde{z},\widetilde{z}'\}$ with $h(\widetilde{z}),\,h(\widetilde{z}')>i+1$ and with at least one end vertex in $N_H^{i+1}(\widetilde{y})$. There are at most $\Delta(\Delta-1)<\Delta^2$ such edges. Note that if both vertices $\widetilde{z}$ and $\widetilde{z}'$ are neighbours of $\widetilde{y}$, i.e., $\widetilde{z},\,\widetilde{z}'\in N_H^{i+1}(\widetilde{y})$, then

$$\max\left\{\operatorname{ldeg}^i(\widetilde{y})+\operatorname{edeg}(\widetilde{y}),\operatorname{ldeg}^i(\widetilde{z})+\operatorname{edeg}(\widetilde{z}),\operatorname{ldeg}^i(\widetilde{z}')+\operatorname{edeg}(\widetilde{z}')\right\}\le\Delta-2\,,$$

by (C) of Lemma 12 and because all three vertices $\widetilde{y}$, $\widetilde{z}$, and $\widetilde{z}'$ have at least two neighbours in $\widetilde{W}_{i+1}\cup\dots\cup\widetilde{W}_t$. From property (b) of $(\mathcal{S}_i)$, and (A) and (D) of Lemma 12 we infer for all $\widetilde{w}\in\{\widetilde{y},\widetilde{z},\widetilde{z}'\}$ that

$$|C_i(\widetilde{w})|\overset{\text{(b)}}{\ge}\left(\frac{dp}{2}\right)^{\operatorname{ldeg}^i(\widetilde{w})}|C(\widetilde{w})|\overset{\text{(A),(D)}}{\ge}\left(\frac{dp}{2}\right)^{\operatorname{ldeg}^i(\widetilde{w})+\operatorname{edeg}(\widetilde{w})}\frac{n}{r}\overset{(46)}{\ge}\xi_{19}p^{\Delta-2}n.$$

Furthermore, Γ has property $\mathrm{D}_p(d,\varepsilon_{i+1},\mu,\varepsilon_i,\xi_{19})$ by assumption. This implies that there are at most $\mu|C_i(\widetilde{y})|$ vertices $v\in C_i(\widetilde{y})$ such that the pair $(N_G(v)\cap C_i(\widetilde{z}),N_G(v)\cap C_i(\widetilde{z}'))$ fails to be (ε_{i+1},d,p)-dense.

If, on the other hand, say, only $\widetilde{z}\in N_H^{i+1}(\widetilde{y})$ and $\widetilde{z}'\notin N_H^{i+1}(y)$, then

$$\begin{aligned}\max\left\{\operatorname{ldeg}^i(\widetilde{y})+\operatorname{edeg}(\widetilde{y}),\operatorname{ldeg}^i(\widetilde{z}')+\operatorname{edeg}(\widetilde{z}')\right\}&\le\Delta-1\\ \text{and}\qquad \operatorname{ldeg}^i(\widetilde{z})+\operatorname{edeg}(\widetilde{z})&\le\Delta-2.\end{aligned}$$

Consequently, similarly as above,

$$\min\Big\{|C_i(\widetilde{y})|\,,|C_i(\widetilde{z}')|\Big\}\ge\xi_{19}p^{\Delta-1}n\quad\text{and}\quad|C_i(\widetilde{z})|\ge\xi_{19}p^{\Delta-2}n$$

and we can appeal to the fact that Γ has property $\mathrm{D}_p(d,\varepsilon_{i+1},\mu,\varepsilon_i,\xi_{19})$ to infer that there are at most $\mu|C_i(\widetilde{y})|$ vertices $v\in C_i(\widetilde{y})$ such that $(N_G(v)\cap C_i(\widetilde{z}),C_i(\widetilde{z}'))$ fails to be (ε_{i+1},d,p)-dense. For a given $v\in C_i(\widetilde{y})$, let $\hat{C}_i(\widetilde{z})=C_i(\widetilde{z})\cap N_G(v)$ if $\widetilde{z}\in N_H^{i+1}(\widetilde{y})$ and $\hat{C}_i(\widetilde{z})=C_i(\widetilde{z})$ if $\widetilde{z}\notin N_H^{i+1}(\widetilde{y})$, and define $\hat{C}_i(\widetilde{z}')$ analogously.

Summarizing the above we infer that there are at least

$$(1-\Delta\varepsilon_t-\Delta^2\mu)|C_i(\widetilde{y})| \tag{48}$$

vertices $v\in C_i(\widetilde{y})$ such that

(b') $|N_G(v)\cap C_i(\widetilde{z})|\ge(dp/2)^{\operatorname{ldeg}^{i+1}(\widetilde{z})}|C(\widetilde{z})|$ for every $\widetilde{z}\in N_H^{i+1}(\widetilde{y})$ (see (47)) and

(c') $(\hat{C}_i(\widetilde{z}),\hat{C}_i(\widetilde{z}'))$ is (ε_{i+1},d,p)-dense for all edges $\{\widetilde{z},\widetilde{z}'\}$ of H with $h(\widetilde{z}),\,h(\widetilde{z}')>i+1$ and $\{\widetilde{z},\widetilde{z}'\}\cap N_H^{i+1}(\widetilde{y})\neq\emptyset$.

Let $C'(\widetilde{y})$ be the set of those vertices v from $C_i(\widetilde{y})$ satisfying properties (b') and (c') above. Recall that $\operatorname{ldeg}^i(\widetilde{y})+\operatorname{edeg}(\widetilde{y})=\operatorname{ldeg}^i(\widetilde{y}')+\operatorname{edeg}(\widetilde{y}')$ for all $\widetilde{y},\,\widetilde{y}'\in\widetilde{W}_{i+1}$ and set

$$k:=\operatorname{ldeg}^i(\widetilde{y})+\operatorname{edeg}(\widetilde{y})\text{ for some }\widetilde{y}\in\widetilde{W}_{i+1}. \tag{49}$$

Since $\widetilde{y}\in\widetilde{W}_{i+1}$ was arbitrary, we infer from property (b) of $(\mathcal{S}_i)$, properties (A) and (D) of Lemma 12, and the choices of μ and ε_t that

$$\begin{aligned}|C'(\widetilde{y})|&\overset{(48)}{\ge}(1-\Delta\varepsilon_t-\Delta^2\mu)|C_i(\widetilde{y})|\overset{\text{(b)}}{\ge}(1-\Delta\varepsilon_t-\Delta^2\mu)\left(\frac{dp}{2}\right)^{\operatorname{ldeg}^i(\widetilde{y})}|C(\widetilde{z})|\\ &\overset{\text{(A),(D)}}{\ge}(1-\Delta\varepsilon_t-\Delta^2\mu)\left(\frac{dp}{2}\right)^{k}\frac{n}{r}\overset{(44),(45)}{\ge}\left(\frac{dp}{10}\right)^{k}\frac{n}{r}\,.\end{aligned} \tag{50}$$

Second step: We now turn to the aforementioned second part of the inductive step. Here we ensure the existence of a system of distinct representatives for the indexed set system

$$\mathcal{C}_{i+1}:=\Big(C'(\widetilde{y})\colon\ \widetilde{y}\in\widetilde{W}_{i+1}\Big)\,.$$

We shall appeal to *Hall's condition* and show that for every subfamily $\mathcal{C}'\subseteq\mathcal{C}_{i+1}$ we have

$$|\mathcal{C}'|\le\left|\bigcup_{C'\in\mathcal{C}'}C'\right|. \tag{51}$$

Because of (50), assertion (51) holds for all families $\mathcal{C}'$ with $1 \le |\mathcal{C}'| \le (dp/10)^k n/r$.

Thus, consider a family $\mathcal{C}' \subseteq \mathcal{C}_i$ with $|\mathcal{C}'| > (dp/10)^k n/r$. For every $\widetilde{y} \in \widetilde{W}_{i+1}$ we have a set $\widetilde{K}(\widetilde{y})$ of $\mathrm{ldeg}^i(\widetilde{y})$ already embedded vertices of H such that $\widetilde{K}(\widetilde{y}) = N_H(\widetilde{y}) \setminus N_H^{i+1}(\widetilde{y})$. Let $K'(\widetilde{y}) := \varphi_i(\widetilde{K}(\widetilde{y}))$ be the image of $\widetilde{K}(\widetilde{y})$ in G under φ_i. Recall that $\widetilde{y}$ is equipped with a set $X_{\widetilde{y}} \subseteq V(\Gamma) \setminus W$ of size $\mathrm{edeg}(\widetilde{y})$ in Lemma 12. We have $\mathrm{ldeg}^i(\widetilde{y}) + \mathrm{edeg}(\widetilde{y}) = k$ by (49). Hence, when we add the vertices of $X_{\widetilde{y}}$ to $K'(\widetilde{y})$ we obtain a set $K(\widetilde{y}) = \{u_1(\widetilde{y}), \dots, u_k(\widetilde{y})\}$ of k vertices in Γ. Note that for two distinct vertices $\widetilde{y}$, $\widetilde{y}' \in \widetilde{W}_{i+1}$ the sets $\widetilde{K}(\widetilde{y})$ and $\widetilde{K}(\widetilde{y}')$ are disjoint. This follows from the fact that the distance in H between $\widetilde{y}$ and $\widetilde{y}'$ is at least four by the 3-independence of $\widetilde{W}_{i+1}$ (cf. (B) of Lemma 12) and if $\widetilde{K}(\widetilde{y}) \cap \widetilde{K}(\widetilde{y}') \neq \emptyset$, then this distance would be at most two. In addition $\big(X_{\widetilde{y}} : \widetilde{y} \in \widetilde{W}_{i+1}\big)$ consists of pairwise disjoint sets by hypothesis. Consequently, the sets $K(\widetilde{y})$ and $K(\widetilde{y}')$ are disjoint as well and, therefore,

$$\mathcal{F} := \{K(\widetilde{y}) \colon\ C'(\widetilde{y}) \in \mathcal{C}'\} \subseteq \{K(\widetilde{y}) \colon\ \widetilde{y} \in \widetilde{W}_{i+1}\} \subseteq \binom{V(\Gamma)}{k}$$

is a family of $|\mathcal{C}'|$ pairwise disjoint k-sets in $V(\Gamma)$. Moreover, $C(\widetilde{y}) = N^{\cap}_{W_i}(X_{\widetilde{y}})$ by definition and so (a) of $(\mathcal{S}_i)$ implies

$$C'(\widetilde{y}) \subseteq C(\widetilde{y}) \cap \bigcap_{v \in K'(\widetilde{y})} N_\Gamma(v) = \bigcap_{v \in K(\widetilde{y})} N_\Gamma(v)\,.$$

Let

$$U = \bigcup_{C'(\widetilde{y}) \in \mathcal{C}'} C'(\widetilde{y}) \subseteq W_{i+1}\,,$$

and suppose for a contradiction that

$$|U| < |\mathcal{C}'| = |\mathcal{F}|. \tag{52}$$

We now use the fact that Γ has property $\mathrm{STAR}_p(k, \xi_{15}, \nu)$ and apply it to U and $\mathcal{F}$ (see Lemma 15). By assumption $|U| < |\mathcal{F}| \le \nu n p^k |\mathcal{F}|$. We deduce that

$$\#\,\mathrm{stars}^\Gamma(U, \mathcal{F}) \le p^k |U||\mathcal{F}| + 6\xi_{15} n p^k |\mathcal{F}|\,.$$

On the other hand, because of (50), we have

$$\#\,\mathrm{stars}^\Gamma(U, \mathcal{F}) \ge \left(\frac{dp}{10}\right)^k \frac{n}{r}|\mathcal{F}|\,.$$

Combining the last two inequalities we infer from property (A) of Lemma 12 that

$$|U| \ge \left(\left(\frac{d}{10}\right)^k \frac{1}{r} - 6\xi_{15}\right) n \overset{(46)}{\ge} \xi_{15} n \overset{(46)}{=} \xi \frac{n}{r} \overset{(A)}{\ge} |\widetilde{W}_{i+1}| \ge |\mathcal{C}'|,$$

which contradicts (52). This contradiction shows that (52) does not hold, that is, Hall's condition (51) does hold. Hence, there exists a system of representatives for $\mathcal{C}_{i+1}$, i.e., an injective mapping $\psi \colon \widetilde{W}_{i+1} \to \bigcup_{\widetilde{y} \in \widetilde{W}_{i+1}} C'(\widetilde{y})$ such that $\psi(\widetilde{y}) \in C'(\widetilde{y})$ for every $\widetilde{y} \in \widetilde{W}_{i+1}$.

Finally, we extend φ_i. For that we set

$$\varphi_{i+1}(\widetilde{w}) = \begin{cases} \varphi_i(\widetilde{w})\,, & \text{if } \widetilde{w} \in \bigcup_{j=1}^{i} \widetilde{W}_j, \\ \psi(\widetilde{w})\,, & \text{if } \widetilde{w} \in \widetilde{W}_{i+1}\,. \end{cases}$$

Note that every $\widetilde{z} \in \bigcup_{j=i+2}^{t} \widetilde{W}_j$ has at most one neighbour in $\widetilde{W}_{i+1}$, as otherwise there would be two vertices $\widetilde{y}$ and $\widetilde{y}' \in \widetilde{W}_{i+1}$ with distance at most 2 in H, which contradicts property (B) of Lemma 12. Consequently, for every $\widetilde{z} \in \bigcup_{j=i+2}^{t} \widetilde{W}_j$ we have

$$C_{i+1}(\widetilde{z}) = \begin{cases} C_i(\widetilde{z})\,, & \text{if } N_H(\widetilde{z}) \cap \widetilde{W}_{i+1} = \emptyset, \\ C_i(\widetilde{z}) \cap N_G(\varphi_{i+1}(\widetilde{y}))\,, & \text{if } N_H(\widetilde{z}) \cap \widetilde{W}_{i+1} = \{\widetilde{y}\}. \end{cases}$$

by (a) of $(\mathcal{S}_{i+1})$. In what follows we show that φ_{i+1} and $C_{i+1}(\widetilde{z})$ for every $\widetilde{z} \in \bigcup_{j=i+2}^{t} \widetilde{W}_j$ have the desired properties and validate $(\mathcal{S}_{i+1})$.

First of all, from (a) of $(\mathcal{S}_i)$, combined with $\varphi_{i+1}(\widetilde{y}) \in C'(\widetilde{y}) \subseteq C_i(\widetilde{y})$ for every $\widetilde{y} \in \widetilde{W}_{i+1}$ and the property that $\big(\varphi_{i+1}(\widetilde{y})\colon\ \widetilde{y} \in \widetilde{W}_{i+1}\big)$ is a system of distinct representatives, we infer that φ_{i+1} is indeed a partial embedding of $H[\bigcup_{j=1}^{i+1} W_j]$.

Next we shall verify property (b) of $(\mathcal{S}_{i+1})$. So let $\widetilde{z} \in \bigcup_{j=i+2}^{t} \widetilde{W}_j$ be fixed. If $N_H(\widetilde{z}) \cap \widetilde{W}_{i+1} = \emptyset$, then $C_{i+1}(\widetilde{z}) = C_i(\widetilde{z})$, $\operatorname{ldeg}^{i+1}(\widetilde{z}) = \operatorname{ldeg}^{i}(\widetilde{z})$, which yields (b) of $(\mathcal{S}_{i+1})$ for that case. If, on the other hand, $N_H(\widetilde{z}) \cap \widetilde{W}_{i+1} \neq \emptyset$, then there exists a unique neighbour $\widetilde{y} \in \widetilde{W}_{i+1}$ of H (owing to the 3-independence of W_{i+1} by property (B) of Lemma 12). As discussed above we have $C_{i+1}(\widetilde{z}) = C_i(\widetilde{z}) \cap N_G(\varphi_{i+1}(\widetilde{y}))$ in this case. Since $\varphi_{i+1}(\widetilde{y}) \in C'(\widetilde{y})$, we infer directly from (b') that (b) of $(\mathcal{S}_{i+1})$ is satisfied.

Finally, we verify property (c) of $(\mathcal{S}_{i+1})$. Let $\{\widetilde{z}, \widetilde{z}'\}$ be an edge of H with $\widetilde{z}, \widetilde{z}' \in \bigcup_{j=i+2}^{t} \widetilde{W}_j$. We consider three cases, depending on the size of $N_H(\widetilde{z}) \cap \widetilde{W}_{i+1}$ and of $N_H(\widetilde{z}') \cap \widetilde{W}_{i+1}$. If $N_H(\widetilde{z}) \cap \widetilde{W}_{i+1} = \emptyset$ and $N_H(\widetilde{z}') \cap \widetilde{W}_{i+1} = \emptyset$, then part (c) of $(\mathcal{S}_{i+1})$ follows directly from part (c) of $(\mathcal{S}_i)$ and $\varepsilon_{i+1} \geq \varepsilon_i$, combined with $C_{i+1}(\widetilde{z}) = C_i(\widetilde{z})$, $C_{i+1}(\widetilde{z}') = C_i(\widetilde{z}')$. If $N_H(\widetilde{z}) \cap \widetilde{W}_{i+1} = \{\widetilde{y}\}$ and $N_H(\widetilde{z}') \cap \widetilde{W}_{i+1} = \emptyset$, then (c) of $(\mathcal{S}_{i+1})$ follows from (c') and the definition of $C_{i+1}(\widetilde{z})$ and $C_{i+1}(\widetilde{z}')$. If $N_H(\widetilde{z}) \cap \widetilde{W}_{i+1} = \{\widetilde{y}\}$ and $N_H(\widetilde{z}') \cap \widetilde{W}_{i+1} = \{\widetilde{y}'\}$, then $\widetilde{y} = \widetilde{y}'$, as otherwise there would be a $\widetilde{y}$-$\widetilde{y}'$-path in H with three edges, contradicting the 3-independence of $\widetilde{W}_{i+1}$. Consequently, (c) of $(\mathcal{S}_{i+1})$ follows from (c') and the definition of $C_{i+1}(\widetilde{z})$ and $C_{i+1}(\widetilde{z}')$.

We have therefore verified (a)–(c) of $(\mathcal{S}_i)$, thus concluding the induction step. The proof of Lemma 12 follows by induction. □

Appendix B. Proofs of auxiliary lemmas

In this section we provide all proofs that were postponed earlier, namely those of Lemma 8, Lemma 15, Lemma 17, and Lemma 20.

B.1. **Proof of Lemma 8.** For the proof of Lemma 8 we need the following lemma which collects some well known facts about the edge distribution in random graphs $\mathcal{G}_{n,p}$ and follows directly from the Chernoff bound for binomially distributed random variables.

Lemma 36. *If $\log^4 n/(pn) = o(1)$ then a.a.s. the random graph $\Gamma = \mathcal{G}_{n,p}$ has the following properties. For all vertex sets X, Y, $Z \subseteq V(\Gamma)$ with $X \cap Y = \emptyset$ and $|X|, |Y| \geq \frac{n}{\log n}$ we have*

- *(i)* $e_\Gamma(X) = (1 \pm \frac{1}{\log n})p\binom{|X|}{2}$,
- *(ii)* $e_\Gamma(X, Y) = (1 \pm \frac{1}{\log n})p|X||Y|$,
- *(iii)* $\sum_{z \in Z} \deg_\Gamma(z) = (1 \pm \frac{1}{\log n})p|Z|n$. □

Proof of Lemma 8. For the proof we will use the sparse regularity lemma (Lemma 5) and the facts about the edge distribution in random graphs given by Lemma 36.

Given α, ε, and r_0 let r_1, ν, and n_0 be as provided by Lemma 5 for input

$$\varepsilon' := \varepsilon^2/1000\,, \quad K := 1 + \varepsilon'\,, \quad \text{and} \quad r_0' := \max\{2r_0, \lceil 1/\varepsilon' \rceil\}\,.$$

Let further d be given and assume that n is such that $n \geq n_0$, $\log n \geq 1/\varepsilon'$, and $\log n \geq 1/\nu$. Let Γ be a typical graph from $\mathcal{G}_{n,p}$ with $\log^4 n/(pn) = o(1)$, by which we mean here that Γ should satisfy properties (i)–(iii) of Lemma 36. We will show that, then, Γ also satisfies the conclusion of Lemma 8.

To this end we consider an arbitrary subgraph $G = (V, E)$ of Γ that satisfies the assumptions of Lemma 8. By property (ii) of Lemma 36 the graph $G \subseteq \Gamma$ is $(1/\log n, 1 + 1/\log n)$-bounded with respect to p. Since we have $1 + 1/\log n \leq 1 + \varepsilon' = K$, the sparse regularity lemma (Lemma 5) with input ε', K, and r_0' asserts that G has an (ε', p)-regular ε-equipartition $V = V_0' \dot\cup V_1' \dot\cup \dots \dot\cup V_{r'}'$ for some $r_0' \leq r' \leq r_1$. Observe that there are at most $r'\sqrt{\varepsilon'}$ clusters in this partition which are contained in more than $r'\sqrt{\varepsilon'}$ pairs that are not (ε', p)-regular. We add all these clusters to V_0', denote the resulting set by V_0 and let the remaining clusters be $V_1, \dots, V_r$. Then $r_0 \leq r'/2 \leq r \leq r_1$. We claim that the partition $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_r$ has the desired properties.

Indeed, $|V_0| \leq \varepsilon' n + r'\sqrt{\varepsilon'}(n/r') \leq \varepsilon n$ and the number of pairs in $V_1 \dot\cup \ldots \dot\cup V_r$ which are not (ε,p)-regular is at most $r \cdot r'\sqrt{\varepsilon'} \leq 2r^2\sqrt{\varepsilon'} \leq \varepsilon\binom{r}{2}$. It follows that $V_1 \dot\cup \ldots \dot\cup V_r$ is an (ε,p)-regular partition. Let R be the (edge maximal) reduced graph for the given paramter d, so that, R has vertex set $[r]$ and edges ij for exactly all the (ε,d,p)-dense pairs (V_i,V_j) with $i,\, j \in [r]$. It remains to show that we have $\delta(R) \geq (\alpha - d - \varepsilon)|R|$.

To see this, define $L := |V_i|$ $(i \in [r])$ and consider arbitrary disjoint sets $X, Y \subseteq V(G)$. Then $\sum_{x\in X} \deg_G(x) = 2e_G(X) + e_G(X,Y) + e_G(X, V \setminus (X \cup Y))$ and therefore

$$e_G(X,Y) \geq \Big(\alpha \sum_{x\in X} \deg_\Gamma(x)\Big) - 2e_\Gamma(X) - e_\Gamma\big(X, V \setminus (X \cup Y)\big).$$

By properties (i)–(iii) of Lemma 36, if $|X| \geq n/\log n$ and $|X \cup Y| \leq n - n/\log n$, then this implies

$$\begin{aligned} e_G(X,Y) &\geq \alpha\Big(1 - \frac{1}{\log n}\Big)p|X|n - 2\Big(1+\frac{1}{\log n}\Big)p\binom{|X|}{2} \\ &\qquad\qquad - \Big(1+\frac{1}{\log n}\Big)p|X|\Big(n - |X| - |Y|\Big) \\ &\geq \big(\alpha(1-\varepsilon')n - (1+\varepsilon')(n-|Y|)\big)p|X|\,. \end{aligned} \tag{53}$$

Now fix $i \in [r]$ and let $\bar{V}_i := V \setminus (V_0 \cup V_i)$. Then

$$e_G(V_i, \bar{V}_i) \leq \big(\deg_R(i) + 2r\sqrt{\varepsilon'}\big)\,(1+\varepsilon')\,pL^2 + \big(r - \deg_R(i)\big)dpL^2,$$

since each cluster is contained in at most $r'\sqrt{\varepsilon'} \leq 2r\sqrt{\varepsilon'}$ (ε',p)-irregular pairs and because R is a maximal (ε',d,p)-reduced graph and $G \subseteq \Gamma$ is $(1/\log n, 1+\varepsilon')$-bounded with respect to p. On the other hand, (53) implies that

$$\begin{aligned} e_G(V_i,\bar{V}_i) &\geq \Big(\alpha(1-\varepsilon')n - (1+\varepsilon')\big(|V_0| + |V_i|\big)\Big)p|V_i| \\ &\geq \Big(\alpha(1-\varepsilon') - (1+\varepsilon')3\sqrt{\varepsilon'}\Big)pLn, \end{aligned}$$

where we used $|V_0| \leq (\varepsilon' + \sqrt{\varepsilon'})n$ and $|V_i| \leq n/r'_0 \leq \varepsilon' n$. We conclude that

$$\Big(\deg_R(i)(1+\varepsilon'-d) + 2r\sqrt{\varepsilon'}(1+\varepsilon') + rd\Big)pL^2 \geq \Big(\alpha(1-\varepsilon') - (1+\varepsilon')3\sqrt{\varepsilon'}\Big)prL^2,$$

since $n/L \geq r$. This gives

$$\begin{aligned} \deg_R(i)(1+\varepsilon'-d) &\geq \Big(\alpha(1-\varepsilon') - (1+\varepsilon')3\sqrt{\varepsilon'} - 2\sqrt{\varepsilon'}(1+\varepsilon') - d\Big)r \\ &\geq \Big(\alpha - \alpha\varepsilon' - 9\sqrt{\varepsilon'} - d\Big)r \geq (\alpha - d - \varepsilon/2)|R|\,. \end{aligned}$$

Thus, $\deg_R(i) \geq (\alpha-d-\varepsilon/2)(1+\varepsilon'-d)^{-1}|R| \geq (\alpha-d-\varepsilon/2)(1+\varepsilon')^{-1}|R| \geq (\alpha-d-\varepsilon/2)(1-\varepsilon')|R| \geq (\alpha-d-\varepsilon)|R|$. Hence the (ε,d,p)-dense partition $V = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_r$ has a reduced graph R with $\delta(R) \geq (\alpha - d - \varepsilon)|R|$. □

B.2. **Proof of Lemma 15.** This proof makes use of a Chernoff bound for the binomially distributed random variable $\#\,\mathrm{stars}^\Gamma(X,\mathcal{F})$ appearing in this lemma (cf. Definition 13 and the discussion below this definition).

Proof of Lemma 15. Given Δ and ξ let ν and c be constants satisfying

$$\begin{aligned} -6\xi\log(2\xi) &\leq -(6\xi - 2\sqrt{\nu})\log\xi, \qquad & 2\nu &\leq (\sqrt{\nu} - 2\nu), \\ \Delta + 1 - 6\xi c^\Delta &\leq -1, \qquad \text{and} & \Delta &\leq \nu c^\Delta. \end{aligned} \tag{54}$$

First we estimate the probability that there are X and $\mathcal{F}$ with $|\mathcal{F}| \geq n/\log n$ fulfilling the requirements of the lemma but violating (2). Chernoff's inequality $\mathbb{P}[Y \geq \mathbb{E}\,Y + t] \leq \exp(-t)$ for a binomially distributed random variable Y and $t \geq 6\,\mathbb{E}\,Y$ (see [22, Chapter 2]): implies

$$\mathbb{P}\Big[\#\,\mathrm{stars}^\Gamma(X,\mathcal{F}) \geq p^\Delta|X||\mathcal{F}| + 6\xi np^\Delta|\mathcal{F}|\Big] \leq \exp(-6\xi np^\Delta|\mathcal{F}|) \leq \exp(-6\xi c^\Delta|\mathcal{F}|\log n)$$

for fixed X and $\mathcal{F}$ since $6\xi np^{\Delta}|\mathcal{F}| \geq 6p^{\Delta}|X||\mathcal{F}|$. As the number of choices for $\mathcal{F}$ and X can be bounded by $\sum_{f=n/\log n}^{\xi n} n^{\Delta f}$ and $2^n \leq \exp(n)$, respectively, the probability we want to estimate is at most

$$\sum_{f=\frac{n}{\log n}}^{\xi n} \exp\Big(\Delta f \log n + n - 6\xi c^{\Delta} f \log n\Big) \leq \sum_{f=\frac{n}{\log n}}^{\xi n} \exp\big(f \log n(\Delta + 1 - 6\xi c^{\Delta})\big),$$

which does not exceed $\xi n \exp(-n)$ by (54) and thus tends to 0 as n tends to infinity.

It remains to establish a similar bound on the probability that there are X and $\mathcal{F}$ with $|\mathcal{F}| < n/\log n$ fulfilling the requirements of the lemma but violating (2). For this purpose we use that

$$\mathbb{P}[Y \geq t] \leq q^t \binom{m}{t} \leq \exp\Big(-t \log \frac{t}{3qm}\Big)$$

for a random variable Y with distribution $\mathrm{Bi}(m,q)$ and infer for fixed X and $\mathcal{F}$

$$\begin{aligned}\mathbb{P}\Big[\#\,\mathrm{stars}^{\Gamma}(X,\mathcal{F}) \geq p^{\Delta}|X||\mathcal{F}| + 6\xi np^{\Delta}|\mathcal{F}|\Big] &\leq \mathbb{P}\Big[\#\,\mathrm{stars}^{\Gamma}(X,\mathcal{F}) \geq 6\xi np^{\Delta}|\mathcal{F}|\Big] \\ &\leq \exp\left(-6\xi np^{\Delta}|\mathcal{F}| \log \frac{2\xi n}{|X|}\right) \leq \exp\left(-2\sqrt{\nu} np^{\Delta}|\mathcal{F}| \log \frac{n}{|X|}\right).\end{aligned}$$

because $-6\xi \log(2\xi) \leq -(6\xi - 2\sqrt{\nu}) \log \xi \leq (6\xi - 2\sqrt{\nu}) \log(n/|X|)$ by (54). The number of choices for $\mathcal{F}$ and X in total can be bounded by

$$\begin{aligned}\sum_{f=1}^{\frac{n}{\log n}} \sum_{x=1}^{\nu np^{\Delta} f} n^{\Delta f} \binom{n}{x} &\leq \sum_{f=1}^{\frac{n}{\log n}} \sum_{x=1}^{\nu np^{\Delta} f} \exp\Big(\Delta f \log n + \nu np^{\Delta} f \log \frac{en}{x}\Big) \\ &\leq \sum_{f=1}^{\frac{n}{\log n}} \sum_{x=1}^{\nu np^{\Delta} f} \exp\Big(2\nu np^{\Delta} f \log \frac{en}{x}\Big) \leq \sum_{f=1}^{\frac{n}{\log n}} \sum_{x=1}^{\nu np^{\Delta} f} \exp\Big(\sqrt{\nu} np^{\Delta} f \log \frac{n}{x}\Big)\end{aligned}$$

where the second inequality follows from $\Delta \log n \leq \nu c^{\Delta} \log n \leq \nu np^{\Delta}$ and the last from $2\nu \log e \leq (\sqrt{\nu} - 2\nu) \log(n/x)$ by (54). Therefore the probability under consideration is at most

$$\sum_{f=1}^{\frac{n}{\log n}} \sum_{x=1}^{\nu p^{\Delta} nf} \exp\Big(\sqrt{\nu} np^{\Delta} f \log \frac{n}{x} - 2\sqrt{\nu} np^{\Delta} f \log \frac{n}{x}\Big) \leq n^2 \exp\left(-\sqrt{\nu} \log n \frac{n}{\log n}\right).$$

□

B.3. **Proof of Lemma 17.** We will use the following simple proposition about cuts in hypergraphs. This proposition generalises the well known fact that any graph G admits a vertex partition into sets of roughly equal size such that the resulting cut contains at least half the edges of G.

Proposition 37. *Let $\mathcal{G} = (V, \mathcal{E})$ be an ℓ-uniform hypergraph with m edges and n vertices such that $n \geq 3\ell$. Then there is a partition $V = V_1 \dot\cup V_2$ with $|V_1| = \lfloor 2n/3 \rfloor$ and $|V_2| = \lceil n/3 \rceil$ such that at least $m \cdot \ell/2^{\ell+2}$ edges in $\mathcal{E}$ are 1-crossing, i.e., they have exactly one vertex in V_2.*

Proof. Let X be the number of $\frac{1}{3}$-cuts of V, i.e., cuts $V = V_1 \dot\cup V_2$ with $|V_1| = \lfloor 2n/3 \rfloor$ and $|V_2| = \lceil n/3 \rceil$. For a fixed edge B there are precisely 2^{ℓ} ways to distribute its vertices over $V_1 \dot\cup V_2$ out of which exactly ℓ are such that B is 1-crossing. Further, for r fixed vertices of B exactly $\binom{n-\ell}{\lceil n/3 \rceil - r}$ of all $\frac{1}{3}$-cuts have exactly these vertices in V_2. It is easy to check that

$$\binom{n-\ell}{\lceil n/3 \rceil - r} \leq 4 \binom{n-\ell}{\lceil n/3 \rceil - 1} \qquad \text{for all } 0 \leq r \leq \ell\,.$$

It follows that B is 1-crossing for at least an $\frac{1}{4}\ell/(2^{\ell})$ fraction of all $\frac{1}{3}$-cuts. Now assume that all $\frac{1}{3}$-cuts have less than $m \cdot \ell/2^{\ell+2}$ edges that are 1-crossing. Then double counting gives

$$m \cdot \frac{\ell}{2^{\ell+2}} \cdot X > \sum_{B \in \mathcal{E}} \#\big\{\tfrac{1}{3}\text{-cuts s.t. } B \text{ is 1-crossing}\big\} \geq m \cdot \frac{1}{4} \frac{\ell}{2^{\ell}} \cdot X$$

which is a contradiction. □

In the proof of Lemma 17 we need to estimate the number of "bad" ℓ-sets in a vertex set X. For this purpose we will use Proposition 37 to obtain a partition of X into sets $X = X_1 \dot\cup X_2$ such that a substantial proportion of all these bad ℓ-sets will be 1-crossing and X_1 is not too small. In this way we obtain many $(\ell-1)$-sets in X_1 most of which will, as we show, be similarly bad as the ℓ-sets we started with. This will allow us to prove Lemma 17 by induction.

Proof of Lemma 17. Let Δ and d be given. Let Γ be an n-vertex graph, let ℓ be an integer, let ε', μ, ε, ξ be positive real numbers, and let $p = p(n)$ be a function. We say that Γ has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$ if Γ has the property stated in Lemma 17 with parameters ε', μ, ε, ξ, $p(n)$ and with parameters and Δ and d. Similarly, Γ has property $D(\varepsilon', \mu, \varepsilon, \xi, p(n))$ if it satisfies the conclusion of Lemma 19 with these parameters and with Δ and $d_0 := d$. For any fixed $\ell > 0$, we denote by $(\mathcal{P}_\ell)$ the following statement.

$(\mathcal{P}_\ell)$ *For all $\varepsilon', \mu > 0$ there is ε such that for all $\xi > 0$ there is $c > 1$ such that a random graph $\Gamma = \mathcal{G}_{n,p}$ with $p > c(\frac{\log n}{n})^{1/\Delta}$ has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$ with probability $1 - o(1)$.*

We prove that $(\mathcal{P}_\ell)$ holds for every fixed $\ell > 0$ by induction on ℓ. The case $\ell = 1$ is an easy consequence of Proposition 7 which states that in *all* (ε, d, p)-dense pairs most vertices have a large neighbourhood.

For the inductive step assume that $(\mathcal{P}_{\ell-1})$ holds. We will show that this implies $(\mathcal{P}_\ell)$. We start by specifying the constants appearing in statement $(\mathcal{P}_\ell)$. Let ε' and μ be arbitrary positive constants. Set $\varepsilon'_{\ell-1} := \varepsilon'$ and $\mu_{\ell-1} := \frac{1}{100}\mu\frac{\ell}{2^{\ell+2}}$. Let $\varepsilon_{\ell-1}$ be given by $(\mathcal{P}_{\ell-1})$ for input parameters $\varepsilon'_{\ell-1}$ and $\mu_{\ell-1}$. Set $\varepsilon'_{19} := \varepsilon_{\ell-1}$ and let ε_{19} be as promised by Lemma 19 with parameters ε'_{19} and $\mu_{19} := \frac{1}{2}$. Define $\varepsilon := \mu_{\ell-1}\varepsilon_{19}\varepsilon'_{\ell-1}$. Next, let ξ be an arbitrary parameter provided by the adversary in Lemma 17 and choose $\xi_{\ell-1} := \xi(d-\varepsilon)$ and $\xi_{19} := \mu_{\ell-1}\xi$. Finally, let $c_{\ell-1}$ and c_{19} be given by $(\mathcal{P}_{\ell-1})$ and by Lemma 19, respectively, for the previously specified parameters together with $\xi_{\ell-1}$ and ξ_{19}. Set $c := \max\{c_{\ell-1}, c_{19}\}$. We will prove that with this choice of ε and c the statement in $(\mathcal{P}_\ell)$ holds for the input parameters ε', μ, and ξ.

Let $\Gamma = \mathcal{G}_{n,p}$ be a random graph. By $(\mathcal{P}_{\ell-1})$ and Lemma 19, and by the choice of the parameters the graph Γ has properties

$$P_{\ell-1}(\varepsilon'_{\ell-1}, \mu_{\ell-1}, \varepsilon_{\ell-1}, \xi_{\ell-1}, p(n)) \qquad \text{and} \qquad D(\varepsilon'_{19}, \mu_{19}, \varepsilon_{19}, \xi_{19}, p(n))$$

with probability $1 - o(1)$ if n is large enough. We will show that a graph Γ with these properties also satisfies $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$. Let $G = (X \dot\cup Y, E)$ be an arbitrary subgraph of such a Γ where $|X| = n_1$ and $|Y| = n_2$ with $n_1 \geq \xi p^{\Delta-1} n$, $n_2 \geq \xi p^{\Delta-\ell} n$, and (X, Y) is an (ε, d, p)-dense pair.

We would like to show that for $\mathcal{B}_\ell := \mathrm{bad}^{G,\ell}_{\varepsilon',d,p}(X, Y)$ we have $|\mathcal{B}_\ell| \leq \mu n_1^\ell$. Assume for a contradiction that this is not the case. By Proposition 37 there is a cut $X = X_1 \dot\cup X_2$ with $|X_1| = \lfloor 2n_1/3 \rfloor$ and $|X_2| = \lceil n_2/3 \rceil$ such that at least $|\mathcal{B}_\ell| \cdot \ell / 2^{\ell+2}$ of the ℓ-sets in $\mathcal{B}_\ell$ are 1-crossing, i.e., have exactly one vertex in X_2. By Proposition 7 there are less than $\varepsilon|X|$ vertices $x \in X_2$ such that $|N_Y(x)| < (d-\varepsilon)pn_2$. We delete all ℓ-sets from $\mathcal{B}_\ell$ that contain such a vertex or are not 1-crossing for $X = X_1 \dot\cup X_2$ and call the resulting set $\mathcal{B}'_\ell$. It follows that

$$|\mathcal{B}'_\ell| \geq |\mathcal{B}_\ell|\tfrac{\ell}{2^{\ell+2}} - \varepsilon|X|n_1^{\ell-1} > \mu n_1^\ell \tfrac{\ell}{2^{\ell+2}} - \varepsilon n_1^\ell \geq 20\mu_{\ell-1}n_1^\ell. \tag{55}$$

Now, for each $v \in X_2$ we count the number of ℓ-sets $B \in \mathcal{B}'_\ell$ containing v. We delete all vertices v from X_2 for which this number is less than $|\mathcal{B}'_\ell|/(10n_1)$ and call the resulting set X'. Observe that the definition of $\mathcal{B}'_\ell$ implies that all vertices x in X' satisfy $|N_Y(x)| \geq (d-\varepsilon)pn_2$. Because $\mathcal{B}'_\ell$ contains only 1-crossing ℓ-sets we get

$$|\mathcal{B}'_\ell| \leq |X_2 \setminus X'|\frac{|\mathcal{B}'_\ell|}{10n_1} + |X'|n_1^{\ell-1} \leq \frac{|\mathcal{B}'_\ell|}{10} + |X'|n_1^{\ell-1}$$

and thus

$$|X'| \geq \frac{9}{10n_1^{\ell-1}}|\mathcal{B}'_\ell| \overset{(55)}{\geq} 10\mu_{\ell-1}n_1.$$

This together with Proposition 6 implies that the pairs (X', Y) and (Y, X_1) are (ε_{19}, d, p)-dense. In addition we have $|X'|, |X_1| \geq \mu_{\ell-1}n_1 \geq \mu_{\ell-1}\xi p^{\Delta-1}n = \xi_{19}p^{\Delta-1}n$ and $|Y| \geq \xi p^{\Delta-\ell}n \geq \xi_{19}p^{\Delta-2}n$.

Because Γ has property $D(\varepsilon'_{19}, \mu_{19}, \varepsilon_{19}, \xi_{19}, p(n))$ we conclude for the tripartite graph $G[X'\dot\cup Y \dot\cup X_1]$ that there are at least $|X'| - \mu_{19}|X'| \geq 1$ vertices x in X' such that $(N_Y(x), X_1)$ is $(\varepsilon'_{19}, d, p)$-dense. Let $x^* \in X'$ be one of these vertices and set $Y' := N_Y(x^*)$. Thus (Y', X_1) is $(\varepsilon'_{19}, d, p)$-dense and since X' only contains vertices with a large neighbourhood in Y we have $|Y'| \geq (d-\varepsilon)pn_2$. Furthermore, let $\mathcal{B}'_\ell(x^*)$ be the family of ℓ-sets in $\mathcal{B}'_\ell$ that contain x^*. Then $\mathcal{B}'_\ell(x^*)$ contains ℓ-sets with $\ell - 1$ vertices in X_1 and with one vertex, the vertex x^*, in X_2 because $\mathcal{B}'_\ell$ contains only 1-crossing ℓ-sets. By definition of X' and because $x^* \in X'$ we have

$$|\mathcal{B}'_\ell(x^*)| \geq |\mathcal{B}'_\ell|/(10n_1) \overset{(55)}{\geq} 2\mu_{\ell-1} n_1^{\ell-1}. \tag{56}$$

For $B \in \mathcal{B}'_\ell(x^*)$ let $\Pi_{\ell-1}(B)$ be the projection of B to X_1. This implies that $\Pi_{\ell-1}(B)$ is an $(\ell-1)$-set in X_1. In addition $N_{Y'}(\Pi_{\ell-1}(B)) = N_Y(B)$ by definition of Y' and hence $\Pi_{\ell-1}(B)$ has less than $(d-\varepsilon')^\ell p^\ell n_2$ common neighbours in Y' because $B \in \mathcal{B}'_\ell(x^*) \subseteq \mathcal{B}_\ell$. Accordingly the family $\mathcal{B}_{\ell-1}$ of all projections $\Pi_{\ell-1}(B)$ with $B \in \mathcal{B}'_\ell(x^*)$ is a family of size $|\mathcal{B}'_\ell(x^*)|$ and contains only $(\ell-1)$-sets B' with

$$|N_{Y'}(B')| \leq (d-\varepsilon')^\ell p^\ell n_2 \leq (d-\varepsilon')^{\ell-1} p^{\ell-1}|Y'| = (d-\varepsilon'_{\ell-1})^{\ell-1}p^{\ell-1}|Y'|.$$

This means $\mathcal{B}_{\ell-1} \subseteq \mathrm{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X_1, Y')$. Recall that (X_1, Y') is $(\varepsilon'_{19}, d, p)$-dense by the choice of x^*. Because $|X| = n_1 \geq \xi p^{\Delta-1} n$ and

$$|Y'| \geq (d-\varepsilon)pn_2 \geq (d-\varepsilon)p \cdot \xi p^{\Delta-\ell} n = \xi_{\ell-1} p^{\Delta-(\ell-1)} n$$

we can appeal to $P_{\ell-1}(\varepsilon'_{\ell-1}, \mu_{\ell-1}, \varepsilon_{\ell-1}, \xi_{\ell-1}, p(n))$ and conclude that

$$|\mathcal{B}'_\ell(x^*)| = |\mathcal{B}_{\ell-1}| \leq |\,\mathrm{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X, Y')| \leq \mu_{\ell-1} n_1^{\ell-1}\,,$$

contradicting (56).

Because G was arbitrary this shows that Γ has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$. Thus $(\mathcal{P}_\ell)$ holds, which finishes the proof of the inductive step. □

B.4. **Proof of Lemma 20.** In this section we provide the proof of Lemma 20 which examines the inheritance of p-density to neighbourhoods of Δ-sets. For this purpose we will first establish a version of this lemma, Lemma 38 below, which only considers Δ-sets that are crossing in a given vertex partition.

We need some definitions. Let $G = (V, E)$ be a graph, X be a subset of its vertices, and $X = X_1 \dot\cup \dots \dot\cup X_T$ be a partition of X. Then, for integers $\ell, T > 0$, we say that an ℓ-set $B \subseteq X$ is *crossing* in $X_1 \dot\cup \dots \dot\cup X_T$ if there are indices $0 < i_1 < \cdots < i_\ell < T$ such that B contains exactly one element in X_{i_j} for each $j \in [\ell]$. In this case we also write $B \in X_{i_1} \times \cdots \times X_{i_\ell}$ (hence identifying crossing ℓ-sets with ℓ-tuples).

Now let p, ε, d be positive reals, and Y, $Z \subseteq V$ be vertex sets such that X, Y, and Z are mutually disjoint. Define

$$\mathrm{bad}^{G,\ell}_{\varepsilon,d,p}(X_1, \dots, X_T; Y, Z)$$

to be the family of all those crossing ℓ-sets B in $X_1 \dot\cup \dots \dot\cup X_T$ that either satisfy $|N^\cap_Y(B)| < (d-\varepsilon)^\ell p^\ell |Y|$ or have the property that $(N^\cap_Y(B), Z)$ is not (ε, d, p)-dense in G. Further, let

$$\mathrm{Bad}^{G,\ell}_{\varepsilon,d,p}(X_1, \dots, X_T; Y, Z)$$

be the family of crossing ℓ-sets B in $X_1 \dot\cup \dots \dot\cup X_T$ that contain an ℓ'-set $B' \subseteq B$ with $\ell' > 0$ such that $B' \in \mathrm{bad}^{G,\ell'}_{\varepsilon,d,p}(X_1, \dots, X_T; Y, Z)$.

Lemma 38. *For all integers $\ell, \Delta > 0$ and positive reals d_0, ε', and μ there is ε such that for all $\xi > 0$ there is $c > 1$ such that if $p > c(\frac{\log n}{n})^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. For $n_1, n_3 \geq \xi p^{\Delta-1} n$ and $n_2 \geq \xi p^{\Delta-\ell-1} n$ let $G = (X \dot\cup Y \dot\cup Z, E)$ be any tripartite subgraph of Γ with $|X| = n_1$, $|Y| = n_2$, and $|Z| = n_3$. Assume further that $X = X_1 \dot\cup \dots \dot\cup X_\ell$ with $|X_i| \geq \lfloor \frac{n_1}{\ell} \rfloor$ and that (X, Y) and (Y, Z) are (ε, d, p)-dense pairs with $d \geq d_0$. Then*

$$\big|\,\mathrm{bad}^{G,\ell}_{\varepsilon',d,p}(X_1, \dots, X_\ell; Y, Z)\big| \leq \mu n_1^\ell.$$

Proof. Let Δ and d_0 be given. For a fixed n-vertex graph Γ, a fixed integer ℓ and fixed positive reals ε', μ, ε, ξ, and a function $p = p(n)$ we say that we say that a graph Γ on n vertices has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$ if Γ has the property stated in the lemma for these parameters and for Δ and d_0, that is, whenever $G = (X \dot\cup Y \dot\cup Z, E)$ is a tripartite subgraph of Γ with the required properties, then G satisfies the conclusion of the lemma. For any fixed $\ell > 0$, we denote by $(\mathcal{P}_\ell)$ the following statement.

$(\mathcal{P}_\ell)$ *For all $\varepsilon', \mu > 0$ there is ε such that for all $\xi > 0$ there is $c > 1$ such that a random graph $\Gamma = \mathcal{G}_{n,p}$ with $p > c(\frac{\log n}{n})^{1/\Delta}$ has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$ with probability $1 - o(1)$.*

We prove that $(\mathcal{P}_\ell)$ holds for every fixed $\ell > 0$ by induction on ℓ.

The case $\ell = 1$ is an easy consequence of Lemma 19 and Proposition 7. Indeed, let ε' and μ be arbitrary, let ε_{19} be as given by Lemma 19 for Δ, d_0, ε', and $\mu/2$ and fix $\varepsilon := \min\{\varepsilon_{19}, \varepsilon', \mu/2\}$. Let ξ be arbitrary and pass it on to Lemma 19 for obtaining c. Now, let $\Gamma = \mathcal{G}_{n,p}$ be a random graph. Then, by the choice of parameters, Lemma 19 asserts that the graph Γ has the following property with probability $1 - o(1)$. Let $G = (X \dot\cup Y \dot\cup Z, E)$ be any subgraph with $X = X_1$ and $|X| = n_1$, $|Y| = n_2$, and $|Z| = n_3$, where $n_1, n_3 \geq \xi p^{\Delta-1} n$ and $n_2 \geq \xi p^{\Delta-2} n$, and (X, Y) and (Y, Z) are (ε, d, p)-dense pairs. Then there are at most $\frac{\mu}{2} n_1$ vertices $x \in X$ such that $(N(x) \cap Y, Z)$ is not an (ε', d, p)-dense pair in G. Because $\varepsilon \leq \mu/2$, Proposition 7 asserts that in every such G there are at most $\frac{\mu}{2} n_1$ vertices $x \in X$ with $|N_Y(x)| < (d - \varepsilon') p |Y|$. This implies that

$$\left| \operatorname{bad}^{G,1}_{\varepsilon',d,p}(X_1; Y, Z) \right| \leq \mu n_1$$

holds with probability $1 - o(1)$ for all such subgraphs G of the random graph Γ. Accordingly we get $(\mathcal{P}_1)$.

For the inductive step assume that $(\mathcal{P}_{\ell-1})$ and $(\mathcal{P}_1)$ hold. We will show that this implies $(\mathcal{P}_\ell)$. Again, let ε' and μ be arbitrary positive constants. Let ε_1 be as promised in the statement $(\mathcal{P}_1)$ for parameters $\varepsilon'_1 := \varepsilon'$ and $\mu_1 := \mu/2$. Set $\varepsilon'_{\ell-1} := \min\{\varepsilon_1, \varepsilon', \frac{\mu}{4}\}$, and let $\varepsilon_{\ell-1}$ be given by $(\mathcal{P}_{\ell-1})$ for parameters $\varepsilon'_{\ell-1}$ and $\mu_{\ell-1} := \frac{\mu}{4}$. We define $\varepsilon := \varepsilon_{\ell-1}/(\ell+1)$. Next, let ξ be an arbitrary parameter and choose

$$\xi_1 := \min\{\xi/(\ell+1), (d_0 - \varepsilon'_{\ell-1})^{\ell-1}\xi\} \quad \text{and} \quad \xi_{\ell-1} := \xi/(\ell+1). \tag{57}$$

Finally, let c_1 and $c_{\ell-1}$ be given by $(\mathcal{P}_1)$ and $(\mathcal{P}_{\ell-1})$, respectively, for the previously specified parameters together with ξ_1 and $\xi_{\ell-1}$. Set $c := \max\{c_1, c_{\ell-1}\}$. We will prove that with this choice of ε and c the statement in $(\mathcal{P}_\ell)$ holds for the input parameters ε', μ, and ξ. For this purpose let $\Gamma = \mathcal{G}_{n,p}$ be a random graph. By $(\mathcal{P}_1)$ and $(\mathcal{P}_{\ell-1})$ and the choice of the parameters the graph Γ has properties $P_1(\varepsilon'_1, \mu_1, \varepsilon_1, \xi_1, p(n))$ and $P_{\ell-1}(\varepsilon'_{\ell-1}, \mu_{\ell-1}, \varepsilon_{\ell-1}, \xi_{\ell-1}, p(n))$ with probability $1 - o(1)$. We will show that a graph Γ with these properties also satisfies $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$. Let $G = (X \dot\cup Y \dot\cup Z, E)$ be an arbitrary subgraph of such a Γ where $X = X_1 \dot\cup \ldots \dot\cup X_\ell$, $|X| = n_1$, $|Y| = n_2$, $|Z| = n_3$, with $n_1, n_3 \geq \xi p^{\Delta-1} n$, $n_2 \geq \xi p^{\Delta-\ell-1} n$, and $|X_i| \geq \lfloor \frac{n_1}{\ell} \rfloor$, and assume that (X, Y) and (Y, Z) are (ε, d, p)-dense pairs for $d \geq d_0$.

We would like to bound $\mathcal{B}_\ell := \operatorname{bad}^{G,\ell}_{\varepsilon',d,p}(X_1, \ldots, X_\ell; Y, Z)$. For this purpose let B' be a fixed $(\ell-1)$-set and define

$$\mathcal{B}_{\ell-1} := \operatorname{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X_1, \ldots, X_{\ell-1}; Y, Z) \cup \operatorname{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X_1, \ldots, X_{\ell-1}; Y, X_\ell) \tag{58a}$$

$$\mathcal{B}_1(B') := \operatorname{bad}^{G,1}_{\varepsilon',d,p}(X_\ell; N^\cap_Y(B'), Z). \tag{58b}$$

For an ℓ-set $B \in X_1 \times \cdots \times X_\ell$ let further $\Pi_{\ell-1}(B)$ denote the $(\ell-1)$-set that is the projection of B to $X_1 \times \cdots \times X_{\ell-1}$ and let $\Pi_\ell(B)$ be the vertex that is the projection of B to X_ℓ. Now, consider an ℓ-set B that is contained in $B \in \mathcal{B}_\ell$ but is such that $B' := \Pi_{\ell-1}(B) \notin \mathcal{B}_{\ell-1}$. Let $v = \Pi_\ell(B) \in X_\ell$ and $Y' := N^\cap_Y(B')$. We will show that then $v \in \mathcal{B}_1(B')$. Indeed, since $B' \notin \mathcal{B}_{\ell-1}$ it follows from (58a) that

$$B' \notin \operatorname{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X_1, \ldots, X_{\ell-1}; Y, Z)$$

and thus $|Y'| \geq (d - \varepsilon'_{\ell-1})^{\ell-1} p^{\ell-1} n_2$. As $N^\cap_Y(B) = N^\cap_{Y'}(v)$ we conclude that

$$v \in \operatorname{bad}^{G,1}_{\varepsilon',d,p}(X_\ell; Y', Z) = \mathcal{B}_1(B')$$

by (58b) because otherwise $(N_Y^{\cap}(B), Z)$ was (ε', d, p)-dense and we had

$$|N_Y^{\cap}(B)| \geq (d-\varepsilon')p|Y'| \geq (d-\varepsilon')p \cdot (d-\varepsilon'_{\ell-1})^{\ell-1}p^{\ell-1}n_2 \geq (d-\varepsilon')^{\ell}p^{\ell}n_2,$$

which contradicts $B \in \mathcal{B}_\ell$. Summarizing, we have

$$\begin{aligned}\mathcal{B}_\ell &= \{B \in \mathcal{B}_\ell \colon \Pi_{\ell-1}(B) \in \mathcal{B}_{\ell-1}\} \cup \{B \in \mathcal{B}_\ell \colon \Pi_{\ell-1}(B) \notin \mathcal{B}_{\ell-1}\} \\ &\subseteq (\mathcal{B}_{\ell-1} \times X_\ell) \cup \bigcup_{B' \notin \mathcal{B}_{\ell-1}} \{B'\} \times \mathcal{B}_1(B'). \end{aligned} \tag{59}$$

For bounding $\mathcal{B}_\ell$ we will thus estimate the sizes of $\mathcal{B}_{\ell-1}$ and $\mathcal{B}_1(B')$ for $B' \notin \mathcal{B}_{\ell-1}$. Let $X' := X_1 \dot\cup \ldots \dot\cup X_{\ell-1}$. Since (X, Y) is (ε, d, p)-dense we conclude from Proposition 6 that (X', Y) and (X_ℓ, Y) are $(\varepsilon_{\ell-1}, d, p)$-dense pairs since $\varepsilon(\ell+1) \leq \varepsilon_{\ell-1}$. Further, by the choice of $\xi_{\ell-1}$ we get $|X'|, |X_\ell| \geq n_1/(\ell+1) \geq \xi_{\ell-1}p^{\Delta-1}n$ since $n_1 \geq \xi p^{\Delta-1}n$ by assumption. Thus we can use the fact that Γ has property $P_{\ell-1}(\varepsilon'_{\ell-1}, \mu_{\ell-1}, \varepsilon_{\ell-1}, \xi_{\ell-1}, p(n))$ once on the tripartite subgraph induced on $X' \dot\cup Y \dot\cup Z$ in G and once on the tripartite subgraph induced on $X' \dot\cup Y \dot\cup X_\ell$ in G and infer that

$$|\mathcal{B}_{\ell-1}| \leq 2 \cdot \mu_{\ell-1} n_1^{\ell-2} = \frac{\mu}{2} n_1^{\ell-2}. \tag{60}$$

For estimating $|\mathcal{B}_1(B')|$ for $B' \notin \mathcal{B}_{\ell-1}$ let $Y' := N_Y^{\cap}(B')$. Observe that this implies that (Y', Z) and (X_ℓ, Y') are (ε_1, d, p)-dense pairs because $\varepsilon'_{\ell-1} \leq \varepsilon_1$, and that

$$|Y'| \geq (d-\varepsilon'_{\ell-1})^{\ell-1}p^{\ell-1}n_2 \geq (d-\varepsilon'_{\ell-1})^{\ell-1}p^{\ell-1} \cdot \xi p^{\Delta-\ell-1}n \overset{(57)}{\geq} \xi_1 p^{\Delta-1}n.$$

By (57) $|X_\ell|, |Z| \geq \xi p^{\Delta-1}n/(\ell+1) \geq \xi_1 p^{\Delta-1}n$. As Γ satisfies $P_1(\varepsilon'_1, \mu_1, \varepsilon_1, \xi_1, p(n))$ we conclude that

$$|\mathcal{B}_1(B')| \overset{(58b)}{=} |\operatorname{bad}^{G,1}_{\varepsilon',d,p}(X_\ell; Y', Z)| \leq \mu_1 n_1 \leq \frac{\mu}{2} n_1. \tag{61}$$

In view of (59), combining (60) and (61) gives

$$\left|\operatorname{bad}^{G,\ell}_{\varepsilon',d,p}(X_1, \ldots, X_\ell; Y, Z)\right| = |\mathcal{B}_\ell| \leq \frac{\mu}{2} n_1^{\ell-1} \cdot n_1 + n_1^{\ell-1} \cdot \frac{\mu}{2} n_1 = \mu n_1^\ell.$$

Because G was arbitrary this shows that Γ has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$. Thus $(\mathcal{P}_\ell)$ holds which finishes the proof of the inductive step. □

In the proof of Lemma 20 we now first partition the vertex set X, in which we count bad ℓ-sets, arbitrarily into T vertex sets of equal size. Lemma 38 then implies that for all $\ell' \in [\ell]$ there are not many bad ℓ'-sets that are crossing in this partition. It follows that only few ℓ-sets in X contain a bad ℓ'-set for some $\ell' \in [\ell]$ (recall that in Definition 16 for $\operatorname{Bad}^{G,\ell}_{\varepsilon,d,p}(X, Y, Z)$ such ℓ'-sets are considered). Moreover, if T is sufficiently large then the number of non-crossing ℓ-sets is negligible. Hence we obtain that there are few bad sets in total.

Proof of Lemma 20. Given $\Delta, \ell, d_0, \varepsilon'$ and μ let T be such that $\mu T \geq 2$, fix $\mu_{38} := \frac{1}{2}\mu/(\ell T^\ell)$. For $j \in [\ell]$ let ε_j be given by Lemma 38 with ℓ replaced by j and for Δ, d_0, ε', and μ_{38} and set $\varepsilon_{38} := \min_{j \in [\ell]} \varepsilon_j$. Define $\varepsilon := \varepsilon_{38}/(T+1)$. Now, in Lemma 20 let ξ be given by the adversary for this ε. Set $\xi_{38} := \xi/(T+1)$, and let c be given by Lemma 38 for this ξ_{38}.

Let $\Gamma = \mathcal{G}_{n,p}$ with $p \geq c(\frac{\log n}{n})^{1/\Delta}$. Then a.a.s. the graph Γ satisfies the statement in Lemma 38 for parameters $j \in [\ell]$, Δ, d_0, ε', μ_{38}, and ξ_{38}. Assume that Γ has this property for all $j \in [\ell]$. We will show that it then also satisfies the statement in Lemma 20.

Indeed, let G and X, Y, Z be arbitrary with the properties as required in Lemma 20. Let $X = X_1 \dot\cup \ldots \dot\cup X_T$ be an arbitrary partition of X with $|X_i| \geq \lfloor \frac{n_1}{T} \rfloor$. We will first show that there are not many bad crossing ℓ-sets with respect to this partition, i.e., we will bound the size of $\operatorname{Bad}^{G,\ell}_{\varepsilon',d,p}(X_1, \ldots, X_T; Y, Z)$. By definition

$$\left|\operatorname{Bad}^{G,\ell}_{\varepsilon',d,p}(X_1, \ldots, X_T; Y, Z)\right| \leq \sum_{j \in [\ell]} \left|\operatorname{bad}^{G,j}_{\varepsilon',d,p}(X_1, \ldots, X_T; Y, Z)\right| \cdot n_1^{\ell-j}.$$

Now, fix $j \in [\ell]$ and an index set $\{i_1, \ldots, i_j\} \in \binom{[T]}{j}$ and consider the induced tripartite subgraph $G' = (X' \dot\cup Y \dot\cup Z, E')$ of G with $X' = X_{i_1} \dot\cup \ldots \dot\cup X_{i_j}$. Observe that $|Y| \geq \xi_{38} p^{\Delta-j-1}n$, $|Z| \geq \xi_{38} p^{\Delta-1}n$, and $n'_1 := |X'| \geq j\lfloor n_1/T \rfloor \geq \xi_{38} p^{\Delta-1}n$. By definition $\varepsilon(T+1)/j \leq \varepsilon_{38} \leq \varepsilon_j$ and so by Proposition 6 the pair (X', Y) is (ε_j, d, p)-dense. Thus, because Γ satisfies the statement in Lemma 38 for parameters j, Δ, d, ε', μ_{38}, and ξ_{38} we have that G' satisfies

$$\left|\operatorname{bad}^{G',j}_{\varepsilon',d,p}(X_{i_1},\dots,X_{i_j};Y,Z)\right| \le \mu_{38}(n'_1)^j.$$

As there are $\binom{T}{j}$ choices for the index set $\{i_1,\dots,i_j\}$ this implies

$$\left|\operatorname{bad}^{G,j}_{\varepsilon',d,p}(X_1,\dots,X_T;Y,Z)\right| \le \binom{T}{j}\mu_{38}(n'_1)^j \le T^j\mu_{38}n_1^j,$$

and thus

$$\left|\operatorname{Bad}^{G,\ell}_{\varepsilon',d,p}(X_1,\dots,X_T;Y,Z)\right| \le \sum_{j\in[\ell]} T^j\mu_{38}n_1^j \cdot n_1^{\ell-j} \le \frac{1}{2}\mu n_1^\ell.$$

The number of ℓ-sets in X that are not crossing with respect to the partition $X = X_1\dot\cup\dots\dot\cup X_T$ is at most $T\binom{n_1/T}{2}\binom{n_1}{\ell-2} \le \frac{1}{T}n_1^\ell \le \frac{1}{2}\mu n_1^\ell$ and so we get $|\operatorname{Bad}^{G,\ell}_{\varepsilon',d,p}(X,Y,Z)| \le \mu n_1^\ell$. □

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