AN APPROXIMATE VERSION OF THE TREE PACKING CONJECTURE

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Abstract. We prove that for any pair of constants $\varepsilon > 0$ and Δ and for n sufficiently large, every family of trees of orders at most n, maximum degrees at most Δ, and with at most $\binom{n}{2}$ edges in total packs into $K_{(1+\varepsilon)n}$. This implies asymptotic versions of the Tree Packing Conjecture of Gyárfás from 1976 and a tree packing conjecture of Ringel from 1963 for trees with bounded maximum degree. A novel random tree embedding process combined with the nibble method forms the core of the proof.

1. Introduction

Graph packing is a concept that generalises the notion of graph embedding to finding several subgraphs in a host graph instead of just one. A family of graphs $\mathcal{H} = (H_1, \ldots, H_k)$ is said to *pack* into a graph G if there exist pairwise edge-disjoint copies of $H_1, \ldots, H_k$ in G, where we allow $H_i = H_j$ for $i \neq j$. Many classical problems in Graph Theory can be stated as packing problems. For example, Mantel's Theorem can be formulated by saying that if G is an n-vertex graph with less than $\binom{n}{2} - \frac{n^2}{4}$ edges, then the family (K_3, G) packs into K_n. pack

Among the best known packing problems, let us for example mention a conjecture of Bollobás, Catlin, and Eldridge [7, 10] that any two n-vertex graphs H_1, H_2 with $(\Delta(H_1) + 1)(\Delta(H_2) + 1) \leq n + 1$ pack into K_n. The asymptotic solution of this conjecture was reported by Gábor Kun around 2006.

Another beautiful packing conjecture was posed by Gyárfás (see [14]) in 1976 and concerns trees. This conjecture is referred to as the Tree Packing Conjecture.

Conjecture 1. *Any family $(T_1, T_2, \ldots, T_n)$ of trees, $j \in [n]$ with $v(T_j) = j$, packs into K_n.*

A related conjecture of Ringel [20] dating back to 1963 deals with packing many copies of the same tree.

Conjecture 2. *Any $2n + 1$ identical copies of any tree of order $n + 1$ pack into K_{2n+1}.*

Note that both conjectures are best possible in the sense that they deal with *perfect packings*, i.e. the total number of edges packed equals the number of edges in the host graph. perfect packings

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Moreover, the fact that two spanning stars do not pack into the complete graph shows that further requirements than this necessary condition are needed.

A slightly outdated survey on packings of trees is by Hobbs [15]. Here, we recall only the most important results concerning the two conjectures above.

A packing of many of the small trees from Conjecture 1 was obtained by Bollobás [6], who showed that any family of trees $T_1, \ldots, T_s$ with $v(T_i) = i$ and $s < n/\sqrt{2}$ can be packed into K_n. He also observed that the validity of the famous conjecture of Erdős and Sós would imply that one can improve the bound to $s < \frac{1}{2}\sqrt{3}n$. The solution of the Erdős–Sós Conjecture for large trees was announced by Ajtai, Komlós, Simonovits, and Szemerédi in the early 1990s. In a similar direction, Yuster [24] proved that any sequence of trees $T_1, \ldots, T_s$, $s < \sqrt{5/8}n$ can be packed into $K_{n-1,n/2}$. This improves upon a result of Caro and Roditty [8] and is related to a conjecture of Hobbs, Bourgeois and Kasiraj [16] (see Conjecture 43 in Section 9). Moreover, a result of Caro and Yuster [9] implies that one can pack perfectly a family of trees into a complete graph K_n, provided that the trees are very small compared to n.

Packing the large trees of Conjecture 1 is a much more challenging task. Balogh and Palmer [3] proved that any family of trees $T_n, T_{n-1}, \ldots, T_{n-\frac{1}{10}n^{1/4}}$, $v(T_i) = i$ packs into K_{n+1}.

Surprisingly few results are known for special classes of tree families. It was proved already in [14] that Conjecture 1 holds when all the trees are stars and paths. Dobson [12] and Hobbs, Bourgeois, and Kasiraj [16] consider packings of trees with small diameter. Moreover, Fishburn [13] proved that it is at least possible to adequately match up the degrees of the trees $T_1, \ldots, T_n$ appearing in Conjecture 1: If we add $n - i$ isolated vertices to the tree T_i and let $d_{i,1}, \ldots, d_{i,n}$ denote the degree sequence of the resulting forest, then there are permutations $\pi_1, \ldots, \pi_n$ such that $\sum_i d_{i,\pi_i(j)} = n - 1$ for all $j \in [n]$.

Our main result, Theorem 3, deals with almost perfect packings of trees of bounded-degree trees into a complete graph. It implies an asymptotic solution of Conjecture 1 and Conjecture 2 for trees of bounded maximum degree.

Theorem 3. *For any $\varepsilon > 0$ and any $\Delta \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ the following holds. Any family of trees $\mathcal{T} = (T_i)_{i\in[k]}$ such that T_i has maximum degree at most Δ and order at most n for each $i \in [k]$, and $\sum_{i\in[k]} e(T_i) \leq \binom{n}{2}$ packs into $K_{(1+\varepsilon)n}$.*

We emphasise that, unlike Conjectures 1 and 2, this theorem only requires the trees to satisfy the obvious upper bound on the total number of edges.

2. Outline of the proof

A very natural approach to pack the trees $T_1, \ldots, T_k$ into $K_{(1+\varepsilon)n}$ is to use a *random embedding process*:

- Start with $G = K_{(1+\varepsilon)n}$. Successively build a packing of the trees, edge by edge, starting with an arbitrary edge in an arbitrary tree and then following the structure of the trees (it is not important which order exactly we choose, but one example would be to use a breadth-first search order; it also should not matter here whether we embed tree by tree, or first embed a few edges of one tree, then a few edges of another tree, and so on, and then return to the first tree).
- In one step of this procedure, when we want to embed an edge xy of some tree T_i, with x already embedded to $h(x)$, choose a random neighbour $v \in V(G)$ of $h(x)$ which is not contained in the set $U_i \subseteq V(G)$ of T_i-images so far, and embed y to $h(y) = v$.

- After embedding xy, remove the edge uv from G and add v to U_i.

Clearly, this process produces a proper packing unless we get stuck, that is, unless the set $\mathrm{N}_G\big(h(x)\big) \setminus U_i$ gets empty. But if, during the evolution, the host graph G always remains sufficiently quasirandom, then with high probability $\mathrm{N}_G\big(h(x)\big) \setminus U_i$ should not get empty (because $e(K_{(1+\varepsilon)n}) - \sum_{i\in[k]} e(T_i) \geq \varepsilon n^2$ implies that G has positive density throughout).

We believe that the host graph does indeed remain quasirandom in this process. Unfortunately, however, graph processes like this are extremely difficult to analyse because of their dynamically evolving environment in each step. A prominent example illustrating the occurrent complexity is that of the random triangle-free graph process: it took more than a decade after the introduction of this process until Bohman [5] gave a detailed analysis. Nonetheless, a related random construction of triangle-free graphs was effectively analysed already much earlier by Kim [18]. This construction was easier to handle because it uses a *nibble* approach.

The nibble method bypasses the difficulties originating from the dynamics of random graph processes by proceeding in constantly many rounds and updating the environment only after each round. This method was used by Rödl [21] to prove the existence of asymptotically optimal Steiner systems (see [1] for an exposition). Since then it has served as an important ingredient for several breakthroughs in combinatorics. In the context of packing problems the nibble method is also used in Kun's announced result on the Bollobás–Catlin–Eldridge Conjecture. In our setting the nibble method amounts to the following approach for embedding $T_1, \dots, T_k$ into $G = K_{(1+\varepsilon)n}$:

- Pack the trees in r rounds (with r big but constant). For this purpose, cut each tree T_i into small equally sized forests F_i^j with $j \in [r]$ and use in each round exactly one forest of each tree.
- In round j, for each i construct a *random homomorphism* from the forest F_i^j to G as follows. First, randomly embed some forest vertex x, then choose a neighbour v uniformly at random in $\mathrm{N}_G(h(x)) \setminus U_i$, where the *forbidden set* $U_i \subseteq V(G)$ are vertices used by T_i in previous rounds. Then continue with the next vertex in F_i^j, following again the structure of T_i.
- After round j, delete all the edges from G to which some forest edges were mapped in this round and add to U_i all images of vertices of F_i^j.

In other words, the difference between this approach and the random process described above is that the host graph G and the sets U_i are not updated after the embedding of each single vertex, but only at the end of each round.

Naturally, this procedure will not produce a proper packing of the trees: Firstly, it will create *vertex collisions*, that is, two vertices of some tree T_i are mapped to the same vertex of the host graph G. Secondly, there will be *edge collisions*, that is, two edges of different trees are mapped to the same edge. However, since all forests F_i^j are small this will create only a small proportion of vertex and edge collisions in each round, and the updates at the end of each round guarantee that there are no collisions between rounds. So our hope is that vertex and edge collisions can be *corrected* at the end.

The difficulty with this construction of random homomorphisms though is that it still leads to lots of small dependencies between embedded vertices, which we found difficult to control. We remark that techniques recently developed by Barber and Long [4] allow to handle these dependencies and show that after each round the host graph is indeed quasirandom. However, applying these techniques to our setting and modifying them so that they also give all the

Theorem 3; proof in Section 3	
⇑	⇑
Packing with a small number of collisions Lemma 6; proof in Section 5	Correcting collisions Lemma 7; proof in Section 6
⇑	
One round: Lemma 21; proof in Section 8. The proof builds on properties of limping homomorphisms derived in Section 7.	

Table 1. Outline of the proof of Theorem 3.

additional properties that we need (such as that there are few collisions; see Lemma 21) would require substantial additional work and probably lead to a significantly longer proof.

Our approach (which was developed before the techniques of Barber and Long) is different. We instead use the following construction of random homomorphisms in round j of the nibble approach described above, which we call *limping homomorphisms*:

- For each i, call one of the colour classes of F_i^j the set of *primary vertices*, and the other the set of *secondary vertices*. Now first map all primary vertices randomly to vertices of $V(G) \setminus U_{i,s}$. Then map each secondary vertex randomly into the common $(G - U_{i,s})$-neighbourhood of the images of its forest neighbours – unless this common neighbourhood is smaller than expected, in which case we simply *skip* this secondary vertex.

Observe that, if our host graph is quasirandom (and the forest has bounded degree), then most common neighbourhoods are big and hence few vertices will get skipped. Of course in this random construction we still have dependencies. But since these occur only between vertices with distance at most 2 in the trees, we now can control them and prove that the host graph is quasirandom after each round and that we get few collisions.

It remains to correct the vertex collisions and edge collisions (and take care of skipped vertices and connections between the different forests of each tree). Before starting the described embedding rounds we put aside $\varepsilon n/2$ reserve vertices of $K_{(1+\varepsilon)n}$. Our random homomorphisms (constructed on the remaining vertices) also guarantee that the collisions are sufficiently well distributed over the host graph so that a simple greedy strategy can be used to relocate vertices in collisions to the reserved vertices, thus obtaining a proper packing of $T_1, \ldots, T_k$.

The organisation of the proof is given in Table 1.

3. Proof of the main theorem (Theorem 3)

Theorem 3 assumes little on the orders of trees $\mathcal{T}$ to be packed. However, as we show as a first step of the proof of Theorem 3, there is a simple transformation of an arbitrary such family into a family of trees whose orders are (with possibly one exception) more than $n/2$. The definition of an (n, Δ)-tree family below formalises this. The fact that the subsequent family is a family of trees of linear orders is crucial for our proof.

Definition 4. *A family of trees $\mathcal{T}$ is called* (n, Δ)-tree family, *if all trees in $\mathcal{T}$ have order at most n, maximum degree Δ and the total number of edges is at most $\binom{n}{2}$. Further all but at most one tree from $\mathcal{T}$ have order more than $n/2$. Observe that the upper bound on the total* (n, Δ)-tree family *number of edges and the lower bound on the number of vertices imply that such a family must contain less than $2n$ trees.*

Indeed, it is easy to show that we can transform any family $\mathcal{T}$ satisfying the requirements of Theorem 3 into an (n, Δ)-tree family (see below).

Our next step will be to relax the requirements of a packing in the sense that we allow an exceptional set R of vertices *not* to be embedded. At the same time, we control both the size of R as well as the number of neighbours of R that get embedded into the same vertex.

Definition 5 (Almost packing)**.** *Let $\mathcal{F} = \{F_i\}_{i\in[k]}$ be a family of graphs. For a graph G, a family of sets $\{R_i\}_{i\in[k]}$ with $R_i \subseteq V(F_i)$ and a family of maps $\{h_i\}_{i\in[k]}$ with $h_i : V(F_i) \setminus R_i \to V(G)$ we say that $\{h_i, R_i\}_{i\in[k]}$ is an ℓ-*almost packing *of $\mathcal{F}$ into G if* ℓ-almost packing

(a) *$\{h_i\}_{i\in[k]}$ is a packing of the family $\{F_i - R_i\}_{i\in[k]}$ into the graph G,*
(b) *we have $|R_i| \le \ell$ for each $i \in [k]$, and*
(c) *for each $v \in V(G)$, $\sum_{j\in[k]} |\{x \in h_j^{-1}(v) : \exists xy \in E(F_j) \text{ such that } y \in R_j\}| \le \ell$.*

*We say that $\mathcal{F}$ ℓ-*almost packs *into a graph G if there exist $\{R_i\}_{i\in[k]}$ and $\{h_i\}_{i\in[k]}$ such that $\{h_i, R_i\}_{i\in[k]}$ is an ℓ-almost packing of $\mathcal{F}$ into G.* ℓ-almost packs

Using this concept, the next two lemmas state that we can always find an almost packing, and that an almost packing can always be turned into a perfect packing.

Lemma 6 (Almost packing lemma)**.** *For any $\varepsilon > 0$ and any $\Delta \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ the following holds. Any (n, Δ)-tree family (εn)-almost packs into $K_{(1+\varepsilon)n}$.*

Lemma 7 (Correction lemma)**.** *Let $\varepsilon > 0$ be arbitrary, and let $\mathcal{T}$ be a family of trees of maximum degrees at most Δ. Suppose that $|\mathcal{T}| \le 2m$, and that $\mathcal{T}$ has an $(\frac{\varepsilon^2 m}{64\Delta^2})$-almost packing into K_m. Then $\mathcal{T}$ packs into $K_{(1+\varepsilon)m}$.*

Lemmas 6 and 7 are proven in Section 5 and Section 6, respectively. Based on these two lemmas, it is now an easy task to prove our main theorem.

Proof of Theorem 3. Let $\varepsilon > 0$ and $\Delta \in \mathbb{N}$ be given. We define $\varepsilon_{\mathrm{L6}} = \varepsilon^2/(256\Delta^2)$ and apply Lemma 6 with parameters $\varepsilon_{\mathrm{L6}}$ and Δ to obtain n_0.

Now we consider a family $\mathcal{T}$ of trees satisfying the requirements of the theorem. If $\mathcal{T}$ contains two trees F_1 and F_2 of orders at most $n/2$, then we can replace them by a single tree of order $v(F_1) + v(F_2) - 1$ that is obtained by identifying a leaf of F_1 and a leaf of F_2. Repeating this step, we arrive at a situation where all but at most one tree in $\mathcal{T}$ have order more than $n/2$. This procedure does not change the maximum degree of the trees nor their total number of edges. Hence we have obtained an (n, Δ)-tree family $\mathcal{T}'$. Observe that now it suffices to pack $\mathcal{T}'$ into $K_{(1+\varepsilon)n}$. Feeding the family $\mathcal{T}'$ to Lemma 6, we obtain an $(\varepsilon_{\mathrm{L6}} n)$-almost packing of $\mathcal{T}'$ into $K_{(1+\varepsilon_{\mathrm{L6}})n}$.

Now we set $m = (1 + \frac{\varepsilon}{4})n \ge (1 + \varepsilon_{\mathrm{L6}})n$ and $\varepsilon_{\mathrm{L7}} = \varepsilon/2$. Since

$$\varepsilon_{\mathrm{L6}} n = \frac{\varepsilon^2 n}{256\Delta^2} \le \frac{\varepsilon^2 m}{256\Delta^2} = \frac{(\varepsilon_{\mathrm{L7}})^2 m}{64\Delta^2},$$

the family $\mathcal{T}'$ also has an $\frac{(\varepsilon_{\mathrm{L7}})^2 m}{64\Delta^2}$-almost packing into K_m. As the number of trees in $\mathcal{T}'$ is bounded by $2n \le 2m$, we can apply Lemma 7 with parameter $\varepsilon_{\mathrm{L7}}$ and obtain a packing of $\mathcal{T}'$ into $K_{(1+\varepsilon_{\mathrm{L7}})m}$. Since $(1 + \varepsilon_{\mathrm{L7}})m = (1 + \frac{\varepsilon}{2})(1 + \frac{\varepsilon}{4})n \le (1 + \varepsilon)n$, this completes the proof. □

4. Notation and preliminaries

4.1. **Basic notation.** Let $G = (V, E)$ be a graph and $V' \subseteq V$ and $E' \subseteq E$. We use the minus symbol to denote both the removal of vertices and edges from a graph, i.e., $G - V' = (V \setminus V', E \cap \binom{V'}{2})$, $G - E' = (V, E \setminus E')$. For vertex sets $U, W \subseteq V$ we let $e(U)$ denote the number of edges with both endvertices in U and let $e(U, W) = |\{(u, w) \in U \times W : uw \in E\}|$. Here, the edges with both endvertices in $U \cap W$ are counted twice. The *neighbourhood* of vertices $v_1, \ldots, v_k$ in the graph G is defined by $\mathrm{N}_G(v_1, \ldots, v_k) = \{u \in V : uv_1, uv_2, \ldots, uv_k \in E\}$. The *codegree* of $v_1, \ldots, v_k$ is then $\mathrm{codeg}_G(v_1, \ldots, v_k) = |\mathrm{N}_G(v_1, \ldots, v_k)|$. In the special case $k = 1$, this quantity is called the *degree* of v_1, $\deg_G(v_1) = \mathrm{codeg}_G(v_1)$. We drop the subscript when the graph G is understood from the context. The *density* of G is defined as $|E|/\binom{|V|}{2}$.

Denote by $\mathrm{dist}(x, y)$ the length of a shortest path between x and y. Here, the distance between vertices lying in different components is defined to be $+\infty$. For two sets U, W of vertices of the same graph we write $\mathrm{dist}(U, W) = \min_{u \in U, w \in W} \mathrm{dist}(u, w)$. In particular, we will use this notation when U and W are edges (i.e., vertex sets of size two).

By a d-th *power* of a graph $G = (V, E)$ we mean its distance-power, that is, a loopless graph, denoted G^d, on the vertex set V where two vertices u and v are adjacent if and only if $\mathrm{dist}_G(u, v) \leq d$. We refer to the cases $d = 2$ and $d = 3$ as *squares* and *cubes*.

Finally, the set of components of G is denoted by $\mathrm{Comp}(G)$.

$e(U), e(U, W)$
neighbourhood
codegree
$\mathrm{codeg}_G(v_1, \ldots, v_k)$
degree
$\deg_G(v)$
density
$\mathrm{dist}(x, y)$
$\mathrm{dist}(U, W)$
power
squares
cubes
$\mathrm{Comp}(G)$

Generally, we shall use letters x, y, and z to denote vertices in trees and forests that we pack. Letters u, v, and w will be used to denote the vertices in the host graph into which we pack.

4.2. **Quasirandomness.** Here, we recall the concept of quasirandom graphs, which goes back to Thomason [23], and Chung, Graham, and Wilson [11].

Definition 8 (Quasirandom graph)**.** *We say that a graph G of order n is* α-quasirandom of density d *if for every $B \subseteq V(G)$ we have $e(B) = d\binom{|B|}{2} \pm \alpha n^2$ edges.*

α-quasirandom of density d

Since $e(A, B) = e(A \cup B) + e(A \cap B) - e(A \setminus B) - e(B \setminus A)$, this definition immediately implies that in a quasirandom graph we also have control over the number of edges between two vertex sets.

Observation 9. *In an α-quasirandom graph G on n vertices, for each pair of sets $A, B \subseteq V(G)$ we have $e(A, B) = d|A||B| \pm 4\alpha n^2 \pm n$.*

Our next easy lemma asserts that induced subgraphs of quasirandom graphs inherit quasirandomness and density.

Lemma 10. *If G is α-quasirandom of density d and order at most $\frac{3}{2}n$, and a set $V' \subseteq V(G)$ has size $|V'| \geq \varepsilon n$, then $G[V']$ is a $(3\alpha/\varepsilon^2)$-quasirandom graph of density $d \pm 3\alpha/\varepsilon^2$.*

Proof. For any $B \subseteq V'$ we have

$$e_G(B) = d\binom{|B|}{2} \pm \alpha(\tfrac{3}{2}n)^2 = d\binom{|B|}{2} \pm \alpha \cdot (\tfrac{3}{2})^2 \cdot \frac{|V'|^2}{\varepsilon^2} = d\binom{|B|}{2} \pm 3\frac{\alpha}{\varepsilon^2}|V'|^2 .$$

Hence $G[V']$ is a $(3\alpha/\varepsilon^2)$-quasirandom graph of density $d \pm 3\alpha/\varepsilon^2$. □

If $G = (V, E)$ is a quasirandom graph with density d, we expect that in G most sets of p vertices have a neighbourhood of order roughly $d^p|V|$. So, we say that a set $\{v_1, \ldots, v_p\} \subseteq V$ is γ-*bad*, if

γ-bad

$$|\mathrm{N}(v_1,\dots,v_p)| \neq (1\pm\gamma)d^p|V|.$$

The next lemma states that most vertices of a quasirandom graph are contained in few bad p-sets. We use the following definitions. For a vertex $v \in V$, let $\mathrm{bad}_{\gamma,p}(v)$

$$\mathrm{bad}_{\gamma,p}(v) = \left|\left\{B \in \tbinom{V}{p-1} : B\cup\{v\} \text{ is } \gamma\text{-bad}\right\}\right| .$$

(In particular, $\mathrm{bad}_{\gamma,1}(v) \in \{0,1\}$, depending on whether $\deg(v) = (1\pm\gamma)d|V|$, or not.) Set $\mathrm{BAD}_{\gamma,\Delta}(G)$

$$\mathrm{BAD}_{\gamma,\Delta}(G) = \left\{v \in V \ :\ \mathrm{bad}_{\gamma,p}(v) > \gamma\tbinom{|V|}{p-1} \text{ for some } p \in [\Delta]\right\} .$$

Lemma 11. *For every $\gamma > 0$ and every integer $\Delta \geq 1$ there is $\alpha > 0$ such that if $G = (V,E)$ is an α-quasirandom graph of density $d \geq \gamma$ and order n, then $|\mathrm{BAD}_{\gamma,\Delta}(G)| \leq \gamma n$.*

Proof. Let $\alpha \leq 1/(10\Delta^2)$ be small enough so that for $\beta = \frac{1}{2}\sqrt{\alpha}$ and $\gamma_1 \leq \dots \leq \gamma_\Delta$ defined by

$$\gamma_p = \begin{cases} \sqrt{\frac{10\beta}{d}} & p = 1 \\ \sqrt{4p\gamma_{p-1} + \frac{20p\beta}{d^p\gamma_{p-1}}} & 1 < p \leq \Delta\,, \end{cases}$$

we have $\gamma_\Delta \leq \min\{\gamma/\Delta, 1/2\}$. Testing over two-element sets in Definition 8, we get that if $n < \max\{2\Delta, \beta^{-1}\}$ then G is either complete or empty. Hence we may assume that $n \geq \max\{2\Delta, \beta^{-1}\}$ in the following.

We prove by induction on p that

$$\text{at most } \gamma_p n \text{ vertices } v \text{ of } G \text{ satisfy } \mathrm{bad}_{\gamma_p,p}(v) > \gamma_p\tbinom{n}{p-1}. \tag{1}$$

Let us first consider the base case $p = 1$. Let V^+ be the set of vertices v with $\deg(v) > (1+\gamma_1)dn$. We have $e(V^+,V) > |V^+|(1+\gamma_1)dn$. But since G is α-quasirandom we have by Observation 9 that $e(V^+,V) \leq d|V^+|n + 4\alpha n^2 + n \leq d|V^+|n + 5\beta n^2$. Putting these bounds together, we get $|V^+| < 5\beta n/(d\gamma_1)$. Similarly for the set V^- of vertices v with $\deg(v) < (1-\gamma_1)dn$ we have $|V^-| < 5\beta n/(d\gamma_1)$. Thus there are at most $10\beta n/(d\gamma_1) = \gamma_1 n$ vertices v with $\mathrm{bad}_{\gamma_1,1}(v) = 1 > \gamma_1\binom{n}{0}$.

Now consider $p > 1$ and assume that (1) holds for $p-1 \geq 1$. The number of γ_{p-1}-bad sets in $\binom{V}{p-1}$ is

$$\frac{1}{p-1}\sum_{v\in V}\mathrm{bad}_{\gamma_{p-1},p-1}(v) \leq \frac{1}{p-1}\left(\gamma_{p-1}n\binom{n}{p-2} + n\gamma_{p-1}\binom{n}{p-2}\right) \leq 4\gamma_{p-1}\binom{n}{p-1}\,, \tag{2}$$

where we used $n/2 \leq n-p+1$. Fix an arbitrary set $\{v_1,\dots,v_{p-1}\}$ in $\binom{V}{p-1}$ that is not γ_{p-1}-bad. Hence for $W = \mathrm{N}(v_1,\dots,v_{p-1})$ we have $|W| = (1\pm\gamma_{p-1})d^{p-1}n$. Let V^+ be the set of vertices $v \in V\setminus\{v_1,\dots,v_{p-1}\}$ with $|\mathrm{N}(v)\cap W| > (1+\gamma_{p-1})d|W|$. We have $|V^+|(1+\gamma_{p-1})d|W| < e(V^+,W) \leq d|V^+||W| + 5\beta n^2$ and hence $|V^+| < 5\beta n^2/(\gamma_{p-1}d|W|) \leq 5\beta n^2/(\gamma_{p-1}d\frac{1}{2}d^{p-1}n) = 10\beta n/(d^p\gamma_{p-1})$. Similarly, for the set V^- of vertices v such that $|\mathrm{N}(v)\cap W| < (1-\gamma_{p-1})d|W|$ we have $|V^-| < 10\beta n/(d^p\gamma_{p-1})$. Let v be an arbitrary vertex in $V\setminus(V^+\cup V^-\cup\{v_1,\dots,v_{p-1}\})$. Then

$$|\mathrm{N}(v,v_1,\dots,v_{p-1})| = (1\pm\gamma_{p-1})d|W| = (1\pm\gamma_p)d^pn\,,$$

and therefore $\{v,v_1,\dots,v_{p-1}\}$ is not γ_p-bad. Hence, by (2), the number of γ_p-bad p-tuples is at most

$$\left(4\gamma_{p-1}\binom{n}{p-1}\right)n + \binom{n}{p-1}\cdot 2\frac{10\beta n}{d^p\gamma_{p-1}} = \frac{\gamma_p^2}{p}n\binom{n}{p-1}\,.$$

Consequently, for at most $\gamma_p n$ vertices $v \in V$, we have $\mathrm{bad}_{\gamma_p,p}(v) > \gamma_p\binom{n}{p-1}$. This gives (1).

The bound $|\mathrm{BAD}_{\gamma,\Delta}(G)| \le \gamma n$ follows by summing (1) over $p = 1, \dots, \Delta$. □

As our next lemma shows, this implies that we need to delete only few vertices from a quasirandom graph to obtain a graph G in which $\mathrm{BAD}_{\gamma,\Delta}(G) = \emptyset$.

Definition 12 (Superquasirandom graph)**.** *We say that a graph G is* (γ,Δ)-superquasirandom *if we have* $\mathrm{BAD}_{\gamma,\Delta}(G) = \emptyset$.

(γ,Δ)-superquasirandom

Lemma 13. *For every $\gamma > 0$ and every integer $\Delta \ge 1$ there is $\alpha > 0$ such that if G is an α-quasirandom graph of density $d > \gamma$ and order m, then G contains an induced (γ,Δ)-superquasirandom subgraph of order at least $(1-\gamma)m$ and density $d \pm \gamma$.*

Proof. We can assume that $\gamma < \frac{1}{2}$. Let α' be given by Lemma 11 for input parameters $\gamma' = \gamma d^{\Delta}/200$ and Δ, and set $\alpha = \min\{\alpha', d\gamma/(800\cdot 2^{\Delta})\}$. Now suppose that G is an α-quasirandom graph of density d and order m. By Lemma 11, we have $|\mathrm{BAD}_{\gamma',\Delta}(G)| \le \gamma' m$.

We claim that the induced subgraph G' on the vertex set $V' = V \setminus \mathrm{BAD}_{\gamma',\Delta}(G)$ satisfies the assertion of the lemma. Indeed, $|V'| \ge (1-\gamma')m$ and since G is α-quasirandom the density d' of G' satisfies $d' = (d\binom{|V'|}{2} \pm \alpha n^2)/\binom{|V'|}{2} = d \pm 4\alpha = d \pm \gamma$. It remains to show that $\mathrm{BAD}_{\gamma,\Delta}(G') = \emptyset$. By the definition of G', for each $v \in V'$ and $p \le \Delta$ all but at most $\gamma'\binom{|V|}{p-1}$ sets $\{v_1,\dots,v_{p-1}\} \in \binom{V'}{p-1}$ are such that $\{v, v_1, \dots, v_{p-1}\}$ is not γ'-bad in G. But such sets $\{v, v_1,\dots,v_{p-1}\}$ are not γ-bad in G' either because

$$\begin{aligned} |\mathrm{N}_{G'}(v,v_1,\dots,v_{p-1})| &= |\mathrm{N}_G(v,v_1,\dots,v_{p-1})| \pm |\mathrm{BAD}_{\gamma',\Delta}(G)| = (1\pm\gamma')d^p m \pm \gamma' m \\ &= (1 \pm \tfrac{1}{100}\gamma)d^p m = (1\pm\tfrac{1}{100}\gamma)(d' \pm 4\alpha)^p(1\pm\gamma')|V'| \\ &= \big(1 \pm 10(\tfrac{1}{100}\gamma + 2^p\cdot 4\alpha\tfrac{1}{d'} + \gamma')\big)(d')^p|V'| = (1\pm\gamma)(d')^p|V'|\,, \end{aligned}$$

where we use $2^p \cdot 4\alpha\frac{1}{d'} \le \gamma/100$. Hence $\mathrm{BAD}_{\gamma,\Delta}(G') = \emptyset$. □

The next easy lemma asserts that very dense graphs are quasirandom.

Lemma 14. *For any $\alpha > 0$ there exist $n_0 = n_{L14}(\alpha)$ such that the following holds for any $n \ge n_0$. Suppose that G was obtained from the complete graph K_n by deleting at most n edges. Then G is α-quasirandom.*

4.3. **Homomorphisms.** Let H and G be graphs. A *homomorphism* h from H to G is an edge-preserving map from $V(H)$ to $V(G)$, i.e., for every $xy \in E(H)$ we have $h(x)h(y) \in E(G)$. By $h\colon H \to G$ or simply $H \to G$ we denote the fact that there is a homomorphism h from H to G. Moreover, we write $V(h) = \{h(v)\colon v \in V(H)\} \subseteq V(G)$ for the image of h, and $E(h) = \{h(x)h(y)\colon xy \in E(H)\} \subseteq E(G)$ for the image of the edges of H.

homomorphism

$V(h)$

$E(h)$

We say that a map h is a *partial homomorphism* of H to G if there exists a set $Y \subseteq V(H)$ such that h is a homomorphism of $H - Y$ to G. The set Y is called vertices *skipped* by h. We define $V(h) = \{h(v)\colon v \in V(H) - Y\} \subseteq V(G)$, and $E(h)$ analogously. We denote the fact that h is a partial homomorphism by $h : H \rightsquigarrow G$.

partial homomorphism

skipped

$V(h)$

$E(h)$

$H \rightsquigarrow G$

In the language of homomorphisms, a packing of a family $(H_1,\dots,H_k)$ of graphs into G is a family of injective homomorphisms $(h_i\colon H_i \to G)_{i\in[k]}$ with mutually disjoint images of the edge sets.

Let $(h_i)_{i\in[k]}$ be a family of homomorphisms $h_i\colon H_i \to G$ with $i \in [k]$ (we assume that the graphs H_i live on different vertex sets). Then the *union* $\bigcup_{i\in[k]} h_i$ of $(h_i)_{i\in[k]}$ is the map $h\colon \bigcup_{i\in[k]} V(H_i) \to G$ defined by $h(x) = h_i(x)$ for all vertices $x \in V(H_i)$ and all $i \in [k]$.

4.4. **Trees.** The pair (F,X) is a *rooted forest* if F is a forest and $X \subseteq V(F)$ contains exactly one vertex of every tree $C \in \mathrm{Comp}(F)$ of F, which we call *root* of C. If F is a tree with root x then we also write (F,x) for $(F,X)=(F,\{x\})$ and say that (F,x) is a *rooted tree*. In a rooted forest (F,x) we can speak of *children*, *parents*, *ancestors*, and *descendants* of vertices. For a vertex y, we let $F(y)$ be the maximal subtree of F with root y.

rooted forest
root
rooted tree
children
parents
ancestors
descendants

4.4.1. *Cutting trees.* The central notion of this section is that of a ϱ-balanced r-level partition defined below.

Definition 15 (balanced level partition)**.** *Given a rooted tree* (T,x)*, we say that a partition* $\mathcal{P}=(L^1,\dots,L^r)$ *of* $V(T)$ *is a* ϱ-balanced r-level partition *if*

(a) $|L^i| = (1\pm\varrho/2)\frac{v(T)}{r}$ *for every* $i\in[r]$*, and*
(b) *for each* $i\in[r]$*, the parent of each vertex in* L^i *lies in the set* $\bigcup_{j\le i} L^j$*.*

ϱ-balanced
r-level
partition

The forest $T[L^i]$ *is called* level i *of the partition* $\mathcal{P}$*. For a vertex* y *of* $T[L^i]$ *or a tree* $C\in \mathrm{Comp}(T[L^i])$ *we say that* y *or* C *are in level* i *of* $\mathcal{P}$*.*

level

The following lemma states that bounded-degree trees have balanced level partitions with a bounded number of components in each level.

Lemma 16. *Let* (T,x) *be a rooted tree with maximum degree at most* Δ *and* $v(T)\ge \frac{4\Delta r}{\varrho}$ *with* $0<\varrho<\frac{1}{4r}$ *and* $r\in\mathbb{N}$*. Then there is a* ϱ*-balanced* r*-level partition of* (T,x) *such that every level has at most* $\frac{8\Delta}{\varrho}$ *components.*

Proof. Let $\xi=\varrho/(2r)$. We first partition T into a family $\mathcal{C}=(C_i)_{i\in[\ell]}$ (for some ℓ) of rooted connected components C_i of T so that

$$v(C_i)\in\left[\tfrac{1}{\Delta}\xi v(T)-1,\xi v(T)\right] \text{ for all } i\in[\ell-1] \qquad\text{and}\qquad v(C_\ell)\le \xi v(T)\,. \tag{3}$$

Clearly, such a partition can be obtained by the following simple algorithm. Starting with the root, always proceed downwards in the tree order, at each step choosing the child y maximising $|F(y)|$ until $|F(y)|\le \xi v(T)$ is satisfied for the first time. This gives the upper bound in (3), and since this upper bound was not satisfied when we were looking at the parent of y, the lower bound in (3) must also be satisfied. In this way, we obtain the first component $C_1=C$, which we cut off from T and then repeat in order to obtain the remaining components.

We now inductively define the sets $L^1,\dots,L^r$ where each set L^i will be the union $L^i=\bigcup_{C\in\mathcal{C}_i}V(C)$ for a suitable set $\mathcal{C}_i$ of components. Suppose we have already chosen $L^1,\dots,L^{i-1}$ together with $\mathcal{C}_1,\dots,\mathcal{C}_{i-1}$. Now choose $\mathcal{C}_i\subseteq\mathcal{C}\setminus\bigcup_{j<i}\mathcal{C}_j$ satisfying the following two properties:

- for every $C\in\mathcal{C}_i$ and for every $C'\in\mathcal{C}\setminus\bigcup_{j<i}\mathcal{C}_j$ that is above C in the tree order, we must have $C'\in\mathcal{C}_i$,
- we have $|L_i|=\sum_{C\in\mathcal{C}_i}|V(C)|=(\frac{1}{r}\pm\xi)v(T)=(1\pm\frac{\varrho}{2})\frac{v(T)}{r}$.

This choice of $\mathcal{C}_i$ is clearly possible by the upper bound given in (3). Both Conditions (a) and (b) in Definition 15 are satisfied by construction and it remains to bound the number $|\mathcal{C}_i|$ of components in each level $T[L^i]$. First observe that due to the assumption $v(T)\ge 4\Delta r/\varrho$, we know that

$$\frac{\xi v(T)}{2\Delta}=\frac{\varrho v(T)}{4\Delta r}\ge 1\,. \tag{4}$$

Therefore we get

$$|\mathcal{C}_i|\le\frac{|L^i|}{\min_{j\in[\ell-1]}|C_j|}+1\le\frac{(\frac{1}{r}+\xi)v(T)}{\frac{1}{\Delta}\xi v(T)-1}+1\overset{(4)}{\le}\frac{(\frac{1}{r}+\xi)v(T)}{\frac{1}{2\Delta}\xi v(T)}+1=\frac{2\Delta}{\xi r}+2\Delta+1\le\frac{8\Delta}{\varrho}\,,$$

and hence the partition $V(T) = L^1 \dot\cup \ldots \dot\cup L^r$ satisfies all requirements of the lemma. □

4.5. **Probabilistic tools.** We write $\mathrm{Be}(p)$ for the Bernoulli distribution with success probability p, and we write $\mathrm{Bin}(p, n)$ for the binomial distribution with n trials and success probability p.

We will use the following two versions of the Chernoff bound [17, (2.9) and (2.12)]. Let $X \in \mathrm{Bin}(n, p)$, and $\mu \geq \mathbb{E}X$, $\delta \in (0, \frac{3}{2})$, $t > 0$. We have that

$$\mathbb{P}\left[X \geq (1+\delta)\cdot\mu\right] \leq 2\exp\left(-\delta^2\mu/3\right) \quad \text{and} \tag{5}$$

$$\mathbb{P}\left[X \geq \mu + t\right] \leq \exp\left(-\frac{2t^2}{n}\right) . \tag{6}$$

Moreover, for every $\delta' > 1$ and every $t \in \mathbb{R}$ with $t \geq \delta'\mathbb{E}X$ there exists $\delta'' > 0$ such that

$$\mathbb{P}\left[X \geq t\right] \leq \exp\left(-\delta'' t\right) . \tag{7}$$

Obviously, these bounds also hold for random variables which are stochastically dominated by X.

Suppose that $\Omega = \prod_{i=1}^k \Omega_i$ is a product probability space. A measurable function $f : \Omega \to \mathbb{R}$ is said to be *C-Lipschitz* if for each $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2, \ldots, \omega_i, \omega_i' \in \Omega_i, \ldots, \omega_k \in \Omega_k$ we have C-Lipschitz

$$|f(\omega_1, \omega_2, \ldots, \omega_i, \ldots, \omega_k) - f(\omega_1, \omega_2, \ldots, \omega_i', \ldots, \omega_k)| \leq C .$$

McDiarmid's Inequality, [19] states that Lipschitz functions are concentrated around their expectation.

Lemma 17 (McDiarmid's Inequality)**.** *Let $f : \Omega \to \mathbb{R}$ be a C-Lipschitz function defined on a product probability space $\Omega = \prod_{i=1}^k \Omega_i$. Then for each $t > 0$ we have*

$$\mathbb{P}\Big[|f - \mathbb{E}[f]| > t\Big] \leq 2\exp\left(-\frac{2t^2}{C^2 k}\right) .$$

Next, we introduce Suen's inequality ([22], see also [1, p. 128]). Let $\{B_i \subseteq \Omega\}_{i\in I}$ be a finite collection of events in an arbitrary probability space Ω. A *superdependency graph* for $\{B_i\}_{i\in I}$ is an arbitrary graph on the vertex set I whose edges satisfy the following. Let $I_1, I_2 \subseteq I$ be two arbitrary disjoint sets with no edge crossing from I_1 to I_2. Then any Boolean combination of the events $\{B_i\}_{i\in I_1}$ is independent of any Boolean combination of the events $\{B_i\}_{i\in I_2}$. In this setting (and only in this setting) we write $i \sim j$ to denote that ij forms an edge. superdependency graph $i \sim j$

Suen's Inequality allows us to approximate $\mathbb{P}[\bigwedge \overline{B_i}]$ by $\prod \mathbb{P}[\overline{B_i}]$.

Lemma 18 (Suen's Inequality)**.** *Using the above notation, and writing $M = \prod \mathbb{P}[\overline{B_i}]$, we have*

$$\left|\mathbb{P}\Big[\bigwedge \overline{B_i}\Big] - M\right| \leq M \cdot \left(\exp\big(\sum_{i\sim j} \nu_{i,j}\big) - 1\right) ,$$

where

$$\nu_{i,j} = \frac{\mathbb{P}[B_i \wedge B_j] + \mathbb{P}[B_i]\mathbb{P}[B_j]}{\prod_{\ell\,\sim\, i \text{ or } \ell\,\sim\, j}(1 - \mathbb{P}[B_\ell])} .$$

5. Almost packings via the nibble method

In this section, we prove the almost packing lemma (Lemma 6).

5.1. **Outline of the proof of Lemma 6.** Given an (n,Δ)-family of trees we want to find an almost packing into $K_{(1+\varepsilon)n}$. Our first step is to prepare the trees (see Section 5.3): We start by grouping all trees but the exceptional tree T_0 according to their sizes into $c=50/\varepsilon$ many groups so that trees in each group have almost the same number of vertices. The reason behind this is that one of our goals is to get good bounds on the quasirandomness of the host graph after each packing round of the nibble method, and for obtaining these bounds we need a very fine-grained control over the sizes of the forests embedded in one round. Since our trees can be very different in size, however, we group them as described and show that quasirandomness is maintained for each group individually (hence also in total). Unfortunately though, even the difference in tree sizes within one group (which are at most $n/2c$) is too big for the precision that we need for our quasirandomness bounds. We resolve this issue by attaching a small path (of length at most $n/2c=\varepsilon n/100$) to each tree, we can guarantee that in each group $i\in[c]$ all trees $T_{i,s}$ with $s\in[k_i]$ are actually of exactly the same size. Observe that in total this adds at most $\varepsilon n^2/50$ edges to our tree family, hence the resulting family in total still has less edges than $K_{(1+\varepsilon)n}$. Next, we use Lemma 16 to obtain a ϱ-balanced r-level partition of each tree $T_{i,s}$ such that each level $F^j_{i,s}$ with $j\in[r]$ forms a forest with constantly many components and all the levels are of similar size. The resulting difference in forest sizes within one group now is sufficiently small for the precision that we need for our quasirandomness bounds.

Our second step (see Section 5.4) is to remove a copy of T_0 from $K_{(1+\varepsilon)n}$. The resulting graph is still α_1-quasirandom for arbitrarily small α_1. Our third step is to almost pack the remaining trees in r rounds. In round j we embed level $F^j_{i,s}$ of tree $T_{i,s}$ for all $i\in[c]$ and $s\in[k_i]$. That this is possible is guaranteed by the nibble lemma, Lemma 21 (see Section 5.5). This lemma states that in an α_j-quasirandom graph G_j we can find partial homomorphisms from our levels $F^j_{i,s}$ to G_j such that these homomorphisms produce an almost packing of $F^j_{i,s}$. At the end of round j we remove from G_j all edges used in images of any $F^j_{i,s}$. Lemma 21 also guarantees that the resulting graph G_{j+1} is still quasirandom (albeit with worse parameters), hence we can continue with the next round.

5.2. **Constants.** We now start the proof of Lemma 6. Suppose that $\varepsilon>0$ and $\Delta\in\mathbb{N}$ are given. Set $c=\frac{50}{\varepsilon}$ and let c, r, β_r

$$r=\frac{1000\Delta^2}{\varepsilon^{10\Delta}}\qquad\text{and}\qquad\beta_r=\varepsilon^2/100. \tag{8}$$

We recursively define $\alpha_r,\beta_{r-1},\alpha_{r-1},\dots,\beta_1,\alpha_1$ by setting α_j, β_j

$$\alpha_j=\alpha_{\text{L21}}(\varepsilon,\beta_j,c,\Delta)\quad\text{and}\quad\beta_{j-1}=\alpha_j\ , \tag{9}$$

using Lemma 21 below. Note that we have that $\alpha_1\le\beta_1=\alpha_2\le\beta_2=\alpha_3\le\dots=\alpha_r\le\beta_r$.

Finally, let ϱ, n_0

$$\varrho=\min\{\frac{1}{4r},\alpha_1\}\qquad\text{and}\qquad n_0=\max\{\frac{8\Delta r}{\varrho\alpha_1},n_{\text{L14}}(\alpha_1),\dots\}\ . \tag{10}$$

5.3. **Preparing the trees.** Now that we have chosen n_0 as required by Lemma 6, consider an (n,Δ)-tree family $\mathcal{T}$ and let $T_0\in\mathcal{T}$ be the exceptional tree of order at most $n/2$ (if it exists). In the following embedding procedure T_0 will be treated separately.

We group the other trees in $\mathcal{T}$ according to their order. For $i\in[c]$ let $T_{i,1},T_{i,2},\dots,T_{i,k_i}$ be the trees of $\mathcal{T}$ whose order is in the interval $\left(\frac{n}{2}+(i-1)\cdot\frac{\varepsilon n}{100},\frac{n}{2}+i\cdot\frac{\varepsilon n}{100}\right]$. We append to an arbitrary leaf of each tree $T_{i,s}$ a path with exactly $\frac{n}{2} + i \cdot \frac{\varepsilon n}{100} - v(T_{i,s})$ edges. As a result, each modified tree $T_{i,s}$ has order exactly $\frac{n}{2} + i \cdot \frac{\varepsilon n}{100}$. Since $\mathcal{T}$ contains at most $2n$ trees, this added at most $\frac{\varepsilon n^2}{50}$ edges to the total number of edges in $\mathcal{T}$ and thus

$$\sum_{i\in[c],s\in[k_i]} e(T_{i,s}) \leq \binom{n}{2} + \frac{\varepsilon n^2}{50}. \tag{11}$$

The order and the maximum degree of the trees are still upper-bounded by n and Δ, respectively. For $i \in [c]$ we now let n_i

$$n_i = \frac{n}{2r} + i\frac{n}{2cr} = \frac{n}{2r} + i\frac{\varepsilon n}{100r} = \frac{v(T_{i,s})}{r}. \tag{12}$$

We slice the trees into r levels as follows. We pick an arbitrary root $x_{i,s}$ for each tree $T_{i,s}$ $x_{i,s}$ with $i \in [c],\ s \in [k_i]$. For all $i \in [c]$, $s \in [k_i]$ we apply Lemma 16 to the rooted tree $(T_{i,s}, x_{i,s})$. Since

$$v(T_{i,s}) > \frac{n}{2} \geq \frac{n_0}{2} \overset{(10)}{\geq} \frac{4\Delta r}{\varrho} \quad \text{and} \quad \varrho \overset{(10)}{\leq} \frac{1}{4r},$$

we obtain a ϱ-balanced r-level partition $\mathcal{P}_{i,s} = (L^1_{i,s}, \dots, L^r_{i,s})$ of $(T_{i,s}, x_{i,s})$ such that every $\mathcal{P}_{i,s} = (L^1_{i,s}, \dots, L^r_{i,s})$ level of $\mathcal{P}_{i,s}$ has at most $8\Delta/\varrho$ components. Finally, we use these partitions to define rooted forests $(F^j_{i,s}, X^j_{i,s})$ with $i \in [c]$, $s \in [k_i]$ and $j \in [r]$ as follows. Let $F^j_{i,s} = T_{i,s}[L^j_{i,s}]$ be the level j $F^j_{i,s}$ of the partition $\mathcal{P}_{i,s}$ and let $n^j_{i,s}$

$$n^j_{i,s} = v(F^j_{i,s}) = |L^j_{i,s}| \overset{\text{Def 15}}{=} (1 \pm \frac{\varrho}{2})\frac{v(T_{i,s})}{r} \overset{(12)}{=} (1 \pm \frac{\varrho}{2})n_i. \tag{13}$$

Using the fact that $\varrho \leq 1/(4r)$ by (10), we obtain that

$$\frac{n}{4r} \leq n^j_{i,s} \leq \frac{2n}{r}\,. \tag{14}$$

Let the root set $X^j_{i,s}$ be obtained by considering $F^j_{i,s}$ as a rooted subforest of the rooted $X^j_{i,s}$ tree $(T_{i,s}, x_{i,s})$, that is, $X^1_{i,s} = \{x_{i,s}\}$, and for $j > 1$, $X^j_{i,s}$ is taken to contain the natural root of every component of $F^j_{i,s}$. Lemma 16 guarantees that for every $i \in [c]$, $s \in [k_i]$ and $j \in [r]$ we have

$$|X^j_{i,s}| \leq \frac{8\Delta}{\varrho} \overset{(10)}{\leq} \alpha_1 \frac{n}{r}\,. \tag{15}$$

5.4. **Embedding T_0.** Our embedding procedure now starts by embedding T_0 arbitrarily into $K_{(1+\varepsilon)n}$. By Lemma 14 the resulting graph

$$G_1 = (V, E_1) = K_{(1+\varepsilon)n} - T_0 \text{ is } \alpha_1\text{-quasirandom}. \tag{16}$$

5.5. **The nibble lemma.** For almost packing the remaining trees we use a nibble method, that is, we proceed in rounds and embed in each round one level of each tree. The setting of Lemma 21, which captures one round of the nibble procedure, is as follows.

We have a quasirandom host graph $G = (V, E)$ and a family $(F_{i,s}, X_{i,s})_{s\in[k_i],\, i\in[c]}$ of rooted forests that we want to pack into G, one sub-forest $F_{i,s}$ for each tree $T_{i,s}$ to be packed. In addition, we are given for each $i \in [c], s \in [k_i]$ a set $U_{i,s} \subseteq V$ of *forbidden vertices* for the embedding of $F_{i,s}$. The set $U_{i,s}$ contains vertices of G that were used for the embedding of vertices of $T_{i,s}$ in earlier rounds.

It is the quasirandomness of G that will enable us to almost pack the forests $F_{i,s}$. While doing so, however, we need to keep in mind that there are future embedding rounds to come.

Therefore we cannot embed the forest just somehow, but we have to assert that certain invariants are maintained. One of these invariants is clearly the quasirandomness of the part of the host graph that remains after the embedding (Property (C7)). In addition we need to guarantee that the embedding of the different forests $F_{i,s}$ is distributed "fairly" over the vertices of G. To this end we require that the sets $U_{i,s}$ are well spread over G and our goal is to maintain this property for the next embedding round (Property (C8)). For this we need a concept which measures whether the sets $U_{i,s}$ are distributed in a sufficiently random-like manner over the vertex set V.

Definition 19 (load)**.** *Consider a graph $G = (V, E)$ with $m = |V|$ and two vertices $v, w \in V$, and let $\mathcal{W} = (W_s)_{s\in[k]}$ be a collection of subsets of V.*

load$(v, w, \mathcal{W})$
$\mu(\mathcal{W})$
$\sigma(\mathcal{W})$

$$\begin{aligned}\mathrm{load}(v, w, \mathcal{W}) &= |\{s \in [k] : \quad W_s \cap \{v, w\} \neq \emptyset\}| ,\\ \mu(\mathcal{W}) &= \frac{1}{\binom{m}{2}} \sum_{\{v',w'\}\in\binom{V}{2}} \mathrm{load}(v', w', \mathcal{W}),\\ \sigma(\mathcal{W}) &= \sum_{\{v',w'\}\in\binom{V}{2}} \big(\mathrm{load}(v', w', \mathcal{W}) - \mu(\mathcal{W})\big)^2 .\end{aligned}$$

*We say that $\mathcal{W}$ is (α, ℓ)-*homogeneous *if $\sigma(\mathcal{W}) \leq \alpha\ell^4$, and for each $s, s' \in [k]$ we have $||W_s| - |W_{s'}|| \leq \alpha\ell$.* (α, ℓ)-homogeneous

In the proof of Lemma 21 we will maintain these invariants by embedding the forests $F_{i,s}$ randomly, that is, we construct random partial homomorphisms $h_{i,s} \colon F_{i,s} \rightsquigarrow G$. The mappings $h_{i,s}$ do not embed the vertices in $X_{i,s}$, and there will be another family of sets, denoted by $Y_{i,s}$ and called the *skipped* vertices, that are left unembedded. Thus the $h_{i,s} \colon F_{i,s} - (X_{i,s} \cup Y_{i,s}) \to G$ are homomorphisms. However, they do not necessarily form a proper packing of $(F_{i,s} - (X_{i,s} \cup Y_{i,s})_{s\in[k_i],\, i\in[c]}$ into G, because they may fail to be injective or pairwise edge-disjoint. In order to measure this shortcoming, we introduce various types of *collisions*, which we describe in the following definition. skipped

Definition 20 (colliding and skipped vertices)**.** *In the setting above, suppose that $h_{i,s}$: $F_{i,s} - (X_{i,s} \cup Y_{i,s}) \to G$ are homomorphisms. We say that a vertex $y \in V(F_{i,s})$ is in a* vertex collision *or that y* colliding*, if there exists a vertex $z \in V(F_{i,s})\setminus\{y\}$ such that $h_{i,s}(y) = h_{i,s}(z)$. We define* vertex collision colliding

$$\mathrm{VC}_{i,s} = \{y \in V(F_{i,s}) \,:\, y \textit{ is colliding}\} .$$

$\mathrm{VC}_{i,s}$

We say that an edge $xy \in E(F_{i,s})$ is colliding *if there is some $(i', s') \neq (i, s)$ with $x'y' \in E(F_{i',s'})$ such that $h_{i,s}(x, y) = h_{i's'}(x', y')$. A vertex $y \in V(F_{i,s})$ is in an* edge collision *if there is $x \in V(F_{i,s}) \setminus \{y\}$ such that xy is colliding. We define* edge collision $\mathrm{EC}_{i,s}$

$$\mathrm{EC}_{i,s} = \{y \in V(F_{i,s}) \,:\, y \textit{ is in an edge collision}\} .$$

We say a vertex $x \in \bigcup_{i,s} V(F_{i,s})$ is faulty *if $x \in \bigcup_{i,s}(\mathrm{VC}_{i,s} \cup \mathrm{EC}_{i,s})$.* faulty

For a vertex $v \in V$ the vertices mapped to v with faulty neighbours *are* FN(v)

$$\mathrm{FN}(v) = \bigcup_{i,s} \big\{x \in h^{-1}_{i,s}(v) \colon \exists xy \in E(F_{i,s}) \textit{ such that } y \textit{ is faulty}\big\} ,$$

the vertices mapped to v with skipped neighbours *are* YN(v)

$$\mathrm{YN}(v) = \bigcup_{i,s} \big\{x \in h^{-1}_{i,s}(v) \colon \exists xy \in E(F_{i,s}) \textit{ such that } y \in Y_{i,s}\big\} ,$$

and the vertices mapped to v with root neighbours *are* XN(v)

$$\mathrm{XN}(v) = \bigcup_{i,s} \left\{ x \in h_{i,s}^{-1}(v)\colon\ \exists xy \in E(F_{i,s}) \text{ such that } y \in X_{i,s} \right\} ,$$

Lemma 21 now asserts that we only have a small number of these collisions. As we will show after stating the lemma, this implies that we get an almost embedding.

Lemma 21 (Nibble Lemma)**.** *For every $\varepsilon, \beta > 0$, and $c, \Delta \in \mathbb{N}$, there exists $0 < \alpha \le \beta$ so that for every integer r there exists n_0 such that for every $n \ge n_0$ the following is true.*

We assume that we are given a family of rooted forests $\mathcal{F} = (F_{i,s}, X_{i,s})_{i\in[c],s\in[k_i]}$ with $n/2 \le \sum_{i=1}^{c} k_i \le 2n$, $|X_{i,s}| \le \alpha\frac{n}{r}$, $n_{i,s} = v(F_{i,s}) = (1 \pm \alpha)n_i$, where, as before, $n_i = \frac{n}{2r} + i\frac{n}{2cr}$. $n_{i,s}$ *Moreover, we assume that $G = (V, E)$ is an α-quasirandom graph with $m = |V| = (1 + \varepsilon)n$ and density $d > \varepsilon$. For each $i \in [c]$, let $\mathcal{U}_i = (U_{i,s})_{s\in[k_i]}$ be an (α, n)-homogeneous family with $|U_{i,s}| < n$ for all $s, s' \in [k_i]$. For all $i \in [c]$, $s \in [k_i]$ set $V_{i,s} = V \setminus U_{i,s}$.*

Then there are sets $Y_{i,s} \subseteq V(F_{i,s})$ and homomorphisms $h_{i,s} : F_{i,s} - (X_{i,s} \cup Y_{i,s}) \to G[V_{i,s}]$ for all $i \in [c]$ and $s \in [k_i]$, with the following properties. For each $i \in [c]$, each $s \in [k_i]$, and each $v \in V(G)$ we have

- (C1) $|Y_{i,s}| \le \beta n/r$,
- (C2) $|\mathrm{VC}_{i,s}| \le 20n/(\varepsilon r^2 d^{\Delta})$,
- (C3) $|\mathrm{EC}_{i,s}| \le 300\Delta n/(\varepsilon^2 r^2 d^{\Delta})$,
- (C4) $|\mathrm{FN}(v)| \le 10^4\Delta^3 n/(\varepsilon^3 r^2 d^{2\Delta})$,
- (C5) $|\mathrm{YN}(v)| \le \beta n/r$,
- (C6) $|\mathrm{XN}(v)| \le \beta n/r$,
- (C7) *the graph $\tilde{G} = (V, E \setminus \bigcup_{i,s} E(h_{i,s}))$ is β-quasirandom, and*
- (C8) *for each $i \in [c]$, the family $\tilde{\mathcal{U}}_i = (\tilde{U}_{i,s})_{s\in[k_i]}$ with $\tilde{U}_{i,s} = U_{i,s} \cup V(h_{i,s})$ is (β, n)-homogeneous.*

5.6. **Applying the Nibble Lemma to obtain an almost-packing.** Let us first recall what we have achieved so far. In Section 5.3 we obtained a family $\mathcal{F}^j = (F_{i,s}^j, X_{i,s}^j)$ of rooted forests for $j \in [r]$. We can assume that $\sum_i k_i \ge n/2$ (as otherwise, we might add dummy trees to be embedded). In Section 5.4 we embedded the tree T_0, deleted its edges and ended up with an α_1-quasirandom graph $G_1 = (V, E_1)$.

Now we set $\mathcal{U}_i^1 = (U_{i,s}^1)_{s\in[k_i]}$ where $U_{i,s}^1 = \emptyset$ for all $i \in [r]$ and $s \in [k_i]$. We perform r embedding rounds. For $j = 1, \dots, r$, we do the following in round j. We apply Lemma 21 with parameters ε, β_j, c, Δ, obtaining α_j and n_0. We then feed to Lemma 21

- (P1)$_j$ the family $\mathcal{F}^j = (F_{i,s}^j, X_{i,s}^j)_{i\in[c],s\in[k_i]}$ of rooted forests,
- (P2)$_j$ an α_j-quasirandom graph $G_j = (V, E_j)$ with $|V| = m = (1+\varepsilon)n$ and $d_j\binom{m}{2} = |E_j| \ge \frac{3}{4}\varepsilon n^2$, which implies $d_j \ge \varepsilon$,
- (P3)$_j$ and for each $i \in [c]$ an (α_j, n)-homogeneous family $\mathcal{U}_i^j = (U_{i,s}^j)_{s\in[k_i]}$.

Let us now check that the conditions required by Lemma 21 are met. By (15) we have $|X_{i,s}^j| \le \alpha_1\frac{n}{r} \le \alpha_j\frac{n}{r}$, by (13) and the definition of ϱ we have $v(F_{i,s}^j) = (1 \pm \alpha_j)n_i$. Hence the conditions of Lemma 21 are satisfied. So we obtain sets $Y_{i,s}^j \subseteq V(F_{i,s}^j)$ and homomorphisms $h_{i,s}^j : F_{i,s}^j - (X_{i,s}^j \cup Y_{i,s}^j) \to G[V_{i,s}^j]$, where $V_{i,s}^j = V \setminus U_{i,s}^j$, with vertex collisions $\mathrm{VC}_{i,s}^j$, edge collisions $\mathrm{EC}_{i,s}^j$, faulty neighbours $\mathrm{FN}^j(v)$, skipped neighbours $\mathrm{YN}^j(v)$, and root neighbours $\mathrm{XN}^j(v)$ for every $v \in V$, such that (C1)–(C8) are satisfied.

We will next argue that we can apply Lemma 21 again in the next round. For this purpose let $G_{j+1} = (V, E_{j+1}) = (V, E_j \setminus \bigcup_{i,s} E(h^j_{i,s}))$. Since $\beta_j = \alpha_{j+1}$ by (9), Conclusion (C7) implies that G_{j+1} is α_{j+1}-quasirandom. Moreover, to check the density requirement in $(\mathrm{P2})_{j+1}$,

$$|E_{j+1}| \geq e(G_1) - \sum_{\substack{i\in[c],s\in[k_i]\\ j\in[r]}} e(F^j_{i,s}) \geq e(K_{(1+\varepsilon)n}) - e(T_0) - \sum_{i\in[c],s\in[k_i]} e(T_{i,s})$$

$$\overset{(11)}{\geq} \binom{(1+\varepsilon)n}{2} - (n-1) - \frac{\varepsilon n^2}{50} - \binom{n}{2} \overset{(10)}{\geq} \frac{3}{4}\varepsilon n^2 \,.$$

Let $\mathcal{U}^{j+1}_i = \tilde{\mathcal{U}}^j_i$. By (C8) the family $\mathcal{U}^{j+1}_i$ is $(\beta_j = \alpha_{j+1}, n)$-homogeneous. We conclude that conditions $(\mathrm{P2})_{j+1}$ and $(\mathrm{P3})_{j+1}$ are again satisfied and hence we can apply Lemma 21 in the next round.

After finishing all r embedding rounds we define the set $R_{i,s}$ that contains all roots, skipped vertices and vertices in vertex or edge collisions in the tree $T_{i,s}$,

$$R_{i,s} = \bigcup_{j\in[r]} \left(X^j_{i,s} \cup Y^j_{i,s} \cup \mathrm{VC}^j_{i,s} \cup \mathrm{EC}^j_{i,s}\right).$$

Let $\tilde{h}^j_{i,s}$ be the restriction of $h^j_{i,s}$ to $V(F^j_{i,s}) \setminus R_{i,s}$ and $\tilde{h}_{i,s} = \bigcup_{j\in[r]} \tilde{h}^j_{i,s}$. We will show that $\{\tilde{h}_{i,s}, R_{i,s}\}_{i\in[c],s\in[k_i]}$ is an (εn)-almost packing of $\mathcal{T}$ into $K_{(1+\varepsilon)n} - T_0$, which will finish the proof of Lemma 6.

Indeed, by the definition of the sets $V^j_{i,s}$, the vertex-images of two homomorphisms $h^j_{i,s}$ and $h^{j'}_{i',s'}$ are disjoint, unless $j = j'$. In other words, vertices of different rounds cannot collide. Moreover, by the definition of G_j, the edges of $K_{(1+\varepsilon)n}$ used for the embedding in some round do not get used again in a later round. Hence edges of different rounds can also not collide. Since $h^j_{i,s}$ is a homomorphism from $F^j_{i,s} - (X^j_{i,s} \cup Y^j_{i,s})$ to $G[V^j_{i,s}]$, the set $\mathrm{VC}^j_{i,s} \cup \mathrm{EC}^j_{i,s}$ contains all vertices in vertex and edge collisions of $F^j_{i,s}$, and $X^j_{i,s}$ contains all roots of trees in $F^j_{i,s}$, we conclude that $\{\tilde{h}_{i,s}\}_{i\in[c],s\in[k_i]}$ is a packing of the family $\{T_{i,s} - R_{i,s}\}_{i\in[c],s\in[k_i]}$ into $K_{(1+\varepsilon)n} - T_0$.

Hence it remains to check conditions (b) and (c) of Definition 5. For condition (b), observe that by (15), (C1), (C2) and (C3) of Lemma 21 we have

$$\begin{aligned}|R_{i,s}| &= \sum_{j\in[r]} \left(|X^j_{i,s}| + |Y^j_{i,s}| + |\mathrm{VC}^j_{i,s}| + |\mathrm{EC}^j_{i,s}|\right)\\ &\leq r\cdot\left(\alpha_1\frac{n}{r} + \beta_r\frac{n}{r} + 20\frac{n}{\varepsilon r^2 d_r^{\Delta}} + \frac{300\Delta n}{\varepsilon^3 r^2 d_r^{2\Delta}}\right)\\ &\leq \left(2\beta_r + \frac{320\Delta}{\varepsilon^3 r d_r^{2\Delta}}\right)n \leq \varepsilon n\,,\end{aligned}$$

where we use $d_r \geq \varepsilon$, and (8). For condition (c), let $v \in V(K_{(1+\varepsilon)n})$ be fixed and define

$$\mathrm{RN}(v) = \bigcup_{i,s} \left\{y \in h^{-1}_{i,s}(v)\colon\ \exists xy \in E(T_{i,s}) \text{ such that } x \in R_{i,s}\right\}.$$

We need to show that $|\mathrm{RN}(v)| \leq \varepsilon n$. The definition of $R_{i,s}$ implies that $\mathrm{RN}(v) = \bigcup_j (\mathrm{FN}^j(v) \cup \mathrm{YN}^j(v) \cup \mathrm{XN}^j(v))$ and thus we infer from (C4), (C5), (C6) of Lemma 21 that

$$|\mathrm{RN}(v)| \leq \left(\frac{10\Delta^3}{\varepsilon^3 r^2 d_r^{2\Delta}} + \frac{\beta_r}{r} + \frac{\beta_r}{r}\right)rn \leq \left(\frac{10\Delta^3\varepsilon^{10\Delta}}{\varepsilon^3\cdot 1000\Delta\cdot\varepsilon^{2\Delta}} + \frac{2\varepsilon^2}{100}\right)n \leq \varepsilon n\,,$$

where again we use $d_r \geq \varepsilon$, and (8).

6. Proof of the Correction Lemma

In this section, we give a proof of Lemma 7. We consider the graph K_m as a subgraph of $K_{(1+\varepsilon)m}$, and set $W = V(K_{(1+\varepsilon)m}) \setminus V(K_m)$. We are given trees $T_1, \dots, T_k$ together with an $(\frac{\varepsilon^2 m}{64\Delta})$-almost packing $(h_i \colon T_i - R_i \to K_m)_{i\in[k]}$ of these trees into K_m.

In each tree T_i we choose a root in $V(T_i) \setminus R_i$ and a breadth-first search ordering of the vertices of T_i starting at this root. We enumerate the vertices $R_i = \{x_{i,1}, \dots, x_{i,\ell_i}\}$ according to this ordering. Our approach now is to proceed tree by tree, starting with T_1, and to embed the vertices of R_i one by one into W (in this order), so that we obtain a packing of all trees into $K_{(1+\varepsilon)m}$ in the end. More precisely, for $i = 1, \dots, k$ and $t = 1, \dots, \ell_i$ we map the vertex $x_{i,t}$ to a vertex $\tilde{h}_i(x_i, t) \in W$ using a greedy algorithm, where $\tilde{h}_i(x_{i,t})$ must avoid certain forbidden sets, which we now define.

Firstly, $x_{i,t}$ should not be embedded on vertices in W which are already images of other vertices of T_i, that is, vertices in

$$X_{i,t} = \bigcup_{s<t} \{\tilde{h}_i(x_{i,s})\} .$$

This will guarantee that $\tilde{h}_i$ is injective. Secondly, $x_{i,s}$ should not be embedded on a vertex in W whose edges to h_i-images of T_i-neighbours of $x_{i,t}$ have been used already by a tree T_j with $j < i$. These forbidden vertex sets are captured below by the sets $Y_{i,t}$ (for T_i-neighbours of $x_{i,t}$ that are not in R_i) and $U_{i,t}$ (for T_i-neighbours of $x_{i,t}$ that are in R_i). Let $A_{i,t} = \mathrm{N}_{T_i}(x_{i,t}) \cap \big(V(T_i) \setminus R_i\big)$ be the neighbours of $x_{i,t}$ that have already been embedded by h_i and set

$$Y_{i,t} = \{w \in W \colon \exists j < i, y \in A_{i,t} \colon h_i(y) = v \text{ and } vw \in E(h_j \cup \tilde{h}_j)\} .$$

Thirdly, we do not want to embed $x_{i,t}$ to vertices contained in "dangerously" many used edges, that is, vertices in the following set Z_i. Let $E_{i,t}$ be the set of edges in $\binom{R}{2}$ that have already been used, that is $E_{i,t} = \bigcup_{j\leq i} E(\tilde{h}_j)$ and set

$$Z_{i,t} = \{w \in W \colon w \text{ is contained in at least } \varepsilon m/2 \text{ edges of } E_{i,t}\} .$$

Embedding $x_{i,t}$ outside $Z_{i,t}$ will guarantee that the embedding process can be continued for R_i-neighbours of $x_{i,t}$.

Finally, let x be the parent of $x_{i,t}$ in T_i. If $x \in R_i$ then we have $x = x_{i,s}$ for some $s < t$. We let

$$U_{i,t} = \big\{w \in W \colon \{\tilde{h}_i(x_{i,s}), w\} \in E_{i,t}\big\} = \big\{w \in W \colon \{\tilde{h}_i(x_{i,s}), w\} \in E_{i,s}\big\} ,$$

that is, the set of vertices in W whose edge to the image of $x_{i,s}$ has been used already. The equality holds because after $x_{i,s}$ and before $x_{i,t}$ we only embed vertices $x_{i,s'}$ of T_i and guarantee that $\tilde{h}_i(x_{i,s'}) \neq \tilde{h}_i(x_{i,s})$. If $x \notin R_i$ we let $U_{i,t} = \emptyset$.

Having defined these forbidden sets we now map $x_{i,t}$ to an arbitrary vertex

$$\tilde{h}_i(x_{i,t}) \in W \setminus \big(X_{i,t} \cup Y_{i,t} \cup Z_{i,t} \cup U_{i,t}\big) .$$

We claim that this set is not empty. Indeed, we have $|X_{i,t}| \leq |R_i| \leq \varepsilon^2 m/(64\Delta^2)$. In addition, in the definition of $Y_{i,t}$ there are at most Δ choices for y and hence for v. For a fixed v, Definition 5(c) states that at most $\varepsilon^2 m/(64\Delta^2)$ vertices z have been mapped by

$\bigcup_{j\le i} h_j$ to v. Each of these vertices $z \in V(T_j)$ has at most Δ neighbours mapped by $\tilde{h}_j$ to some $w \in W$. Hence $|Y_{i,t}| \le \Delta \cdot \Delta \cdot \varepsilon^2 m/(64\Delta^2)$. To get a bound on $|Z_{i,t}|$ we observe that

$$|E_{i,t}| \le \sum_{j\le i} e\big(T_j[R_j]\big) \le \sum_{j\le i} |R_j| \le k\frac{\varepsilon^2 m}{64\Delta^2}\,.$$

Hence, since $k \le 2m$ we obtain

$$|Z_{i,t}| \le \frac{2|E_{i,t}|}{\varepsilon m/2} \le \frac{4k\varepsilon^2 m}{64\Delta^2 \varepsilon m} \le \frac{\varepsilon m}{8}\,.$$

Moreover, $|U_{i,t}| \le \varepsilon m/2$ because $\tilde{h}_i(x_{i,s}) \notin Z_{i,s}$. We conclude that

$$\big|W \setminus \big(X_{i,t} \cup Y_{i,t} \cup Z_{i,t} \cup U_{i,t}\big)\big| \ge \varepsilon m - \frac{\varepsilon^2 m}{64\Delta^2} - \Delta^2\frac{\varepsilon^2 m}{64\Delta^2} - \frac{\varepsilon m}{8} - \frac{\varepsilon m}{2} > 0\,.$$

It remains to check that, at the end of this procedure, the mappings $(h_i \cup \tilde{h}_i)_{i\in[k]}$ form a packing of $\mathcal{T}$ into $K_{(1+\varepsilon)m}$. Firstly, each $h_i \cup \tilde{h}_i$ is injective, because h_i is injective, $\tilde{h}_i$ is injective by the definition of $X_{i,t}$, and $V(h_i) \cap V(\tilde{h}_i) = \emptyset$. Secondly, $h_i \cup \tilde{h}_i$ is edge-preserving because we embed into a complete graph. Thirdly, we have $E(h_i \cup \tilde{h}_i) \cap E(h_j \cup \tilde{h}_j) = \emptyset$ for each $i > j$. Indeed, $E(h_i)$ and $E(h_j)$ are disjoint by assumption. $E(h_i)$ and $E(\tilde{h}_j)$ (and similarly $E(\tilde{h}_i)$ and $E(h_j)$) are disjoint by the definition of $Y_{i,t}$. Finally, $E(\tilde{h}_i)$ and $E(\tilde{h}_j)$ are disjoint by the definition of $U_{i,t}$.

7. Limping homomorphisms on quasirandom graphs

Let F be a forest with maximum degree Δ and a given bipartition into *primary vertices* and *secondary vertices*. Let $G = (V, E)$ be an (α, Δ)-superquasirandom graph of density d. We now define a *limping homomorphism h from F to G*. This is a random partial homomorphism from F to G whose distribution is described by the following two-step procedure.

primary v.
secondary v.
limping homomorphism

1. For each primary vertex $x \in V(F)$ we choose uniformly at random (u.a.r.) a vertex $h(x) \in V$.
2. For each secondary vertex $y \in V(F)$ we choose u.a.r. a real number $\tau(y) \in [0, 1)$. To pick $h(y)$, consider the set $\{u_1, \dots, u_p\} = h(\mathrm{N}_F(y))$.[1]
   (a) If $\{u_1, \dots, u_p\}$ is α-bad then h does not map y anywhere. We say that h *skips* y. skips
   (b) If y is not skipped, let $i = \lfloor \tau(y) \cdot \mathrm{codeg}(u_1, \dots, u_p)\rfloor + 1$ and define $h(y)$ to be the i-th vertex in $\mathrm{N}(u_1, \dots, u_p)$ (for which an order was fixed prior to the experiment). In other words, we choose $h(y)$ u.a.r. in $\mathrm{N}(u_1, \dots, u_p)$.
   Modelling this uniform random choice by $\tau(y)$ will help in the analysis.

Hence, if we denote the set of primary vertices by P and the set of secondary vertices by S, the limping homomorphism is determined by an element of the probability space

$$\Omega_F = V^P \times [0, 1]^S\,. \tag{17}$$

This is the product space that we shall use in applications of McDiarmid's Inequality later.

Observe that a limping homomorphism implicitly depends on the parameter α. This parameter will always be clear from the context.

The next three lemmas establish some fundamental properties of limping homomorphisms.

[1]Note that p can be strictly smaller than $\deg_F(y)$; this happens when h is not injective on $\mathrm{N}_F(y)$.

Lemma 22. *Suppose that we are given $\alpha \in (0, \frac{1}{4})$, a tree F of maximum degree at most Δ with a bipartition into primary and secondary vertices, and an (α, Δ)-superquasirandom graph $G = (V, E)$ of density d and with $|V| \geq 4\Delta/d$.*

Let h be the limping homomorphism from F to G. Let $uv \in E$ be an arbitrary edge of G, let $x \in V(F)$ be an arbitrary primary vertex, let $y \in V(F)$ be an arbitrary secondary vertex and let $\mathcal{H}$ be an arbitrary event describing the placements of all vertices except y. Then the following statements hold.

(a) $\mathbb{P}[h(x) = v] = \frac{1}{|V|}$.
(b) $\mathbb{P}[y \text{ is skipped} \mid h(x) = v] \leq \alpha$.
(c) $\mathbb{P}[y \text{ is skipped}] \leq \alpha$.
(d) *Suppose that $xy \in E(F)$. Then* $\mathbb{P}[h(x) = u \text{ and } h(y) = v] = \frac{\left(1 \pm \alpha\left(\frac{2}{d}\right)^{\Delta}\right)^{\Delta+2}}{d|V|^2}$.
(e) $\mathbb{P}[h(y) = v] = \frac{\left(1 \pm \alpha\left(\frac{2}{d}\right)^{\Delta}\right)^{\Delta+3}}{|V|}$.
(f) $\mathbb{P}[h(y) = v \mid y \text{ not skipped}] = \frac{\left(1 \pm \alpha\left(\frac{2}{d}\right)^{\Delta}\right)^{\Delta+5}}{|V|}$.
(g) $\mathbb{P}[h(y) = v \mid \mathcal{H}] \leq \frac{2}{d^{\Delta}|V|}$.

Proof. (a) This follows immediately from the definition of limping homomorphisms.

(b) The statement is trivially true when $\mathrm{N}_F(y) = \{x\}$. Indeed, then y is never skipped. So, let us assume that $|\mathrm{N}_F(y) \setminus \{x\}| \geq 1$.

Let us expose the placement of all the primary vertices of F. Let $\{u_1, \ldots, u_p\} = h(\mathrm{N}_F(y)) \setminus \{v\}$. Note that $p \geq 1$ almost surely. As G is (α, Δ)-superquasirandom $\mathrm{bad}_{\alpha,p}(v) \leq \alpha\binom{|V|}{p-1}$ and so we have

$$\mathbb{P}\left[y \text{ is skipped} \mid h(x) = v\right] = \mathbb{P}\left[\{u_1, \ldots, u_p, v\} \text{ is } \alpha\text{-bad}\right] \leq \alpha\,.$$

(c) We have $\mathbb{P}[y \text{ is skipped}] = \sum_{w \in V} \mathbb{P}[y \text{ is skipped} \mid h(x) = w] \cdot \mathbb{P}[h(x) = w] \leq \alpha$, by (a) and (b).

(d) Let A be the event that x gets mapped to u, let B be the event that y gets mapped to v, let C be the event that y is not skipped, and let D be the event that v is in the common neighbourhood of $h(\mathrm{N}_F(y) \setminus \{x\})$. Note that $B \subseteq C \cap D$. Let $\mathcal{E}_q$ be the event that $|h(\mathrm{N}_F(y))| = q + 1$. As D and A are independent even if we condition on $\mathcal{E}_q$, we have

$$\begin{aligned} \mathbb{P}\left[A \cap B | \mathcal{E}_q\right] &= \mathbb{P}\left[A \cap B \cap C \cap D | \mathcal{E}_q\right] \\ &= \mathbb{P}\left[A | \mathcal{E}_q\right] \cdot \mathbb{P}\left[D | A \cap \mathcal{E}_q\right] \cdot \mathbb{P}\left[C | \mathcal{E}_q \cap D \cap A\right] \cdot \mathbb{P}\left[B | \mathcal{E}_q \cap C \cap D \cap A\right] \\ &= \mathbb{P}\left[A | \mathcal{E}_q\right] \cdot \mathbb{P}\left[D | \mathcal{E}_q\right] \cdot \mathbb{P}\left[C | \mathcal{E}_q \cap D \cap A\right] \cdot \mathbb{P}\left[B | \mathcal{E}_q \cap C \cap D \cap A\right] . \end{aligned} \tag{18}$$

We have $\mathbb{P}\left[A | \mathcal{E}_q\right] = \mathbb{P}\left[A\right] = \frac{1}{|V|}$. As $\mathrm{bad}_{\alpha,1}(v) = 0$, we get that $\deg(v) = (1 \pm \alpha)d|V|$. Consequently, $\mathbb{P}\left[D | \mathcal{E}_q\right] = \left((1 \pm \alpha)d\right)^q$. The number of α-bad $(q+1)$-sets that contain u and have the remaining vertices inside $\mathrm{N}(v)$ is at most $\alpha\binom{|V|}{q}$. As $|\mathrm{N}(v)| \geq (1 - \alpha)d|V|$, the total number of $(q+1)$-sets that contain u and have the remaining vertices inside $\mathrm{N}(v)$ is at least $\binom{(1-\alpha)d|V|}{q}$. We thus get

$$1 \geq \mathbb{P}\left[C | \mathcal{E}_q \cap D \cap A\right] \geq 1 - \frac{\alpha\binom{|V|}{q}}{\binom{(1-\alpha)d|V|}{q}} \geq 1 - \alpha\left(\frac{2}{d}\right)^{\Delta}\ ,$$

where we use $(1-\alpha)d|V|-q \geq \frac{1}{2}d|V|$, which follows from $|V| \geq 4\Delta/d$. Finally, if y is not skipped, then the set $h(\mathrm{N}_F(y))$ is not α-bad, implying that

$$\mathbb{P}\left[B|\mathcal{E}_q \cap C \cap D \cap A\right] = ((1\pm\alpha)d^{q+1}|V|)^{-1}.$$

The claimed bound now follows by substituting the above estimates into (18).

(e) Fix an arbitrary neighbour z of y. Since z is primary, we have

$$\mathbb{P}[h(y)=v] = \sum_{w\in V\,:\,vw\in E} \mathbb{P}[h(y)=v \text{ and } h(z)=w].$$

The above sum has $(1\pm\alpha)d|V|$ summands. The statement then follows from (d).

(f) We have

$$\mathbb{P}[h(y)=v \mid y \text{ not skipped}] = \frac{\mathbb{P}[h(y)=v \text{ and } y \text{ not skipped}]}{\mathbb{P}[y \text{ not skipped}]} = \frac{\mathbb{P}[h(y)=v]}{\mathbb{P}[y \text{ not skipped}]}.$$

Hence we get the claimed bound from (c) and (e).

(g) We can expose the entire embedding of $F-y$, and condition on the event $\mathcal{H}$. Now, either the image of the neighbours of y form an α-bad tuple, or they do not. In the former case, y is skipped, and the event $h(y)=v$ does not occur. In the latter case, y is chosen uniformly at random inside a set of size at least $d^{\Delta}|V|/2$. □

Lemma 23. *Suppose that we are given $\alpha \in (0,\frac{1}{4})$, a forest F of maximum degree at most Δ with a bipartition into primary and secondary vertices, and an (α,Δ)-superquasirandom graph $G=(V,E)$ of density d and with $|V| \geq 4\Delta/d$.*

Let h be the limping homomorphism from F to G. Let $x,y \in V(F)$ be two distinct vertices, and $u,v \in V$ be not necessarily distinct. Then we have

$$\mathbb{P}[h(x)=u \text{ and } h(y)=v] < \left(\frac{2}{d}\right)^{4\Delta^2}\frac{1}{|V|^2}.$$

Proof. If x and y form an edge, then this follows from Lemma 22(d) because

$$\frac{\left(1+\alpha\left(\frac{2}{d}\right)^{\Delta}\right)^{\Delta+2}}{d} = \frac{(d^{\Delta}+\alpha 2^{\Delta})^{\Delta+2}}{d^{\Delta(\Delta+2)+1}} \leq \frac{2^{\Delta(\Delta+2)}}{d^{4\Delta^2}} \leq \left(\frac{2}{d}\right)^{4\Delta^2}.$$

If x and y are in different components, or the path from x to y contains at least two primary vertices, then $h(x)$ and $h(y)$ are independent, and thus the claim follows from Lemma 22(a) and (e) and a similar calculation as in the previous case.

Thus the only remaining case is that x and y are both secondary and at distance two. We now first expose the entire embedding of $F-\{x,y\}$. Then either the image of $\mathrm{N}(x)$ forms an α-bad tuple, or it does not. In the former case x is not mapped at all. In the latter case, x is chosen uniformly among the at least $(1-\alpha)d^{\Delta}|V|$ vertices in $U_x = \mathrm{N}_G\big(h(\mathrm{N}_F(x))\big)$. Likewise, we have that y is either not mapped, or it is mapped to a vertex selected uniformly in a set U_y with $|U_y| \geq (1-\alpha)d^{\Delta}|V|$. Hence (even though the sets U_x are and U_y are not independent), we get $\mathbb{P}[h(x)=u \text{ and } h(y)=v] \leq (\frac{1}{(1-\alpha)d^{\Delta}|V|})^2 \leq (\frac{2}{d})^{4\Delta^2}$. □

Lemma 24. *Suppose that we are given $\alpha \in (0,\frac{1}{4})$, a forest F of maximum degree at most Δ with a bipartition into primary and secondary vertices, and an (α,Δ)-superquasirandom graph $G=(V,E)$ of density d.*

Let h be the limping homomorphism of F to G. Suppose that $v\in V$ is arbitrary, $x \in V(F)$ is an arbitrary primary vertex, and $y \in V(F)$ is an arbitrary secondary vertex. Then we have:

(a) $\mathbb{P}\left[\exists z \in V(F) \setminus \{x\} \;:\; h(x) = h(z)\right] \leq \frac{v(F)}{|V|}$ *and*
$\mathbb{P}\left[\exists z \in V(F) \setminus \{y\} \;:\; h(x) = h(z) \mid h(y) = v\right] \leq \frac{2v(F)}{d^{\Delta}(1-\alpha(\frac{2}{d})^{\Delta})^{\Delta+3}|V|}$.

(b) $\mathbb{P}\left[\exists z \in V(F) \setminus \{y\} \;:\; h(y) = h(z)\right] \leq \frac{2v(F)}{d^{\Delta}|V|}$ *and*
$\mathbb{P}\left[\exists z \in V(F) \setminus \{y\} \;:\; h(y) = h(z) \mid h(x) = v\right] \leq \frac{2v(F)}{d^{\Delta}|V|}$.

(c) *For the number of colliding vertices* $\mathrm{VC} = \{z \in V(F) : \exists z' \;:\; h(z) = h(z')\}$ *and every* $t > 0$ *we have* $\mathbb{P}\left[|\mathrm{VC}| \geq \frac{2v(F)^2}{d^{\Delta}|V|} + t\right] \leq 2\exp(-\frac{t^2}{2(\Delta+1)^2 v(F)})$.

Proof. (a) We expose the entire embedding of $F - \big(\{x\} \cup \mathrm{N}_F(x)\big)$. This is compatible with the order of embedding in the definition of limping homomorphisms because all vertices in $\mathrm{N}_F(x)$ are secondary, and they are the only secondary vertices whose embedding depends on the embedding of x. Let W be the image of the vertices in $F - \big(\{x\} \cup \mathrm{N}_F(x)\big)$. Observe that the event $\mathcal{E}$ that there is $z \in V(F) \setminus \{x\}$ with $h(x) = h(z)$ occurs if and only if the event $\mathcal{E}'$ that $h(x) \in W$ occurs. But, no matter which vertices ended up in the set W, the probability of $\mathcal{E}'$ (conditioned on W) is $\frac{|W|}{|V|} \leq \frac{v(F)}{|V|}$. Hence $\mathbb{P}\left[\mathcal{E}\right] \leq \frac{v(F)}{|V|}$.

The second part of (a) follows from

$$\mathbb{P}\left[\mathcal{E} \mid h(y) = v\right] = \frac{\mathbb{P}\left[h(y) = v \mid \mathcal{E}\right] \cdot \mathbb{P}\left[\mathcal{E}\right]}{\mathbb{P}[h(y) = v]} \leq \frac{\frac{2}{d^{\Delta}|V|} \cdot \frac{v(F)}{|V|}}{\frac{\left(1-\alpha\left(\frac{2}{d}\right)^{\Delta}\right)^{\Delta+3}}{|V|}} \,,$$

where we use Lemma 22(e) and Lemma 22(g).

(b) We expose the entire embedding of $F - \{y\}$. Let W be the image of the vertices in $F - \{y\}$. Then we either know that y is skipped, or we place y u.a.r. in a set of size at least $(1-\alpha)d^{\Delta}|V|$. Similarly as in (a) the event we are interested in occurs if and only if $h(y) \in W$, which (conditioned on W) has probability at most $\frac{|W|}{(1-\alpha)d^{\Delta}|V|} \leq \frac{2v(F)}{d^{\Delta}|V|}$. This reasoning is valid even in the conditional space $h(x) = v$.

(c) Using the bounds from (a) and (b), we get $\mathbb{E}\left[|\mathrm{VC}|\right] \leq \frac{2v(F)^2}{d^{\Delta}|V|}$. We would now like to apply McDiarmid's inequality, Lemma 17, to show concentration of $|\mathrm{VC}|$. For this purpose we consider the product space Ω_F from (17) and view $|\mathrm{VC}|$ as a function from Ω_F to $\mathbb{R}$. We claim that $|\mathrm{VC}|$ is $2(\Delta+1)$-Lipschitz. Indeed, consider first the case that for a single secondary vertex y the random real $\tau(y)$ changes. This only effects the embedding of y and hence $|\mathrm{VC}|$ changes by 2 at most. If, on the other hand, for a single primary vertex x the random choice of $h(x)$ changes, then only the embedding of x and possibly its neighbours is effected. Hence in this case $|\mathrm{VC}|$ changes by at most $2(\Delta+1)$, as claimed. Therefore McDiarmid's Inequality (Lemma 17) implies that

$$\mathbb{P}\left[|\mathrm{VC}| \geq \tfrac{2v(F)^2}{d^{\Delta}|V|} + t\right] \leq \mathbb{P}\Big[|\mathrm{VC}| \geq \mathbb{E}\big[|\mathrm{VC}|\big] + t\Big] \leq 2\exp\left(-\frac{2t^2}{(2(\Delta+1))^2 v(F)}\right) . \qquad \square$$

8. Proof of the Nibble Lemma (Lemma 21)

Suppose that the numbers $\varepsilon, \beta, c, \Delta$ are given. Let us take

$$0 < \alpha \ll \alpha_{\mathsf{A}} \ll \alpha_{\mathsf{B}} \ll \alpha_{\mathsf{C}} \ll \alpha_{\mathsf{D}} \ll \alpha_{\mathsf{E}} \ll \beta \,.$$

That is we fix (in this order) α_{E}, α_{D}, α_{C}, α_{B}, α_{A}, and α sufficiently small as a function of $\varepsilon, \beta, c, \Delta$, and of the previously fixed constants. Given r, let n_0 be sufficiently large. Let $\mathcal{F}$, G and $\mathcal{U}_i$ be as in the setting of Lemma 21.

For each $i \in [c]$ and each $s \in [k_i]$, the graph $G[V_{i,s}]$ has order at least εn, and hence, by Lemma 10, it is a $(3\alpha/\varepsilon^2)$-quasirandom graph of density $d \pm 3\alpha/\varepsilon^2$. By Lemma 13, this implies that $G[V_{i,s}]$ contains an almost spanning induced subgraph $G_{i,s}$ that is $(\alpha_{\mathsf{A}}, \Delta)$-superquasirandom and has order $m_{i,s}$

$$m_{i,s} \geq (1 - \alpha_{\mathsf{A}})|V_{i,s}| > \varepsilon n/2 \tag{19}$$

and density $d_{i,s} = d \pm \alpha_{\mathsf{A}}$. Since $\mathcal{U}_i$ is (α, n)-homogeneous, we have that $||U_{i,s}| - |U_{i,s'}|| \leq \alpha n$ for each $s, s' \in [k_i]$. Consequently, $m_{i,s} = (1 \pm 2\alpha_{\mathsf{A}})m_{i,s'}$. Thus, we can choose numbers $m_i > \varepsilon n/2$ such that

$$m_{i,s} = (1 \pm \alpha_{\mathsf{A}})m_i. \tag{20}$$

Finally, we recall that

$$(1-\alpha)\frac{n}{2r} \leq n_{i,s} = v(F_{i,s}) \leq \frac{2n}{r}. \tag{21}$$

We now define the limping homomorphism $h_{i,s}$ of $(F_{i,s} - X_{i,s})$ to $G_{i,s}$ so that the vertices of $V(F_{i,s}) \setminus X_{i,s}$ of odd distance from $X_{i,s}$ are the primary vertices and the ones at even distance are the secondary vertices. We denote the set of the primary and the secondary vertices in $F_{i,s}$ by $\mathsf{prim}_{i,s}$, and by $\mathsf{sec}_{i,s}$, respectively. Let $\mathsf{prim} = \bigcup_{i,s} \mathsf{prim}_{i,s}$ and $\mathsf{sec} = \bigcup_{i,s} \mathsf{sec}_{i,s}$. Let $Y_{i,s}$ denote the set of vertices skipped by $h_{i,s}$. Notice that $\mathsf{prim}_{i,s}$ $\mathsf{sec}_{i,s}$

$$X_{i,s} \cap Y_{i,s} = \emptyset \qquad \text{and} \qquad F_{i,s}[X_{i,s} \cup Y_{i,s}] \text{ is an independent set .}$$ prim sec $Y_{i,s}$

Let $h : \bigcup_{i,s} F_{i,s} \to G$ be the union of the homomorphisms $h_{i,s}$, and let $H \subseteq G$ denote the image of the edges of the graphs $F_{i,s}$ under h, i.e. $H = \bigcup_{i,s} E(h_{i,s})$. h

It is our goal to show that the random partial homomorphisms $h_{i,s}$ satisfy the assertions of the lemma with positive probability. We will show that each of the assertions is actually met with high probability. The following table shows lemmas corresponding to individual assertions: H

(C1)	(C2)	(C3)	(C4)	(C5)	(C6)	(C7)	(C8)
Lem 27	Lem 28	Lem 30	Lem 31	Lem 32	Lem 33	Lem 39	Lem 40 and Lem 41

In addition to the parameters controlled by the lemma, we need to control the following quantities. For $v \in V$, define $D_P(v)$ and $D_S(v)$ to be the number of all primary and secondary vertices, respectively, that are mapped to v, $D_P(v)$ $D_S(v)$

$$D_P(v) = \left|h^{-1}(v) \cap \mathsf{prim}\right| \quad \text{and} \quad D_S(v) = \left|h^{-1}(v) \cap \mathsf{sec}\right| .$$

Lemma 25. *We have*

$$\mathbb{P}\left[\exists v \in V\colon D_P(v) > \frac{15n}{\varepsilon r}\right] \leq \exp(-\sqrt{n}) \text{ , and} \tag{22}$$

$$\mathbb{P}\left[\exists v \in V\colon D_S(v) > \frac{15n}{\varepsilon r}\right] \leq \exp(-\sqrt{n}) \text{ .} \tag{23}$$

Further, the same bounds hold, if we condition on $h(z) = u$ for an arbitrary $z \in V(F_{i,s})$ with $i \in [c]$ and $s \in [k_i]$ and $u \in V(G_{i,s})$.

Proof. We fix a vertex $v \in V$ and first compute the expected number of primary vertices mapped to v. For every $i \in [c]$ and $s \in [k_i]$, we embed at most $v(F_{i,s}) \leq 2n/r$ primary vertices into the set $V(G_{i,s})$ with $m_{i,s} \geq \varepsilon n/2$ vertices. Since there are at most $2n$ choices of pairs (i,s), this gives that $\mathbb{E}[D_P(v)] \leq \frac{8n}{\varepsilon r}$. The Chernoff bound (5) with $\mu = 8n/(\varepsilon r)$ and $\delta = \frac{1}{2}$ and a union bound over all choices of v gives (22).

To prove (23), let us again fix a vertex $v \in V$. Lemma 22(e) gives that for a fixed secondary vertex y,

$$\mathbb{P}[h(y) = v] \leq \frac{\left(1 + \alpha_{\mathsf{A}}\left(\frac{2}{d \pm 2\alpha_{\mathsf{A}}}\right)^{\Delta}\right)^{\Delta+3}}{m_{i,s}} \overset{(19)}{\leq} \frac{3}{\varepsilon n}. \tag{24}$$

For each (i,s) consider the square $F^2_{i,s}[\mathsf{sec}_{i,s}]$ of the graph $F_{i,s}[\mathsf{sec}_{i,s}]$. This graph has maximum degree at most Δ^2, and thus is (Δ^2+1)-colourable by Brooks' Theorem. Let $V(F_{i,s}) = C^1_{i,s} \dot\cup \ldots \dot\cup C^{\Delta^2+1}_{i,s}$ be a colouring of $F^2_{i,s}[\mathsf{sec}_{i,s}]$. Note that the events $h(x) = v$ and $h(x') = v$ for $x \neq x' \in C^{\ell}_{i,s}$ are independent, because the unique x,x'-path in $F_{i,s}$ contains at least two primary vertices. The same reasoning gives that the events $\{h(x) = v\}_{x \in C^{\ell}_{i,s}}$ are in fact mutually independent.

We let $C^{\ell} = \bigcup_{i,s} C^{\ell}_{i,s}$ and $Z^{\ell} = \left|C^{\ell} \cap h^{-1}(v)\right|$. Since we have at most $2n$ forests $F_{i,s}$, it follows from (21) that

$$|C^{\ell}| \leq \sum_{\ell'} |C^{\ell'}| \leq \sum_{i,s} v(F_{i,s}) \leq 4n^2/r\,. \tag{25}$$

Thanks to the bound in (24) and the mutual independence described above, the random variable Z^{ℓ} is stochastically dominated by a random variable $Z \in \mathrm{Bin}(|C^{\ell}|, 3/(\varepsilon n))$. We would like to apply the Chernoff bound in (7) with

$$\mu = |C^{\ell}|3/(\varepsilon n) \quad \text{and} \quad \delta' = 1 + \frac{1}{10^3(\Delta^2+1)} \quad \text{and} \quad t = \mu + \frac{n}{10(\Delta^2+1)\varepsilon r}\,.$$

We check the condition of (7),

$$\delta'\mu = \left(1 + \frac{1}{10^3(\Delta^2+1)}\right)\mu = \mu + \frac{3|C^{\ell}|}{10^3(\Delta^2+1)\varepsilon n} \overset{(25)}{\leq} \mu + \frac{12n}{10^3(\Delta^2+1)\varepsilon r} \leq t.$$

Hence we can indeed apply (7) and obtain $\delta'' > 0$ (independent of n) for which

$$\mathbb{P}\left[Z^{\ell} \geq \mu + \frac{n}{10(\Delta^2+1)\varepsilon r}\right] \leq \exp\left(-\delta'' \frac{n}{10(\Delta^2+1)\varepsilon r}\right).$$

By a union bound over all $\ell \in [\Delta^2+1]$ we get that with probability at least $1 - \exp(-n^{2/3})$

$$\begin{aligned} D_S(v) &= \sum_{\ell=1}^{\Delta^2+1} Z^{\ell} \leq \sum_{\ell=1}^{\Delta^2+1} \left(\mu + \frac{n}{10(\Delta^2+1)\varepsilon r}\right) \\ &= \sum_{\ell=1}^{\Delta^2+1} \left(|C^{\ell}|3/(\varepsilon n) + \frac{n}{10(\Delta^2+1)\varepsilon r}\right) \overset{(25)}{\leq} \frac{3}{\varepsilon n}\frac{4n^2}{r} + \frac{1}{10}\frac{n(\Delta^2+1)}{(\Delta^2+1)\varepsilon r} \leq \frac{14n}{\varepsilon r}. \end{aligned}$$

Finally, another union bound over all $v \in V$ shows that (23) is satisfied.

Since the placement of all but at most Δ^2+1 of the forest vertices is independent of the placement of z we also get the bounds from (22) and (23) if we condition on $h(z) = u$. □

Lemma 26. *Let $z \in V(F_{i,s})$ with $i \in [c]$ and $s \in [k_i]$ and $v \in V(G_{i,s})$ be arbitrary.*

$$\mathbb{P}\left[\Delta(H) > \frac{30\Delta n}{\varepsilon r}\right] \le 2\exp(-\sqrt{n}) \quad \textit{and} \quad \mathbb{P}\left[\Delta(H) > \frac{30\Delta n}{\varepsilon r} \mid h(z) = v\right] \le 2\exp(-\sqrt{n})\ .$$

Proof. This follows from the fact that $\Delta(H) \le \Delta \cdot \max_v(D_P(v) + D_S(v))$ and from Lemma 25. □

Lemma 27. *We have*

$$\mathbb{P}\left[\forall i \in [c] \quad \forall s \in [k_i]\colon \quad |Y_{i,s}| \le \frac{\beta n}{r}\right] \ge 1 - \exp(-\sqrt{n}).$$

Proof. Fix $i \in [c]$ and $s \in [k_i]$. By Lemma 22(c), for the number of vertices skipped by $h_{i,s}$ we have $\mathbb{E}\left[|Y_{i,s}|\right] \le \alpha_{\mathsf{A}}\frac{2n}{r}$. Note that the number of skipped vertices is Δ-Lipschitz. McDiarmid's Inequality (Lemma 17) with $t = \alpha_{\mathsf{A}}2n/r$ and $k = v(F_{i,s}) \le 2n/r$ gives that $\mathbb{P}\left[|Y_{i,s}| > 2 \cdot \frac{\alpha_{\mathsf{A}}2n}{r}\right] \le 2 \cdot \exp(\frac{-8\alpha_{\mathsf{A}}^2 n^2 r}{r^2 2n\Delta^2}) = 2\exp(-\frac{4\alpha_{\mathsf{A}}^2 n}{r\Delta^2})$. Hence using the union bound over all choices of (i,s) we obtain

$$\mathbb{P}\left[\exists i, s\colon \quad |Y_{i,s}| > \frac{\beta n}{r}\right] \quad \le \quad \mathbb{P}\left[\exists i, s\colon \quad |Y_{i,s}| > \frac{4\alpha_{\mathsf{A}} n}{r}\right] \quad \le \quad \exp(-\sqrt{n})\ .$$ □

Lemma 28. *We have*

$$\mathbb{P}\left[\forall i \in [c] \quad \forall s \in [k_i]\colon \quad |\mathrm{VC}_{i,s}| \le \frac{20n}{\varepsilon r^2 d^{\Delta}}\right] \ge 1 - \exp(-\sqrt{n})\ .$$

Proof. Fix $i \in [c]$ and $s \in [k_i]$. We first observe that

$$\frac{2v(F_{i,s})^2}{d_{i,s}^{\Delta} m_{i,s}} + \frac{n}{\varepsilon r^2 d^{\Delta}} \quad \le \quad \frac{4(2n/r)^2}{\frac{9}{10} d^{\Delta}\varepsilon n} + \frac{n}{\varepsilon r^2 d^{\Delta}} \quad \le \quad \frac{20n}{\varepsilon r^2 d^{\Delta}}\ .$$

Hence Lemma 24(c) with $t = \frac{n}{\varepsilon r^2 d^{\Delta}}$ gives that

$$\begin{aligned}\mathbb{P}\left[|\mathrm{VC}_{i,s}| \ge \frac{20n}{\varepsilon r^2 d^{\Delta}}\right] &\le \mathbb{P}\left[|\mathrm{VC}_{i,s}| \ge \frac{2v(F_{i,s})^2}{d_{i,s}^{\Delta} m_{i,s}} + \frac{n}{\varepsilon r^2 d^{\Delta}}\right] \\ &\le 2\exp\left(-\frac{n^2}{2\varepsilon^2 r^4 d^{2\Delta}(\Delta+1)^2 n_{i,s}}\right) \le 2\exp\left(-\frac{n}{4\varepsilon^2 r^3 d^{2\Delta}(\Delta+1)^2}\right)\ .\end{aligned}$$

Using a union bound over all choices (i,s), we get the statement of the lemma. □

Recall that $\mathrm{EC}_{i,s}$ contains all the vertices of $F_{i,s}$ that are contained in an edge collision. We define $\mathrm{EC}^*_{i,s} = \{xy \in E(F_{i,s})\colon xy \text{ is colliding}\}$. Notice that $|\mathrm{EC}_{i,s}| \le 2|\mathrm{EC}^*_{i,s}|$. $\mathrm{EC}^*_{i,s}$

Lemma 29. *Let $xy \in E(F_{i,s})$ be an edge with $x \in \mathsf{prim}_{i,s}$ and $y \in \mathsf{sec}_{i,s}$. Let $z \in V(F_{i,s}) \setminus \{y\}$ and $v \in V(G_{i,s})$. Then we have*

$$\mathbb{P}\left[xy \in \mathrm{EC}^*_{i,s} \mid h(z) = v\right] \le \frac{61\Delta}{\varepsilon^2 r d^{\Delta}} \quad \textit{and} \quad \mathbb{P}\left[y \in \mathrm{EC}_{i,s} \mid h(z) = v\right] \le \frac{61\Delta^2}{\varepsilon^2 r d^{\Delta}}\ ,$$

and hence also $\mathbb{P}\left[xy \in \mathrm{EC}^*_{i,s}\right] \le \frac{61\Delta}{\varepsilon^2 r d^{\Delta}}$.

Proof. Let $u = h(x)$ and z be an arbitrary vertex in $F_{i,s} - y$. Let $\{u_1, \ldots, u_p\} = h(\mathrm{N}_{F_{i,s}}(y) \setminus \{x\})$. We denote by $\mathcal{B}$ the event that $\{u, u_1, \ldots, u_p\}$ forms an α_{A}-bad set. First observe that

$p_1 = \mathbb{P}[xy \in \mathrm{EC}^*_{i,s} | h(z) = v \text{ and } \mathcal{B}] = 0$, because the event $\mathcal{B}$ implies that y is skipped and thus the edge xy cannot collide. On the other hand, if $\mathcal{B}$ does not occur, then

$$\left|\mathrm{N}_{G_{i,s}}(u, u_1, \ldots, u_p)\right| \geq (1 - \alpha_{\mathsf{A}})(d - \alpha_{\mathsf{A}})^{\Delta} m_{i,s} \geq \frac{1}{2} d^{\Delta} \varepsilon n. \tag{26}$$

Next we define

$$\tilde{\mathrm{N}}(u) = \left\{w \in \mathrm{N}_G(u) : \exists i \in [c]\ \exists s \in [k_i]\ \exists x'y' \in E(F_{i,s}) \text{ with } xy \neq x'y' \text{ and } h(x'y') = uw\right\} .$$

This means that the edge xy is colliding only if y is mapped to $\tilde{\mathrm{N}}(u)$. By Lemma 26 we have

$$p_2 = \mathbb{P}\Big[|\tilde{\mathrm{N}}(u)| > \frac{30\Delta n}{\varepsilon r}\ \Big|\ h(z) = v\Big] \leq 2\exp(-\sqrt{n}).$$

Moreover, because $z \neq y$ we have

$$p_3 = \mathbb{P}\left[xy \in \mathrm{EC}^*_{i,s}\ \middle|\ h(z) = v \text{ and } \overline{\mathcal{B}} \text{ and } |\tilde{\mathrm{N}}(u)| \leq \frac{30\Delta n}{\varepsilon r}\right] \overset{(26)}{\leq} \frac{30\Delta n}{\varepsilon r \cdot d^{\Delta}\varepsilon n/2} = \frac{60\Delta}{\varepsilon^2 r d^{\Delta}} .$$

Since $\mathbb{P}[xy \in \mathrm{EC}^*_{i,s} \mid h(z) = v] \leq p_1 + p_2 + p_3$, we obtain that $\mathbb{P}[xy \in \mathrm{EC}^*_{i,s} \mid h(z) = v] \leq \frac{61\Delta}{\varepsilon^2 r d^{\Delta}}$. In addition,

$$\mathbb{P}\left[y \in \mathrm{EC}_{i,s} \mid h(z) = v\right] \leq \sum_{x \in \mathrm{N}_{F_{i,s}}(y)} \mathbb{P}[xy \in \mathrm{EC}^*_{i,s} \mid h(z) = v] \leq \Delta \frac{61\Delta}{\varepsilon^2 r d^{\Delta}} .$$ □

Lemma 30. *We have*

$$\mathbb{P}\left[\exists i \in [c], s \in [k_i] \colon |\mathrm{EC}_{i,s}| > \frac{300\Delta n}{\varepsilon^2 r^2 d^{\Delta}}\right] \leq \exp(-\sqrt{n}) .$$

Proof. Fix an arbitrary $i \in [c]$ and an arbitrary $s \in [k_i]$. Consider the line graph $L(F_{i,s})$ of $F_{i,s}$, and let $L(F_{i,s})^3$ be its cube. The maximum degree of $L(F_{i,s})$ is at most $2(\Delta - 1)$, and hence the maximum degree of $L(F_{i,s})^3$ is less than $8\Delta^3$. By Brooks' Theorem, there exists an $(8\Delta^3)$-colouring $E_1 \dot\cup \ldots \dot\cup E_{8\Delta^3}$ of $L(F_{i,s})^3$ which we view as an edge-colouring of $F_{i,s}$. Observe that edges of the same colour have mutual distances at least 3 in $F_{i,s}$, and hence their placements are mutually independent. For each $j \in [8\Delta^3]$ denote by C_j the number of colliding edges in E_j. We have that $\sum_{j=1}^{8\Delta^3} C_j = |\mathrm{EC}^*_{i,s}|$ and that $\sum_{j=1}^{8\Delta^3} |E_j| = |E(F_{i,s})| \leq n_{i,s} \leq \frac{2n}{r}$.

Fix a number $j \in [8\Delta^3]$. By the first assertion of Lemma 29 and by the mutual independence described above, the random variable C_j is stochastically dominated by $\mathrm{Bin}(|E_j|, \frac{61\Delta}{\varepsilon^2 r d^{\Delta}})$. Thus by Chernoff's inequality (6) with $\mu = \frac{61\Delta}{\varepsilon^2 r d^{\Delta}}|E_j|$ and $t = n/(4\varepsilon^2 d^{\Delta}\Delta^3 r^2)$ we get

$$\mathbb{P}\left[C_j \geq \frac{61\Delta}{\varepsilon^2 r d^{\Delta}}|E_j| + \frac{n}{4\varepsilon^2 d^{\Delta}\Delta^3 r^2}\right] \leq \exp\left(-\frac{2n^2}{16\varepsilon^4 d^{2\Delta}\Delta^6 r^4 |E_j|}\right) \leq \exp\left(-n^{2/3}\right) .$$

Using the union bound over all $j \in [8\Delta^3]$ we get that $\mathbb{P}[|\mathrm{EC}_{i,s}| > \frac{124\Delta}{\varepsilon^2 r d^{\Delta}} \cdot \frac{2n}{r}] \leq \mathbb{P}[|\mathrm{EC}^*_{i,s}| > \frac{62\Delta}{\varepsilon^2 r d^{\Delta}} \cdot \frac{2n}{r}] < 8\Delta^3 \cdot \exp\left(-n^{2/3}\right)$. The lemma follows by a union bound over all pairs (i, s). □

Lemma 31. *We have*

$$\mathbb{P}\left[\exists v \in V \colon |\mathrm{FN}(v)| > \frac{10^4\Delta^3 n}{\varepsilon^3 r^2 d^{2\Delta}}\right] \leq \exp(-\sqrt{n}) .$$

Proof. Fix a vertex $v \in V$. Fix $i \in [c]$ and $s \in [k_i]$.

Claim 31.1. Let $xy \in E(F_{i,s})$. Then $\mathbb{P}[h(x) = v \text{ and } y \text{ is faulty}] \leq \frac{10^3\Delta^2}{\varepsilon^3 r d^{2\Delta} n}$.

Proof of Claim 31.1. We shall use

$$\mathbb{P}[h(x)=v \text{ and } y \text{ is faulty}] = \mathbb{P}[h(x)=v]\cdot\mathbb{P}[y \text{ is faulty} \mid h(x)=v]\,. \tag{27}$$

First consider the case that x is primary. The secondary vertex y is faulty if it is colliding, or if it is in an edge collision (due to either xy colliding, or yz colliding with $z\in \mathrm{N}_{F_{i,s}}(y)\setminus\{x\}$). For vertex collisions, by Lemma 24(b), we have

$$\mathbb{P}[y\in \mathrm{VC}_{i,s}\mid h(x)=v] \le \frac{2n_{i,s}}{d^{\Delta}m_{i,s}} \overset{(19),(21)}{\le} \frac{8}{\varepsilon r d^{\Delta}}\,.$$

For edge collisions, Lemma 29 gives $\mathbb{P}[y\in \mathrm{EC}_{i,s}\mid h(x)=v]\le \frac{61\Delta^2}{\varepsilon^2 r d^{\Delta}}$. Hence

$$\mathbb{P}[y \text{ is faulty} \mid h(x)=v] \le \frac{8}{\varepsilon r d^{\Delta}} + \frac{61\Delta^2}{\varepsilon^2 r d^{\Delta}} \le \frac{70\Delta^2}{\varepsilon^2 r d^{\Delta}}\,.$$

Since $\mathbb{P}[h(x)=v]=\frac{1}{m_{i,s}}\le\frac{2}{\varepsilon n}$ by Lemma 22(a), we get together with (27) that

$$\mathbb{P}[h(x)=v \text{ and } y \text{ is faulty}] \le \frac{140\Delta^2}{\varepsilon^3 r d^{\Delta} n}\,,$$

which gives the claim in this case.

Next, consider the case that x is secondary. We denote by $\mathcal{A}_1$ the event that y is in a vertex collision. We denote by $\mathcal{A}_2$ the event that y together with some vertex $z\in\mathrm{N}(y)\setminus\{x\}$ forms a colliding edge. We denote by $\mathcal{A}_3$ the event that xy is colliding. The primary vertex y is faulty if at least one of the events $\mathcal{A}_1$, $\mathcal{A}_2$, or $\mathcal{A}_3$ occurs.

By Lemma 24(a) we have

$$\mathbb{P}[\mathcal{A}_1\mid h(x)=v] \le \frac{2n_{i,s}}{d^{\Delta}\frac{1}{2}m_{i,s}} \le \frac{16}{r d^{\Delta}\varepsilon}\,. \tag{28}$$

For $z\in\mathrm{N}(y)\setminus\{x\}$ fixed, Lemma 29 gives $\mathbb{P}[yz\in\mathrm{EC}_{i,s}\mid h(x)=v]\le\frac{61\Delta}{\varepsilon^2 r d^{\Delta}}$. Hence we obtain

$$\mathbb{P}[\mathcal{A}_2\mid h(x)=v] \le \Delta\frac{61\Delta}{\varepsilon^2 r d^{\Delta}}\,. \tag{29}$$

In order to obtain a similar bound for the event $\mathcal{A}_3$, let $\mathcal{A}_3'$ be the event that xy is in an edge collision with an edge from a different forest $F_{i',s'}$. Let H' be the graph formed by the images of all forests but $F_{i,s}$, that is, $H-E(h_{i,s})$. Now fix a mapping of all forests but $F_{i,s}$. By Lemma 26, with probability at least $1-2\exp(-\sqrt{n})$ we have $\Delta(H')\le\frac{30\Delta n}{\varepsilon r}$ (and this is independent of the event $h(x)=v$). Assume that this is the case and let $\mathbb{P}_{i,s,H'}$ be (the measure on) the conditional probability space associated with the limping homomorphism for $F_{i,s}$. We have,

$$\begin{aligned}\mathbb{P}_{i,s,H'}[\mathcal{A}_3'\mid h(x)=v] &= \mathbb{P}_{i,s,H'}[h(y)\in\mathrm{N}_{H'}(v)\mid h(x)=v]\\ &\le \sum_{u\in\mathrm{N}_{H'}(v)} \mathbb{P}_{i,s,H'}[h(y)=u\mid h(x)=v]\,.\end{aligned}$$

For a fixed vertex $u\in V$ we have

$$\mathbb{P}_{i,s,H'}[h(y)=u\mid h(x)=v] = \mathbb{P}[h(y)=u \text{ and } h(x)=v]/\mathbb{P}[h(x)=v]\,,$$

which by Lemma 22(d) and Lemma 22(e) is at most $(\frac{2}{dm_{i,s}^2})/\frac{1}{2m_{i,s}}=\frac{4}{dm_{i,s}}$. Hence,

$$\mathbb{P}_{i,s,H'}[\mathcal{A}_3'\mid h(x)=v] \le \frac{30\Delta n}{\varepsilon r}\cdot\frac{4}{dm_{i,s}} \le \frac{120\Delta}{\varepsilon^2 d r}\,.$$

Returning to our original probability space we thus obtain $\mathbb{P}[\mathcal{A}_3' \mid h(x)=v] \leq 2\exp(-\sqrt{n}) + \frac{120\Delta}{\varepsilon^2 dr}$. Since

$$\mathbb{P}[\mathcal{A}_3 \mid h(x)=v] \leq \mathbb{P}[\mathcal{A}_3' \mid h(x)=v] + \mathbb{P}[y \in \mathrm{VC}_{i,s} \mid h(x)=v]\,,$$

we conclude from Lemma 24(a) that

$$\mathbb{P}[\mathcal{A}_3 \mid h(x)=v] \leq \frac{121\Delta}{\varepsilon^2 dr} + \frac{2n_{i,s}}{d^\Delta \frac{1}{2} m_{i,s}} \leq \frac{121\Delta}{\varepsilon^2 dr} + \frac{16}{rd^\Delta \varepsilon} \leq \frac{150\Delta}{\varepsilon^2 rd^\Delta}\,. \tag{30}$$

Finally, since $\mathbb{P}[h(x)=v] \leq \frac{2}{m_{i,s}} \leq \frac{4}{\varepsilon n}$ by Lemma 22(e), we get from (27), (28), (29) and (30) that

$$\mathbb{P}[h(x)=v \text{ and } y \text{ is faulty}] \leq \frac{4}{\varepsilon n}\Big(\frac{16}{\varepsilon rd^\Delta} + \frac{61\Delta^2}{\varepsilon^2 rd^\Delta} + \frac{150\Delta}{\varepsilon^2 rd^\Delta}\Big) \leq \frac{4}{\varepsilon n} \cdot \frac{200\Delta^2}{\varepsilon^2 rd^\Delta}\,,$$

which also gives the claim in this case. □

For $x \in V(F_{i,s})$ let $\mathcal{E}_{x,v}$ be the event that $h(x)=v$ and that there exists a vertex $y \in \mathrm{N}_{F_{i,s}}(x)$ such that y is faulty.

Claim 31.2. For each $x \in V(F_{i,s})$ we have $\mathbb{P}[\mathcal{E}_{x,v}] \leq \frac{10^3\Delta^3}{\varepsilon^3 rd^{2\Delta} n}$.

Proof of Claim 31.2. This follows immediately from $\Delta(F_{i,s}) \leq \Delta$ and Claim 31.1. □

We now return to the proof of Lemma 31 and recall that $\mathrm{FN}(v) = \bigcup_{i,s}\{x \in V(F_{i,s})\colon \mathcal{E}_{x,v}\}$. Observe first that for x and x' in $\bigcup_{i,s} F_{i,s}$ of distance at least 7 the events $\mathcal{E}_{x,v}$ and $\mathcal{E}_{x',v}$ are independent. Indeed, $\mathcal{E}_{x,v}$ (in the "worst case") could occur because of a neighbour y of x which is in a colliding edge yz, and $\mathcal{E}_{x',v}$ because of a neighbour y' of x' which is in a colliding edge $y'z'$. But since x and x' are of distance at least 7 they are either in different components of $\bigcup_{i,s} F_{i,s}$ or there are at least two primary vertices on the unique path between z and z', which implies independence.

So consider the 7-th power F^7 of $\bigcup_{i,s} F_{i,s}$, which has maximum degree less than Δ^7. Thus, by Brooks' Theorem there is a Δ^7-colouring $\bigcup_{i,s} V(F_{i,s}) = C^1 \dot\cup \dots \dot\cup C^{\Delta^7}$ of F^7. Denote by Z^ℓ the number of $x \in C^\ell$ such that $\mathcal{E}_{x,v}$ holds. By Claim 31.2 the random variable Z^ℓ is stochastically dominated by $\mathrm{Bin}(|C^\ell|, \frac{10^3\Delta^3}{\varepsilon^3 rd^{2\Delta} n})$. We now apply Chernoff's inequality (7) with

$$\mu = \frac{10^3\Delta^3}{\varepsilon^3 rd^{2\Delta} n}|C^\ell|\,, \quad \delta' = 1 + \frac{\varepsilon^3 d^{2\Delta}}{10^4\Delta^{10}} \quad \text{and} \quad t = \mu + \frac{n}{r^2\Delta^7}\,.$$

This is possible because

$$\delta'\mu = \mu + \frac{1}{10r\Delta^7 n}|C^\ell| \leq \mu + \frac{4n}{10r^2\Delta^7} \leq t\,.$$

We conclude that there is a constant $\delta'' > 0$ such that

$$\mathbb{P}\Big[Z^\ell \geq \frac{10^3\Delta^3}{\varepsilon^3 rd^{2\Delta} n}|C^\ell| + \frac{n}{r^2\Delta^7}\Big] \leq \exp\Big(-\delta''\frac{n}{r^2\Delta^7}\Big)\,.$$

A union bound over all $\ell \in [\Delta^7]$ thus gives that with probability at least $1 - \Delta^7\exp\big(-\delta'' n/(r^2\Delta^7)\big)$ we have

$$|\mathrm{FN}(v)| = \sum_{\ell \in [\Delta^7]} Z^\ell \leq \frac{10^3\Delta^3}{\varepsilon^3 rd^{2\Delta} n} \cdot \frac{4n^2}{r} + \Delta^7 \cdot \frac{n}{r^2\Delta^7} \leq \frac{10^4\Delta^3 n}{\varepsilon^3 r^2 d^{2\Delta}}\,,$$

that is, the desired bound for $|\mathrm{FN}(v)|$. The lemma then follows by a union bound over all vertices $v \in V$. □

Lemma 32. *We have*

$$\mathbb{P}\left[\exists v \in V \colon |\mathrm{YN}(v)| > \frac{\beta n}{r}\right] \leq \exp(-\sqrt{n}) .$$

Proof. We proceed similarly as in the proof of the previous lemma. Fix $v \in V$. For $x \in V(F_{i,s})$ denote by $\mathcal{E}_{x,v}$ the event that $h(x) = v$ and that there exists a vertex in $\mathrm{N}_{F_{i,s}}(x)$ that is skipped. By Lemma 22(a) and Lemma 22(b) we have

$$\mathbb{P}[\mathcal{E}_{x,v}] \leq \sum_{y \in \mathrm{N}_{F_{i,s}}(x)} \mathbb{P}[y \text{ is skipped} \mid h(x) = v] \cdot \mathbb{P}[h(x) = v] \leq \Delta \cdot \alpha_{\mathsf{A}} \cdot \frac{2}{\varepsilon n} . \tag{31}$$

Observe that $\mathrm{YN}(v) = \bigcup_{i,s}\{x \in V(F_{i,s}) \colon \mathcal{E}_{x,v}\}$. Moreover, for $x, x' \in \bigcup_{i,s} V(F_{i,s})$ of distance at least 6 the events $\mathcal{E}_{x,v}$ and $\mathcal{E}_{x',v}$ are independent. Therefore we consider the 6-th power F^6 of $\bigcup_{i,s} F_{i,s}$. Since F^6 has maximum degree less than Δ^6 this graph has a Δ^6-colouring $\bigcup_{i,s} V(F_{i,s}) = C^1 \dot\cup \ldots \dot\cup C^{\Delta^6}$ by Brooks' Theorem. For $\ell \in \Delta^6$ let Z^ℓ be the number of $x \in C^\ell$ such that $\mathcal{E}_{x,v}$ holds. By (31) the random variable Z^ℓ is stochastically dominated by $\mathrm{Bin}(|C^\ell|, \frac{2\Delta\alpha_{\mathsf{A}}}{\varepsilon n})$. Thus we can apply Chernoff's inequality (7) with

$$\mu = \frac{2\Delta\alpha_{\mathsf{A}}}{\varepsilon n}|C^\ell| , \quad \delta' = 1 + \frac{\varepsilon}{8\Delta^7} \quad \text{and} \quad t = \mu + \frac{\alpha_{\mathsf{A}} n}{r\Delta^6} ,$$

which is possible because $\delta'\mu \leq \mu + \frac{\varepsilon}{8\Delta^7} \cdot \frac{2\Delta\alpha_{\mathsf{A}}}{\varepsilon n} \cdot \frac{4n^2}{r} = t$. We conclude that there is $\delta'' > 0$ such that

$$\mathbb{P}\Big[Z^\ell \geq \frac{2\Delta\alpha_{\mathsf{A}}}{\varepsilon n}|C^\ell| + \frac{\alpha_{\mathsf{A}} n}{r\Delta^6}\Big] \leq \exp\Big(-\delta''\frac{\alpha_{\mathsf{A}} n}{r\Delta^6}\Big) .$$

Hence with probability at least $1 - \Delta^6 \cdot \exp\big(-\delta''\frac{\alpha_{\mathsf{A}} n}{r\Delta^6}\big)$ we have

$$|\mathrm{YN}(v)| = \sum_{\ell \in [\Delta^6]} Z^\ell \leq \frac{2\Delta\alpha_{\mathsf{A}}}{\varepsilon n} \cdot \frac{4n^2}{r} + \Delta^6\frac{\alpha_{\mathsf{A}} n}{r\Delta^6} \leq 9\frac{\Delta\alpha_{\mathsf{A}} n}{\varepsilon r} < \frac{\beta n}{r} .$$

The lemma follows by a union bound over $v \in V$. □

Lemma 33. *We have*

$$\mathbb{P}\left[\exists v \in V \colon |\mathrm{XN}(v)| > \frac{\beta n}{r}\right] \leq \exp(-\sqrt{n}) .$$

Proof. Fix $v \in V$. For $i \in [c]$ and $s \in [k_i]$ let $Q_{i,s} = \bigcup_{x \in X_{i,s}} \mathrm{N}_{F_{i,s}}(x)$. By definition each $y \in Q_{i,s}$ is primary, hence y gets mapped to v with probability at most $\frac{2}{\varepsilon n}$ by Lemma 22(a). These events are independent, and thus the number of vertices in $\bigcup_{i,s} Q_{i,s}$ which are mapped to v is stochastically dominated by $\mathrm{Bin}(\frac{2}{\varepsilon n}, \sum_{i,s}|Q_{i,s}|)$. We have $\sum_{i,s}|Q_{i,s}| \leq \Delta\sum_{i,s}|X_{i,s}| \leq \Delta \cdot 2n\frac{\alpha n}{r}$. Thus, by Chernoff's inequality (5) applied with $\mu = \frac{2}{\varepsilon n} \cdot \frac{2\Delta\alpha n^2}{r} = \frac{4\Delta\alpha n}{\varepsilon r}$ and $\delta = 1$ we have

$$\mathbb{P}\Big[|\mathrm{XN}(v)| > \frac{\beta n}{r}\Big] \leq \mathbb{P}\Big[|\mathrm{XN}(v)| > \frac{8\Delta\alpha n}{\varepsilon r}\Big] \leq 2\exp\Big(-\frac{4\Delta\alpha n}{3\varepsilon r}\Big) .$$

The lemma follows by taking the union bound over all choices of v. □

We now prepare for the proof of (C7).

Definition 34 (important group)**.** *We say that* $i \in [c]$ *is an* important group *if* $k_i > \frac{\sqrt{\alpha} n r}{2}$*. The set of important groups is denoted by* $\mathsf{IG} \subseteq [c]$*.* important g. IG

Lemma 35. *The total number of edges in forests* $(F_{i,s})_{i,s}$ *from non-important groups is less than* $\beta n^2/16$*.*

Proof. By definition there are at most $\frac{\sqrt{\alpha} n r}{2}$ forests in each non-important groups and each such forest has at most $\frac{2n}{r}$ edges. The number of non-important groups is at most c. As $c\sqrt{\alpha} < \beta/16$, the claim follows. □

Definition 36 (typical)**.** *Let* $i \in [c]$*. A pair* $uv \in \binom{V}{2}$ *is called* i-typical *if* $(\mathrm{load}(u, v, \mathcal{U}_i) - \mu(\mathcal{U}_i))^2 \leq \sqrt{\alpha} n^2$*, and* i-atypical*, otherwise. An edge* $uv \in E$ *is called* typical*, if it is* i*-typical for each* $i \in [c]$*, and* atypical *otherwise.* i-typical i-atypical typical atypical

Lemma 37. *For each* $i \in [c]$ *there are at most* $\sqrt[4]{\alpha} n^2$ *pairs in* $\binom{V}{2}$ *that are* i*-atypical. Consequently, there are at most* $\beta n^2/16$ *atypical edges in the graph* G*.*

Proof. For each group $i \in [c]$, we have $\sigma(\mathcal{U}_i) < \alpha n^4$ by assumption, and thus at most $\sqrt[4]{\alpha} n^2$ pairs satisfy $(\mathrm{load}(u, v, \mathcal{U}_i) - \mu(\mathcal{U}_i))^2 > \sqrt{\alpha} n^2$ and are thus i-atypical. As $c\sqrt[4]{\alpha} < \beta/16$, the assertion second follows. □

For showing the quasirandomness of $\tilde{G}$ we shall use the following easy error bound.

Lemma 38. *For each* $M \in (0, 1]$ *and each* $a \in (-0.5, \infty)$*, we have* $M - |a| \leq M^{1+a} \leq M + |a|$*.*

Proof. Suppose that M is fixed. The claim holds trivially for $a = 0$. Thus it suffices to prove that within the range of a, the derivative of M^{1+a} with respect to a is at most 1 in absolute value. We have $|\frac{\mathrm{d}}{\mathrm{d}a} M^{1+a}| = |M^{1+a} \ln M| \leq |\sqrt{M} \ln M|$. It can be numerically checked, that for each $M \in (0, 1]$, we have $\sqrt{M} \ln M \in (-0.8, 0]$. The claim follows. □

Lemma 39. *With probability at least* $1 - \exp(-\sqrt{n})$*, we have that* $\tilde{G}$ *is* β*-quasirandom.*

Proof. By the definition of quasirandomness (Definition 8) we need to show that with high probability there exists a number $p_{\tilde{G}}$ such that for each set $B \subseteq V$, we have that

$$e(\tilde{G}[B]) = p_{\tilde{G}} \binom{|B|}{2} \pm \beta n^2 .$$

As G is α-quasirandom, it is enough to show that with high probability there is a number p_h such that each set $B \subseteq V$ satisfies

$$\left| E(h) \cap \binom{B}{2} \right| = p_h \binom{|B|}{2} \pm \frac{\beta n^2}{2} .$$

Let us fix a set $B \subseteq V$. We first show that with high probability $|E(h) \cap \binom{B}{2}|$ is close to its expectation $\lambda_B = \mathbb{E}[|E(h) \cap \binom{B}{2}|]$. Note that the random variable $|E(h) \cap \binom{B}{2}|$ is Δ^2-Lipschitz. McDiarmid's Inequality, Lemma 17, gives that

$$\mathbb{P}\left[\left| \left| E(h) \cap \binom{B}{2} \right| - \lambda_B \right| \geq \frac{\beta n^2}{8} \right] \leq 2 \exp\left(-\frac{2\beta^2 n^4}{64 \Delta^4} \cdot \frac{r}{4n^2} \right) \leq \exp(-n^{19/10}) .$$

Since there are $2^m \leq 4^n$ choices of the set B, the lemma will follow from a union bound, if we show that for each set B we have

$$\mathbb{E}\left[\left| E(h) \cap \binom{B}{2} \right| \right] = p_h \binom{|B|}{2} \pm \frac{\beta n^2}{4} . \tag{32}$$

By Lemmas 35 and 37, the total contribution to the number of edges in $E(h)$ from non-important groups and from atypical edges is at most $\beta n^2/8$. Thus (32) follows if for each typical edge uv of G the probability that there is an edge of a forest in an important group that gets mapped to uv is $p_h \pm \alpha_{\mathsf{D}}$. We shall prove that this is the case in Claim 39.2, which will conclude the prove of the lemma.

Before turning to this claim, we consider a fixed important group i and bound the probability that a typical edge uv is the image of any edge of a forest of this group. Observe that it suffices to consider forests $F_{i,s}$ with $U_{i,s} \cap \{u,v\} = \emptyset$. Let $xy \in E(F_{i,s})$ for such a forest. Denote by $A(x,y,u,v)$ the event that $h(x) = u$ and $h(y) = v$. Then by Lemma 22(d) we have

$$\mathbb{P}\left[A(x,y,u,v)\right] = \left(1 \pm \alpha_{\mathsf{A}}(\frac{1}{d})^{\Delta}\right)^{\Delta+2} \frac{1}{dm_{i,s}^2} \overset{(20)}{=} (1 \pm \alpha_{\mathsf{B}}) \frac{1}{dm_i^2} \,. \tag{33}$$

Let H^i_{uv} be the set of all ordered pairs (x,y) such that $xy \in E(F_{i,s})$ for s with $U_{i,s} \cap \{u,v\} = \emptyset$ and

$$M_i(u,v) = \prod_{(x,y)\in H^i_{uv}} \mathbb{P}\left[\overline{A(x,y,u,v)}\right] \,. \tag{34}$$

Note that $M_i(u,v)$ is the probability that uv is not used by any forest from group i in an alternative random experiment where the forest edges are mapped to G independently. Our next goal is to show that in our random experiment the corresponding probability does not deviate much from $M_i(u,v)$.

Claim 39.1. For each $uv \in E(G)$ and each important group i we have

$$\mathbb{P}[h^{-1}(uv) \cap \bigcup_{s\in[k_i]} E(F_{i,s}) = \emptyset] = (1 \pm \alpha) M_i(u,v) \,.$$

Proof of Claim 39.1. We want to use Suen's inequality. Let $uv \in E$ be fixed and abbreviate $A(x,y) = A(x,y,u,v)$. We set up a superdependency graph for the events $\{A(x,y)\}_{(x,y)\in H^i_{uv}}$ as follows. For $(x,y),(x',y') \in H^i_{uv}$, define $(x,y) \sim (x',y')$ if $\mathrm{dist}(xy,x'y') \le 4$. Notice that the embedding of a primary vertex influences only the embedding of the vertices in its neighbourhood (and itself). The embedding of a secondary vertex on the other hand is independent of the embedding of all vertices of distance at least 3. As a consequence, we get that $\sim$ indeed defines a superdependency graph for the events $A(x,y)$. The degrees in the superdependency graph are at most $1 + 4\Delta^5 \le 5\Delta^5$. For $(x,y),(x',y') \in H^i_{uv}$, set

$$\nu_{xy,x'y'} = \frac{\mathbb{P}[A(x,y) \cap A(x',y')] + \mathbb{P}[A(x,y)] \cdot \mathbb{P}[A(x',y')]}{\prod(1 - \mathbb{P}[A(\tilde{x},\tilde{y})])} \,, \tag{35}$$

where the product in the denominator ranges through all $(\tilde{x},\tilde{y}) \in H^i_{uv}$ such that $(x,y) \sim (\tilde{x},\tilde{y})$ or $(x',y') \sim (\tilde{x},\tilde{y})$. We next upper-bound (35) in the case that $(x,y) \neq (x',y')$ are such that $(x,y) \sim (x',y')$. The denominator in (35) has at most $10\Delta^5$ factors, each of which is at least $1 - \frac{1+\alpha_{\mathsf{B}}}{dm_i^2}$ by (33). Similarly, by (33) the terms $\mathbb{P}[A(x,y)]$ and $\mathbb{P}[A(x',y')]$ are at most $\frac{1+\alpha_{\mathsf{B}}}{dm_i^2}$.

The event $A(x,y) \cap A(x',y')$ is empty when $x' = y$, or $x = y'$. If $x = x' \in \mathsf{sec}$ or $y = y' \in \mathsf{sec}$, the event $A(x,y) \cap A(x',y')$ puts requirements on the placement of two primary vertices and one secondary vertex $t \in \{x',y'\}$. Analogously to the proof of Lemma 22(d), we can show that in this case this event has probability

$$\mathbb{P}[A(x,y) \cap A(x',y')] = \frac{\left(1 \pm \alpha_{\mathsf{A}}(\frac{2}{d})^{\Delta}\right)^{\Delta+2}}{dm_{i,s}^3} \overset{(20)}{=} \frac{1 \pm \alpha_{\mathsf{B}}}{dm_i^3} \,.$$

It remains to consider the case when $\{x', y'\} \setminus \{x, y\}$ contains a secondary vertex. Without loss of generality assume that y' is secondary. We first expose the limping homomorphism entirely, except for y'. Two cases may occur: either y' is skipped, and therefore, $A(x', y')$ cannot occur, or the image of y' is selected uniformly among at least $d^{\Delta} m_i/2$ vertices. Using (33) we have

$$\begin{aligned}\mathbb{P}[A(x',y') \cap A(x,y)] &= \mathbb{P}[A(x',y')|A(x,y)] \cdot \mathbb{P}[A(x,y)] \leq \mathbb{P}[h(y') = v|A(x,y)] \cdot \mathbb{P}[A(x,y)] \\ &\leq \frac{2}{d^{\Delta} m_i} \cdot \frac{1+\alpha_{\mathsf{B}}}{d m_i^2} \leq \frac{3}{d^{\Delta+1} m_i^3} \,.\end{aligned}$$

Thus, for all $(x, y) \neq (x', y')$ with $(x, y) \sim (x', y')$ we have

$$\nu_{xy,x'y'} \leq \frac{4}{d^{\Delta+1} m_i^3} \cdot \frac{1}{\left(1 - \frac{1+\alpha_{\mathsf{B}}}{d m_i^2}\right)^{10\Delta^5}} \leq \frac{5}{d^{\Delta+1} (\varepsilon n)^3} \,. \tag{36}$$

Suen's inequality (Lemma 18) states that

$$\begin{aligned}\left|\mathbb{P}\Big[h^{-1}(uv) \cap \bigcup_{s \in [k_i]} E(F_{i,s}) = \emptyset\Big] - M_i(u,v)\right| &= \left|\mathbb{P}\Big[\bigwedge_{(x,y) \in H^i_{uv}} \overline{A(x,y)}\Big] - M_i(u,v)\right| \\ &\leq M_i(u,v)\Bigg(\exp\Big(\sum_{(x,y)\sim(x',y')} \nu_{xy,x'y'}\Big) - 1\Bigg) \,.\end{aligned} \tag{37}$$

We use (36), the bound $5\Delta^5$ on the degrees in the superdependency graph, and the fact that we have at most $4n^2/r$ edges in $\bigcup_s E(F_{i,s})$ to obtain that

$$\sum_{xy \sim x'y'} \nu_{xy,x'y'} \leq \frac{5}{d^{\Delta+1} \varepsilon^3 n^3} \cdot \frac{4n^2}{r} \cdot 5\Delta^5 = \frac{100\Delta^5}{d^{\Delta+1}\varepsilon^3 r n} \,.$$

In particular, as $n \geq n_0$ is large, we get $\sum_{xy\sim x'y'} \nu_{xy,x'y'} < \frac{\alpha}{2} < 1$. We use that $\exp(a) - 1 \leq 2a$ for each $a \in (0,1)$ and get $\mathbb{P}[h^{-1}(uv) \cap \bigcup_{s\in[k_i]} E(F_{i,s}) = \emptyset] = (1 \pm \alpha) M_i(u,v)$. □

Claim 39.2. There exists $p_h > 0$ such that for each typical edge $uv \in E$ we have

$$\mathbb{P}[h^{-1}(uv) \cap \bigcup_{i \in \mathsf{IG}} \bigcup_{s \in [k_i]} E(F_{i,s}) = \emptyset] = p_h \pm \alpha_{\mathsf{D}} \,.$$

Proof of Claim 39.2. First fix $i \in \mathsf{IG}$ and a typical edge $uv \in E$. Let $S = \big\{s \in [k_i] \colon U_{i,s} \cap \{u, v\} = \emptyset\big\}$. Observe that $|S| = k_i - \mathrm{load}(u, v, \mathcal{U}_i)$ and

$$|H^i_{u,v}| = \sum_{s \in S} 2(n_{i,s} - 1) = |S| 2(1 \pm \alpha)(n_i) - 1 = 2\big(k_i - \mathrm{load}(u,v,\mathcal{U}_i)\big)(n_i - 1) \pm 3n\alpha n_i \,.$$

Let us write $\ell_{uv} = 2\big(k_i - \mathrm{load}(u,v,\mathcal{U}_i)\big)(n_i - 1)$. Further, we write $\ell_i = 2(k_i - \mu(\mathcal{U}_i))(n_i - 1)$, and $M_i = (1 - \frac{1}{d m_i^2})^{\ell_i}$. Note that $\varepsilon^2 n^2 \leq \ell_{uv} \leq 2nn_i$ because $k_i \geq \sqrt{\alpha} nr/2$ as $i \in \mathsf{IG}$.

Plugging (33) into (34), we get

$$\begin{aligned} M_i(u,v) &= \left(1-\frac{1\pm\alpha_{\mathsf{B}}}{dm_i^2}\right)^{|H^i_{uv}|} = \exp\left((\ell_{uv}\pm 3\alpha n n_i)\cdot\ln\left(1-\frac{1\pm\alpha_{\mathsf{B}}}{dm_i^2}\right)\right) \\ &= \exp\left((\ell_{uv}\pm 3\alpha n n_i)\cdot(1\pm 2\alpha_{\mathsf{B}})\cdot\ln\left(1-\frac{1}{dm_i^2}\right)\right) \\ &= \exp\left((\ell_i\pm\alpha_{\mathsf{C}} n n_i)\cdot\ln\left(1-\frac{1}{dm_i^2}\right)\right) = \exp\left(\ell_i(1\pm\alpha_{\mathsf{D}})\cdot\ln\left(1-\frac{1}{dm_i^2}\right)\right) \\ &= \left(1-\frac{1}{dm_i^2}\right)^{(1\pm\alpha_{\mathsf{D}})\ell_i} = M_i^{1\pm\alpha_{\mathsf{D}}}\ , \end{aligned} \tag{38}$$

where the third equality follows by Taylor expansion of $\ln(1-x)$ and the fourth equality uses that uv is typical. In total we get that for each typical edge $uv\in E$ we have

$$\mathbb{P}\left[h^{-1}(uv)\cap\bigcup_{i\in\mathsf{IG}}\bigcup_{s\in[k_i]}E(F_{i,s})=\emptyset\right]=\prod_{i\in\mathsf{IG}}M_i^{1\pm\alpha_{\mathsf{D}}}=\prod_{i\in\mathsf{IG}}M_i\pm\alpha_{\mathsf{D}}\ , \tag{39}$$

where we used Lemma 38. The claim follows by setting $p_h=\prod_{i\in\mathsf{IG}}M_i$. □

This finishes the proof of Lemma 39. □

Recall that $\tilde{\mathcal{U}}_i=(\tilde{U}_{i,s})_{s\in[k_i]}$ with $\tilde{U}_{i,s}=U_{i,s}\cup V(h_{i,s})$.

Lemma 40. *We have*

$$\mathbb{P}[\forall i\in[c]\colon \sigma(\tilde{\mathcal{U}}_i)\le\beta n^4]\ge 1-\exp(-\sqrt{n}).$$

Proof. Fix $i\in[c]$. Let $L_i^*(u,v)=\mathrm{load}(u,v,\tilde{\mathcal{U}}_i)-\mathrm{load}(u,v,\mathcal{U}_i)$ and $\mu_i^*=\mu(\tilde{\mathcal{U}}_i)-\mu(\mathcal{U}_i)$.

Claim 40.1. With probability at least $1-\frac{1}{c}\exp(-\sqrt{n})$ we have that

$$\sum_{uv\in\binom{V}{2}}(L_i^*(u,v)-\mu_i^*)^2\le\alpha_{\mathsf{E}}n^4\ . \tag{40}$$

Proof of Claim 40.1. Fix an arbitrary i-typical pair $uv\in\binom{V}{2}$. Let s be such that $U_{i,s}\cap\{u,v\}=\emptyset$ and let $x\in V(F_{i,s})$ be arbitrary. Denote by A_x the event that $h(x)\in\{u,v\}$. Lemma 22(a) and (e) and (20) give that

$$\mathbb{P}[A_x]=\frac{2(1\pm\alpha_{\mathsf{B}})}{m_i}\le\frac{3}{m_i}\ . \tag{41}$$

Set $M=\prod_{x\in V(F_{i,s})}(1-\mathbb{P}[A_x])$. We have $M=\left(1-\frac{2\pm 2\alpha_{\mathsf{B}}}{m_i}\right)^{n_{i,s}}$. Recall that $n_{i,s}=(1\pm\alpha)n_i$. We can now manipulate the error bounds as in (38), (39) and get

$$M=\left(1-\frac{2}{m_i}\right)^{n_i}\pm\alpha_{\mathsf{C}}\ . \tag{42}$$

We shall approximate the values $p_{i,s}(u,v)=\mathbb{P}[h^{-1}(\{u,v\})\cap V(F_{i,s})=\emptyset]$ using Suen's Inequality, similarly as in the proof of Claim 39.1. We define a superdependency graph on vertex set $V(F_{i,s})$ for the events $\{A_x\}_{x\in V(F_{i,s})}$ by letting $x\sim y$ whenever $\mathrm{dist}(x,y)\le 3$. Let

$$\nu_{xy}=\frac{\mathbb{P}[A_x\cap A_y]+\mathbb{P}[A_x]\cdot\mathbb{P}[A_y]}{\prod(1-\mathbb{P}[A_z])}\ , \tag{43}$$

where the product in the denominator is over all z with $z \sim x$ or $z \sim y$. By Lemma 23 we have $\mathbb{P}[A_x \cap A_y] \leq 4\big(\frac{3}{d}\big)^{4\Delta^2}\frac{1}{m_{i,s}^2} \leq 5\big(\frac{3}{d}\big)^{4\Delta^2}\frac{1}{m_i^2}$ Using (41) and that the product has at most $2\Delta^3$ factors we get for each $x \sim y$ that

$$\nu_{xy} \leq \frac{5(\frac{3}{d})^{4\Delta^2}\frac{1}{m_i^2} + \frac{9}{m_i^2}}{(1-\frac{3}{m_i})^{2\Delta^3}} \leq 10\Big(\frac{3}{d}\Big)^{4\Delta^2}\frac{1}{m_i^2}\,.$$

Note that for each $s \in [k_i]$, there are at most $(\Delta+1)^3 m_{i,s}$ pairs $x, y \in V(F_{i,s})$ with $x \sim y$. Hence Suen's Inequality (Lemma 18) gives

$$\begin{aligned} p_{i,s}(u,v) = \mathbb{P}\Big[\bigwedge_{x\in V(F_{i,s})}\overline{A_x}\Big] &= M \pm M\cdot\left(\exp\left(\frac{(\Delta+1)^3 11(\frac{3}{d})^{4\Delta^2}}{m_i}\right)-1\right) \\ &\overset{(42)}{\leq} \left(1-\frac{2}{m_i}\right)^{n_i} \pm 2\alpha_{\mathsf{C}}\,. \end{aligned} \tag{44}$$

Set $p_i = \Big(1-\frac{2}{m_i}\Big)^{n_i}$. We will show that

$$\mathbb{P}[L_i^*(u,v) \geq (k_i - \mu(\mathcal{U}_i))\cdot p_i + 4\alpha_{\mathsf{C}} n] \leq \exp(-\alpha_{\mathsf{C}}^2 n)\,. \tag{45}$$

The random variable $L_i^*(u,v)$ has law $\sum_{s\,:\,U_{i,s}\cap\{u,v\}=\emptyset}\mathrm{Be}(p_{i,s}(u,v))$. Using (44), we get that $L_i^*(u,v)$ is stochastically dominated by $\sum \mathrm{Be}(p_i + 2\alpha_{\mathsf{C}})$, where the sum runs through all s such that $U_{i,s}\cap\{u,v\} = \emptyset$. The number of summands is $k_i - \mathrm{load}(u,v,\mathcal{U}_i)$, which is at most $k_i - \mu(\mathcal{U}_i) + \sqrt[4]{\alpha}n$, as uv is i-typical. Observe that $(k_i - \mu(\mathcal{U}_i) + \sqrt[4]{\alpha}n)(p_i + 2\alpha_{\mathsf{C}}) \leq (k_i - \mu(\mathcal{U}_i))\cdot p_i + 3\alpha_{\mathsf{C}} n$. By Chernoff's inequality (6) and because $k_i \leq 2n$, we obtain (45). The computation that $\mathbb{P}[L_i^*(u,v) \leq (k_i-\mu(\mathcal{U}_i))\cdot p_i - 4\alpha_{\mathsf{C}} n] \leq \exp(-\alpha_{\mathsf{C}}^2 n)$ is done analogously. So with probability at least $1 - 2\binom{m}{2}\cdot\exp(-\alpha_{\mathsf{C}}^2 n) \geq 1 - \frac{1}{c}\exp(-\sqrt{n})$, all i-typical pairs uv satisfy $L_i^*(u,v) = (k_i - \mu(\mathcal{U}_i))p_i \pm 4\alpha_{\mathsf{C}} n$. Suppose this is the case. Then

$$\begin{aligned} \mu_i^* = \frac{1}{\binom{m}{2}}\sum_{uv\in\binom{V}{2}} &= \frac{1}{\binom{m}{2}}\sum_{uv\ i\text{-typical}}((k_i-\mu(\mathcal{U}_i))p_i \pm 4\alpha_{\mathsf{C}} n) + \frac{1}{\binom{m}{2}}\sum_{uv\ i\text{-atypical}} L_i^*(u,v) \\ &= (k_i-\mu(\mathcal{U}_i))p_i \pm 4\alpha_{\mathsf{C}} n \pm \frac{\sqrt[4]{\alpha}n^2\cdot 2n}{\binom{m}{2}} = (k_i-\mu(\mathcal{U}_i))p_i \pm 5\alpha_{\mathsf{C}} n\,, \end{aligned}$$

where we used Lemma 37. So with probability at least $1-\frac{1}{c}\exp(-\sqrt{n})$ we have

$$\begin{aligned} \sum_{uv\in\binom{V}{2}}(L_i^*(u,v)-\mu_i^*)^2 &\leq \sum_{uv\ i\text{-typical}}(9\alpha_{\mathsf{C}} n)^2 + \sum_{uv\ i\text{-atypical}} n^2 \\ &\leq n^2\cdot 100\alpha_{\mathsf{C}}^2 n^2 + \sqrt[4]{\alpha}n^2\cdot n^2 \leq \alpha_{\mathsf{E}} n^4\,, \end{aligned}$$

where we used Lemma 37 again. □

Claim 40.2. If (40) holds, then $\sigma(\tilde{\mathcal{U}}_i) \leq \beta n^4$.

Proof of Claim 40.2. Let W_i be the set of all pairs $uv \in \binom{V}{2}$ such that uv is i-atypical or $(L_i^*(u,v)-\mu_i^*)^2 > \sqrt{\alpha_{\mathsf{E}}}n^2$. From Lemma 37 and (40) we get that $|W_i| \leq \sqrt[4]{\alpha}n^2 + \sqrt{\alpha_{\mathsf{E}}}n^2 <$

$2\sqrt[4]{\alpha_{\mathsf{E}}}n^2$. So,

$$\begin{aligned}\sigma(\tilde{\mathcal{U}}_i) &= \sum_{uv\in\binom{V}{2}} (\mathrm{load}(u,v,\mathcal{U}_i) + L_i^*(u,v) - \mu(\mathcal{U}_i) - \mu_i^*)^2 \\ &= \sum_{uv\in\binom{V}{2}} (\mathrm{load}(u,v,\mathcal{U}_i) - \mu(\mathcal{U}_i))^2 + \sum_{uv\in\binom{V}{2}} (L_i^*(u,v) - \mu_i^*)^2 \\ &\qquad + \sum_{uv\in\binom{V}{2}} 2(\mathrm{load}(u,v,\mathcal{U}_i - \mu(\mathcal{U}_i))(L_i^*(u,v) - \mu_i^*) \\ &\overset{(40)}{\leq} \alpha n^4 + \alpha_{\mathsf{E}} n^4 + 2\Bigg(\sum_{uv\in W_i} n^2 + \sum_{uv\in\binom{V}{2}\setminus W_i} (\sqrt[4]{\alpha}n \cdot \sqrt[4]{\alpha_{\mathsf{E}}}n) \Bigg) \\ &\leq \alpha n^4 + \alpha_{\mathsf{E}} n^4 + 2(2\sqrt[4]{\alpha_{\mathsf{E}}}n^2 \cdot n^2 + n^2 \cdot \sqrt{\alpha_{\mathsf{E}}}n^2) < \beta n^4 \,.\end{aligned}$$

□

Claims 40.1 and 40.2 and a union bound over all $i \in [c]$ imply Lemma 40. □

Lemma 41. *With probability at least $1 - \exp(-\sqrt{n})$ we have for each $i \in [c]$ that $\big||\tilde{U}_{i,s}| - |\tilde{U}_{i,s'}|\big| \leq \beta n$, for all $s, s' \in [k_i]$.*

Proof. We first compute the expected size of the image $V(h_{i,s})$. More than the exact value of the expected size, we need to show that it does not depend much on $s \in [k_i]$. This is done in (48). Then we show the concentration.

Fix (i,s) and fix $v \in V(G_{i,s})$. For $x \in V(F_{i,s})$ denote by A_x the event that x is mapped to v. By Lemma 22(a) and (e) we have that

$$\mathbb{P}[A_x] = \frac{(1 \pm \alpha_{\mathsf{A}})^{\Delta+3}}{m_{i,s}} \overset{(20)}{=} \frac{1 \pm \alpha_{\mathsf{B}}}{m_i} \,. \tag{46}$$

Using Suen's Inequality, we shall approximate $\mathbb{P}[h_{i,s}^{-1}(v) \cap V(F_{i,s}) = \emptyset] = \mathbb{P}\big[\bigwedge_{x\in V(F_{i,s})} \overline{A_x}\big]$ by $M = \prod_{x\in V(F_{i,s})} \mathbb{P}[\overline{A_x}]$. Manipulating the error bounds same as in (42), we have that

$$M = \left(1 - \frac{1 \pm \alpha_{\mathsf{B}}}{m_i}\right)^{n_{i,s}} = \left(1 - \frac{1}{m_i}\right)^{n_i} \pm \alpha_{\mathsf{C}} \,. \tag{47}$$

For $x, y \in V(F_{i,s})$, we write $x \sim y$ if $\mathrm{dist}(x,y) \leq 2$. Note that this defines a superdependency graph for the events $\{A_x\}_{x\in V(F_{i,s})}$. Let

$$\nu_{xy} = \frac{\mathbb{P}[A_x \cap A_y] + \mathbb{P}[A_x] \cdot \mathbb{P}[A_y]}{\prod(1 - \mathbb{P}[A_z])},$$

where the product in the denominator is over all z with $z \sim x$ or $z \sim y$. The product in the denominator has at most $2(\Delta^2 + 1)$ terms. We infer from (46) that the denominator is at least $1/2$ and that $\mathbb{P}[A_x] \cdot \mathbb{P}[A_y]$ is at most $(1 + 3\alpha_{\mathsf{B}})/m_i^2$. Lemma 23 and (20) give $\mathbb{P}[A_x \cap A_y] \leq (\frac{3}{d})^{4\Delta^2} \frac{1}{m_{i,s}^2} \leq (\frac{4}{d})^{4\Delta^2} \frac{1}{m_i^2}$. Thus, we get that $\nu_{xy} \leq (\frac{5}{d})^{4\Delta^2} \frac{1}{m_i^2}$. Suen's Inequality

(Lemma 18) gives that

$$\mathbb{P}[h_{i,s}^{-1}(v) \cap V(F_{i,s}) = \emptyset] = M\Big(1 \pm \Big(\exp(\Delta^2 n_{i,s} \cdot \Big(\frac{5}{d}\Big)^{4\Delta^2} \frac{1}{m_i^2}\Big) - 1\Big)\Big)$$
$$\overset{(47)}{=} \Big(\Big(1 - \frac{1}{m_i}\Big)^{n_i} \pm \alpha_{\mathsf{C}}\Big)\Big(1 \pm \Big(\exp\Big(\Delta^2\Big(\frac{5}{d}\Big)^{4\Delta^2} \cdot \frac{8}{r\varepsilon^2 n}\Big) - 1\Big)\Big) = \Big(1 - \frac{1}{m_i}\Big)^{n_i} \pm 2\alpha_{\mathsf{C}} \,.$$

Therefore the expected size of the image $V(h_{i,s})$ is

$$\begin{aligned}\mathbb{E}[|V(h_{i,s})|] &= \sum_{v \in V(G_{i,s})} \mathbb{P}[h_{i,s}^{-1}(v) \cap V(F_{i,s}) \neq \emptyset] = m_{i,s} \cdot \Big(1 - \Big(1 - \frac{1}{m_i}\Big)^{n_i} \pm 2\alpha_{\mathsf{C}}\Big) \\ &\overset{(20)}{=} m_i \Big(1 - \Big(1 - \frac{1}{m_i}\Big)^{n_i} \pm 3\alpha_{\mathsf{C}}\Big) \,.\end{aligned} \tag{48}$$

Now we use McDiarmid's Inequality to show the concentration of $|V(h_{i,s})|$. Note that $|V(h_{i,s})|$ is $(\Delta+1)$-Lipschitz. Hence by McDiarmid's Inequality, Lemma 17, we have

$$\mathbb{P}[\big|\mathbb{E}[|V(h_{i,s})|] - |V(h_{i,s})|\big| > \beta n/4] \leq 2\exp\Big(-\frac{2\beta^2 n^2}{16(\Delta+1)^2 n_{i,s}}\Big) = \exp\Big(-n^{2/3}\Big) \,.$$

Set $H_i = m_i\Big(1 - \Big(1 - \frac{1}{m_i}\Big)^{n_i}\Big)$. Then $\mathbb{E}[|V(h_{i,s})|] = H_i \pm 6\alpha_{\mathsf{C}} n$. As $\big||U_{i,s}| - |U_{i,s'}|\big| \leq \alpha n$, by a union bound over all $s \in [k_i]$ we obtain that

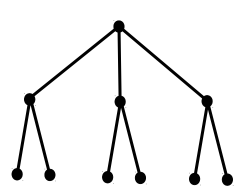
$$\mathbb{P}[\exists s, s' \in [k_i] \,:\, \big||\tilde{U}_{i,s}| - |\tilde{U}_{i,s'}|\big| > \beta n] \leq \tfrac{1}{c}\exp(-\sqrt{n}) \,.$$

A union bound over all $i \in [c]$ leads to the statement of the lemma. □

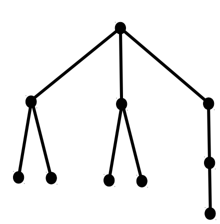
9. Concluding remarks

In this section we discuss various ways how our main result, Theorem 3, could be extended.

9.1. **Strengthening Theorem 3: approximation.** Theorem 3 does not hold for $\varepsilon = 0$. To see this, fix $\Delta \geq 3$ odd, let $\ell \geq 2$ be arbitrarily large, and consider the full Δ-regular tree of depth ℓ as in Figure 1(a), that is, each vertex in this tree has degree either 1 or Δ. This tree has an even number of leaves and an even number of internal vertices, hence its order n is even. Consider a family of $\frac{n}{2}$ copies of this tree. This family has $\binom{n}{2}$ edges in



(a) The 3-regular tree of depth 2.



(b) An example of the modified 3-regular tree.

Figure 1. Regular trees and modified regular trees

total. If it does not pack into K_n we are done. Otherwise, in any such packing a vertex v of

K_n accommodates exactly c_1 leaves and c_2 internal vertices of the trees, where c_1 and c_2 are integral and determined by the system

$$\begin{aligned} c_1 + c_2 &= \tfrac{n}{2} && \text{(each tree uses } v\text{)},\\ c_1 + \Delta c_2 &= n-1 && \text{(each edge incident with } v \text{ is used)}. \end{aligned}$$

This system has a unique solution (and thus the same for all vertices v) where c_1 is half the number of leaves of one tree and c_2 is half the number of internal vertices.

Now, we modify one of the trees by chopping off one leaf and appending it to another leaf; see Figure 1(b) (the resulting tree is not uniquely determined). This modified family does not pack into K_n. Indeed, if it did then the vertex of K_n hosting the unique vertex of degree 2 would have to host $\tilde{c}_1$ leaves and $\tilde{c}_2$ vertices of degree Δ, with

$$\begin{aligned} 1 + \tilde{c}_1 + \tilde{c}_2 &= \tfrac{n}{2}\ ,\\ 2 + \tilde{c}_1 + \Delta\tilde{c}_2 &= n-1\ . \end{aligned}$$

The integrality of the solution of the original system implies that the current one is not integral, contradiction.

On the other hand, the following strengthening of Theorem 3 may be true: Any family of trees of orders at most n and maximum degrees at most Δ whose total number of edges is at most $\binom{n}{2}$ packs into K_{n+C_Δ}, for a suitable constant C_Δ depending on Δ only.

9.2. **Strengthening Theorem 3: maximum degree.** We are convinced that at an expense of a more involved analysis, our techniques would allow to prove a version of Theorem 3 (for each fixed $\varepsilon > 0$) for Δ growing with n, possibly as big as $\Delta = O(\log^\alpha n)$ for some $\alpha > 0$.

We believe that Theorem 3 holds even for $\Delta = \frac{n}{2}$. (New techniques would be necessary for a proof.) The following example shows that the $\frac{n}{2}$ barrier can essentially not be exceeded. Suppose that $\varepsilon \in (0, 10^{-3})$ is fixed. Let us consider a family of $\ell = \left\lfloor \binom{n}{2} / ((\frac{1}{2} + 2\sqrt{\varepsilon})n) \right\rfloor$ copies of the star of order $(\frac{1}{2} + 2\sqrt{\varepsilon})n + 1$. Note that $\ell < (1 - 3\sqrt{\varepsilon})n$. The total number of edges in this family is between $\binom{n}{2} - n$ and $\binom{n}{2}$. We claim it does not pack into $K_{(1+\varepsilon)n}$. Suppose it does, and let us fix a packing. Let $W \subseteq V(K_{(1+\varepsilon)n})$ be the vertices that do not host the centres of the stars. Observe that $|W| > 3\sqrt{\varepsilon}n$. Observe also that no edge of the packing lies inside W. That means that all the edges of the stars must be accommodated in the set $E(K_{(1+\varepsilon)n}) \setminus \binom{W}{2}$. We have

$$\binom{n}{2} - n > \binom{(1+\varepsilon)n}{2} - \binom{3\sqrt{\varepsilon}n}{2}\ ,$$

a contradiction.

Note that if the orders of the trees are at most half of the order of the host graph, no example analogous to that in Section 9.1 can be found. Moreover, in Ringel's Conjecture (Conjecture 2), it follows from the assumption on the order of the tree that its maximum degree is at most half of the order of the host graph. Thus we propose the following strengthening of Conjecture 2.

Conjecture 42. *Any family of trees of individual orders at most $n+1$ and total number of edges at most $\binom{2n+1}{2}$ packs into K_{2n+1}.*

9.3. **Different host graphs in Theorem 3.** Hobbs, Bourgeois, and Kasiraj [16] modified Conjecture 1 to the setting of complete bipartite graphs.

Conjecture 43. *If n is even then any family of n trees $(T_j)_{j\in[n]}$ with $v(T_j) = j$ packs into $K_{n-1,n/2}$. If n is odd then any family of n trees $(T_j)_{j\in[n]}$ with $v(T_j) = j$ packs into $K_{n,(n-1)/2}$.*

Our proof of Theorem 3 can be adjusted with only minor modifications to the bipartite setting. Thus, the very same method yields an asymptotic solution of Conjecture 43 for trees of bounded maximum degree. In that setting, the ratio of the host graph's colour classes does not have to be 1 : 2; one just needs them to be of the same order of magnitude.

Theorem 44. *For any $\varepsilon > 0$ and any $\Delta \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ such that for any $a, b \geq n_0$, $\frac{a}{b} \in \left(\varepsilon, \varepsilon^{-1}\right)$ the following holds. Any family of trees $(T_i)_{i\in[t]}$ with maximum degree at most Δ and order at most $\min\{a, b\}$ satisfying $\sum_{i=1}^{t} e(T_i) \leq ab$ packs into $K_{(1+\varepsilon)a,(1+\varepsilon)b}$.*

Also, it is clear that the proof of Theorem 3 goes through when the graph $K_{(1+\varepsilon)n}$ is replaced by an arbitrary dense quasirandom graph (and the condition on the total number of edges in the family of trees is adjusted accordingly). Packing in random and quasirandom graphs is an important direction of research for its own sake, see e.g. [2].

9.4. **The tree-packing process.** We expect that the random embedding process described in Section 2 performs well even as a dynamic process on an evolving graph. That is, we believe that the quasirandomness of the host graph is also maintained by a sequential random embedding of the trees, where we forbid the edges (globally) and vertices (just for that particular tree) immediately after they are used. This would yield another proof of Theorem 3, but we believe the analysis of this process would also be interesting in its own right.

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