Numerical modelling of non-Newtonian shear-thinning viscoelastic fluids with application in hemodynamics.[1]

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Description of blood flow in human arteries is a very complex process. In recent years there is the growing interest in the use of mathematical models and numerical methods arising from other fields of computational fluid dynamics also in the hemodynamics, see, e.g., [2], [5], [8] just to mention some of them.

There are many numerical methods used in the blood flow simulation, which are based on the Newtonian models using the Navier-Stokes equations. This is effective and useful, especially if the flow in large arteries should be modelled. However, in small arteries blood cannot be considered as the Newtonian fluid anymore and more precise models need to be used. Blood is a complex rheological mixture showing shear-thinning as well as viscoelastic properties. In fact, there are several approaches presented in literature which can capture shear-thinning behaviour of blood. However, there are still relatively few models, which deal with both shear-thinning as well as viscoelastic properties, see, e.g., [1]. The aim of our presentation is to report on recent results concerning numerical modelling of shear-thinning viscoelastic flows with application in hemodynamics.

The motion of incompressible shear-thinning viscoelastic flows can be described by the so-called generalized Oldroyd-B fluids, see, e.g., [1]. In the non-dimensional form they read

$$\operatorname{div}\boldsymbol{u} = 0 \tag{1}$$

$$Re\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \nabla p - (1-\alpha)\Delta\boldsymbol{u} = \operatorname{div}\boldsymbol{\tau} + \boldsymbol{f} \tag{2}$$

$$We\left(\frac{\partial \boldsymbol{\tau}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\tau} - \boldsymbol{\tau}\nabla\boldsymbol{u}^T - \nabla\boldsymbol{u}\boldsymbol{\tau}\right) + \boldsymbol{\tau} = 2\left(\frac{1}{\mu_0}\mu(|\boldsymbol{D}(\boldsymbol{u})|) - (1-\alpha)\right)\boldsymbol{D}(\boldsymbol{u}), \tag{3}$$

where $\boldsymbol{u}$ is the velocity vector, $\boldsymbol{\tau}$ is the extra elastic stress tensor, p stays for pressure, $\boldsymbol{f}$ describes the vector of outer forces. In order to model shear-thinning behaviour we assume that the global viscosity $\mu(|\boldsymbol{D}(\boldsymbol{u})|)$ depends nonlinearly on the norm of deformation tensor $|\boldsymbol{D}(\boldsymbol{u})|$. In our experiments we have tested the Carreau model $\mu(|\boldsymbol{D}(\boldsymbol{u})|) = \mu_\infty + (\mu_0 - \mu_\infty)(1+\lambda|\boldsymbol{D}(\boldsymbol{u})|^a)^{q/a}$ and the Yeleswarapu model $\mu(|\boldsymbol{D}(\boldsymbol{u})|) = \mu_\infty + (\mu_0 - \mu_\infty)\left(\frac{1+\ln(1+\lambda|\boldsymbol{D}(\boldsymbol{u})|)}{1+\lambda|\boldsymbol{D}(\boldsymbol{u})|}\right)$. Here $q \in [-1, 0]$, $\lambda, a \in \mathbb{R}$ are material constants and μ_0, μ_∞ are asymptotic viscosities

$$\mu_0 := \lim_{|\boldsymbol{D}(\boldsymbol{u})|\to 0} \mu(|\boldsymbol{D}(\boldsymbol{u})|), \quad \mu_\infty := \lim_{|\boldsymbol{D}(\boldsymbol{u})|\to \infty} \mu(|\boldsymbol{D}(\boldsymbol{u})|).$$

The amount of viscous and elastic effects is modelled by the Reynolds Re and the Weissenberg We numbers, respectively. Further the parameter α in (2), (3) is given as $1-\alpha = \lambda_2/(\lambda_1\mu_0)$, $\lambda_{1,2} \geq 0$ are elastic constants (relaxation and retardation time).

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In the recent years there have been many publications on numerical treatment of the problem with viscoelastic fluids obeying the Oldroyd-B constitutive equations, see, e.g., [6] and the references therein. The additional difficulty of our model is that the viscosity behaves nonlinearly with respect to $|\boldsymbol{D}(\boldsymbol{u})|$. It should be pointed out that the momentum equation (2) is a nonlinear parabolic equation for velocity having singular behaviour for large Reynolds numbers Re. Equation (3) is a linear transport equation for elastic stress tensor, which again behaves singularly when $We \to \infty$. Our numerical approach reflects this different character of the partial differential equations (1-3) and is based on the combined finite volume-finite element scheme, cf., [3], [4] and the references therein. Instead of approximating the complex problem (1-3), say on the time interval $[t_n, t_{n+1}]$, we use the operator splitting approach and divide the system (1-3) into two subproblems. First, the fluid equations (1-2) are approximated by the non-conforming finite elements (Q_1^{rot}, Q_0) using rotated bilinear elements for velocity and constant elements for pressure, cf. [7]. The computational domain is dived into quadrilaterals. In order to solve numerically the flow equations (1-2) we apply the fractional-θ scheme in time and the Chorin projection algorithm. In our time stepping scheme in order to find $\boldsymbol{u}^{n+1}, p^{n+1}$ at new time step we use the elastic stress tensor $\boldsymbol{\tau}^n$ at the old time step. Having solved the flow equations (1-2) we substitute new velocity $\boldsymbol{u}^{n+1}$ into the transport equation (3) and solve it by the finite volume scheme to get $\boldsymbol{\tau}^{n+1}$. The discretization mesh for the finite volume method is a dual mesh to the original finite element mesh. The dual volume D_i corresponding to a vertex P_i of the original mesh is obtained by connecting the centers of the edges having the vertex P_i in common. Thus, the elastic stress tensors $\boldsymbol{\tau}$ is approximated by piecewise constants on each finite volume. Since the equation (3) is linear it is straightforward to use the upwinding to approximate fluxes on cell interfaces. In contrast to the finite element scheme for (1-2), which is time implicit scheme, the finite volume scheme for (3) is explicit in time. Therefore some CFL stability condition is necessary in order to guarantee the stability of the scheme.

The present research is in progress. First numerical experiments on simple geometries are encouraging. In order to specify the range of appropriate parameters such as Re and We extensive numerical treatment on complex geometries is necessary.

References

[1] Arada, N.; Sequeira, A.; *Strong steady solutions for a generalized Oldroyd-B model with shear-dependent viscosity in a bounded domain*, Math. Models Methods Appl. Sci. 13(9) (2003), pp. 1303-1323.

[2] Čanič, S.; *Blood flow through compliant vessels after endovascular repair: Wall deformations induced by the discontinuous wall properties*, Comput. Vis. Sci. 4 (2002), pp. 147-155.

[3] Feistauer, M.; Felcman, J.; Lukáčová-Medviďová, M.; Warnecke, G.; *Error estimates for a combined finite volume-finite element method for nonlinear convection-diffusion problems*, SIAM J. Numer. Anal. 36(5) (1999), pp. 1528-1548.

[4] Lukáčová-Medviďová, M.; *Combined finite element-finite volume method (Convergence analysis)*, Comment. Math. Univ. Carolinae 38(4) (1997), pp. 717-741.

[5] Perktold, K; Rappitsch. G.; *Mathematical modelling of local arterial flow and vessel mechanics*, In Comput. Methods for Fluid-Structure Interaction, Longman, Harlow, 1994, pp. 230-245.

[6] Phillips, T.N.; Williams, A.J.; *Viscoelastic flow through a planar contraction using a semi-Lagrangian finite volume method*, J. Non-Newtonian Fluid Mech. 87 (1999), pp. 215-246.

[7] Turek, S., *Efficient Solver for Incompressible Flow Problems*, Springer 1999.

[8] Quarteroni, A.; Tuveri, M.; Veneziani, A.; *Computational vascular fluid dynamics: Problems, models and methods. Survey article*, Comput. Vis. Sci. 2 (2000), pp. 163-197.