Exploiting model-order reduction techniques for solving symmetric rational eigenproblems

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Abstract. We consider large and sparse rational eigenproblems where the rational term is of low rank. Exploiting model-order reduction techniques, namely the Padé approximation via the block Lanczos method, problems of this type can be solved very efficiently.

1 Introduction

In this contribution we consider the rational eigenvalue problem

$$T(\lambda)x := -Kx + \lambda Mx + CD(\lambda)C^T x = 0, \tag{1}$$

where $K \in \mathbb{R}^{N\times N}$ and $M \in \mathbb{R}^{N\times N}$ are sparse symmetric and positive (semi-) definite matrices, $C \in \mathbb{R}^{N\times k}$ is a rectangular matrix of low rank k, and $D(\lambda) \in \mathbb{R}^{k\times k}$ is a real diagonal matrix depending rationally on a real parameter λ. Problems of this type arise in (finite element models of) vibrations of fluid–solid structures and of plates with elastically attached loads.

Problem (1) has a countable set of eigenvalues which can be characterized as minmax values of a Rayleigh functional [9], and its eigenpairs can be determined by iterative projection methods of Arnoldi [7] or Jacobi–Davidson type [1].

In this contribution we take advantage of the fact that problem (1) can be interpreted as a rational perturbation of a linear eigenproblem of small rank k. Decomposing $x \in \mathbb{R}^N$ into its component in the null space of C^T and its orthogonal complement, (1) can be rewritten as

$$\tilde{T}(\lambda)\tilde{x} := D(\lambda)^{-1}\tilde{x} + C^T(-K + \lambda M)^{-1}C\tilde{x} = 0, \tag{2}$$

which is a rational eigenvalue problem of much smaller dimension k.

The eigenproblem (2) retains the symmetry properties of problem (1), and hence, in principle it can be solved efficiently by safeguarded iteration. However, every step of safeguarded iteration requires the evaluation of $\tilde{T}(\lambda)$ for some λ, i.e. of $C^T(-K + \lambda M)^{-1}C$, which is too expensive because the dimension N is very large.

The term $C^T(-K + \lambda M)^{-1}C$ appears in transfer functions of time invariant linear systems, and in systems theory techniques have been developed to reduce the order of this term considerably. Taking advantage of these techniques,

namely of the Padé approximation by the block Lanczos method, problem (2) is replaced by a problem of the same structure of much smaller order, which is solved efficiently by safeguarded iteration.

This paper is organized as follows. Section 2 briefly summarizes model-order reduction of time-invariant systems by Padé approximation of the transfer function using the block Lanczos method. In Section 3 we present a rational eigenproblem governing free vibrations of a tube bundle immersed in an inviscid compressible fluid, and its transformation to a suitable nonlinear eigenproblem of type (2), and Section 4 contains numerical results demonstrating that model-order reduction techniques for this type of problems are superior to iterative projection methods like Arnoldi's method.

2 Padé approximation via block Lanczos

In this section we briefly sketch the Padé via Lanczos method for order reduction of linear time-invariant systems following a survey article of Roland Freund [5].

The basic idea of reduced-order modeling is to replace a given linear time invariant-system

$$E\dot{z}(t) = Az(t) + Bu(t), \quad y(t) = C^T z(t) + Du(t), \tag{3}$$

with $E, A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times k}$, and $C \in \mathbb{R}^{p\times N}$ by a system of the same type, but with much smaller dimension $n \ll N$ of the state space. Applying the Laplace transform to system (3) we obtain the algebraic equations

$$sEX(s) = AX(s) + BU(s), \quad Y(s) = C^T X(s) + DU(s), \tag{4}$$

and eliminating $X(s)$ we arrive at the frequency-domain input-output relation $Y(s) = H(s)U(s)$ where

$$H(s) := D + C^T(sE - A)^{-1}B, \ s \in \mathbb{C}, \tag{5}$$

is the transfer function of the system. Notice, that $H \in \mathbb{C}^{p\times k}$ is a rational matrix function.

The reduced-order problem will have a transfer function of the same type, but with smaller matrices. One way to define such a model is by means of an Padé approximation of $H(s)$. Let $s_0 \in \mathbb{C}$ be a shift which is not a pole of H. Then H has a Taylor series about s_0

$$H(s) = D + \sum_{j=0}^{\infty} \mu_j (s - s_0)^j \tag{6}$$

where the moments μ_j are $p \times k$ matrices. A reduced-order model of state-space dimension n is called an n-th Padé model of system (3), if the Taylor expansions of the transfer function H of the original problem and H_n of the reduced model agree in as many leading terms as possible, i.e.

$$H(s) = H_n(s) + \mathcal{O}\big((s - s_0)^{q(n)}\big), \tag{7}$$

where $q(n)$ is as large as possible, and which was proved by Freund [4] to satisfy

$$q(n) \geq \lfloor \frac{n}{m} \rfloor + \lfloor \frac{n}{p} \rfloor.$$

Although the Padé approximation is determined via a local property (6) it usually has excellent approximation properties in large domains which may even contain poles. As introduced by Feldmann and Freund [3] the Padé approximation H_n can be evaluated via the Lanczos process.

To this end assume that s_0 is a suitable shift, and let $A - s_0E = F_1F_2$, $F_1, F_2 \in \mathbb{C}^{N\times N}$ be a factorization of $A - s_0E$. Then with $G := F_1^{-1}EF_2^{-1}$, $R := F_1^{-1}B$, and $L := F_2^{-T}C$ the transfer function H can be rewritten as

$$H(s) = D - L^T(I - (s - s_0)G)^{-1}R = D - \sum_{j=0}^{\infty} L^T G^j R(s - s_0)^j, \qquad (8)$$

and the leading moments

$$\mu_j = -L^T G^j R = -((G^{j-i})^T L)^T (G^i R), \quad i = 0, 1, \ldots, j \qquad (9)$$

defining the Padé approximation are blocks of the right and left block Krylov matrices

$$[R, GR, G^2R, \ldots, G^ir, \ldots] \quad \text{and} \quad [L, G^TL, (G^T)^2L, \ldots, (G^T)^iL, \ldots]. \qquad (10)$$

Hence, all information needed to generate the Padé reduced-order model is contained in the block Krylov matrices (10).

For SISO (single-input-single-output) systems, i.e. where $m = p = 1$, the leading moments μ_j can be obtained easily by the two-sided Lanczos method which constructs bases $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ of the right Krylov space

$$\mathcal{K}_n(G, r) := \text{span}\{r, Gr, \ldots, G^{n-1}r\} \qquad (11)$$

induced by G and the right starting vector $r := R$, and the left Krylov space

$$\mathcal{K}_n(G^T, \ell) := \text{span}\{\ell, G^T\ell, \ldots, (G^T)^{n-1}\ell\}, \qquad (12)$$

induced by G^T and the left starting vector $\ell := L$, respectively, in a stable way from the so called modified moments $w_j^Tv_j$ and $w_j^TGv_j$, $j = 1, 2, \ldots, n$.

For multiple-input-multiple-output (MIMO) systems the situation is more involved since linear dependence of columns might appear in the block Krylov sequence. Deleting linear dependent columns in the right and left block Krylov matrix one obtains the n-th right and left block Krylov subspaces $\mathcal{K}_n(G, R)$ and $\mathcal{K}_n(G^T, L)$, respectively, which are spanned by the first n columns of the deflated right and left block Krylov matrix (10). [5] contains the band Lanczos method for constructing suitable basis vectors of these right and left block Krylov subspaces

For the rational eigenproblem (1) and its transformed form (2) the term $C^T(-K + \lambda M)^{-1}C$ that we want to deal with can be shifted and factorized such that the transfer function the order of which we want to reduce is symmetric

$$\tilde{H}(s) = \tilde{B}^T(I - s\tilde{A})\tilde{B}. \qquad (13)$$

If no deflation is necessary the order can be reduced by block Lanczos method, and the following theorem holds. A more general version taking into account deflation is proved in [4]. Note that we will consider only real shifts and therefore all appearing matrices can be assumed to be real.

Theorem 1. *Let $V_m \in \mathbb{R}^{N\times mk}$ be an orthonormal basis of the block Krylov space $\mathcal{K}_m(\tilde{A},\tilde{B}) := span\{\tilde{B},\tilde{A}\tilde{B},\dots,\tilde{A}^{m-1}\tilde{B}\}$ generated by the block Lanczos process such that the following recursion holds*

$$\tilde{A}V_m = V_m\tilde{A}_m + [O,\dots,O,\hat{V}_{m+1}\beta_{m+1}] \tag{14}$$

where $\hat{V}_{m+1}\in\mathbb{R}^{N\times k}$, $\beta_{m+1}\in\mathbb{R}^{k\times k}$, and $\tilde{A}_m\in\mathbb{R}^{mk\times mk}$.

Then with $\tilde{B}=V_1\Phi$, and $E_1=[I_k,O,\dots,O]^T$ the moments are given by

$$\tilde{B}\tilde{A}^i\tilde{B} = \Phi^T E_1^T\tilde{A}_m^i E_1\Phi, \quad i=0,1,\dots,2m-1, \tag{15}$$

and it holds

$$\tilde{B}^T(I-s\tilde{A})^{-1}\tilde{B} = \Phi^T E_1^T(I-s\tilde{A}_m)^{-1}E_1\Phi + \mathcal{O}(|s|^{2m}). \tag{16}$$

3 A fluid–structure interaction problem

Free vibrations of a tube bundle immersed in a slightly compressible fluid under the following simplifying assumptions are governed by a rational eigenproblem: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

Mathematically this problem can be described in the following way (cf. [2]). Let $\Omega\subset\mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ. We assume that there exists a family $\Omega_j\neq\emptyset$, $j=1,\dots,p$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j\subset\Omega$ for every j, $\bar{\Omega}_j\cap\bar{\Omega}_i=\emptyset$ for $j\neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set $\Omega_0 := \Omega\setminus\bigcup_{j=1}^K\bar{\Omega}_j$. Then the boundary of Ω_0 consists of $p+1$ connected components which are Γ and Γ_j, $j=1,\dots,p$.

We denote by $H^1(\Omega_0)=\{u\in L^2(\Omega_0) : \nabla u\in L^2(\Omega_0)^2\}$ the standard Sobolev space equipped with the usual scalar product

$$(u,v) := \int_{\Omega_0}(u(x)v(x)+\nabla u(x)\cdot\nabla v(x))\,dx.$$

Then the eigenfrequencies and the eigenmodes of the fluid-solid structure are governed by the following variational eigenvalue problem (cf. [2])

Find $\lambda \in \mathbb{R}$ *and* $u \in H^1(\Omega_0)$ *such that for every* $v \in H^1(\Omega_0)$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega_0} uv \, dx + \sum_{j=1}^{p} \frac{\lambda \rho_0}{k_j - \lambda m_j} \int_{\Gamma_j} un \, ds \cdot \int_{\Gamma_j} vn \, ds. \tag{17}$$

Here u is the potential of the velocity of the fluid, c denotes the speed of sound in the fluid, ρ_0 is the specific density of the fluid, k_j represents the stiffness constant of the spring system supporting tube j, m_j is the mass per unit length of the tube j, and n is the outward unit normal on the boundary of Ω_0.

Discretizing problem (17) by finite elements yields a rational eigenproblem

$$T(\lambda)\tilde{x} := -K\tilde{x} + \lambda M\tilde{x} + CD(\lambda)C^T\tilde{x} = 0, \tag{18}$$

where $K \in \mathbb{R}^{N\times N}$ and $M \in \mathbb{R}^{N\times N}$ are sparse symmetric matrices, M is positive definite and K is positive semidefinite, $C \in \mathbb{R}^{N\times k}$ is a rectangular matrix of low rank k, and $D(\lambda) \in \mathbb{R}^{k\times k}$ is a real diagonal matrix depending rationally on a real parameter λ.

Decomposing $\tilde{x} = Cy + z$ with $y \in \mathbb{R}^k$ and $z \in \text{range}\{C\}^\perp$, and multiplying equation (18) by $C^T(-K + \lambda M)^{-1}$ one obtains

$$C^T(Cy + z) + C^T(-K + \lambda M)^{-1}CD(\lambda)C^T(Cy + z) = 0$$

which is equivalent to

$$S(\lambda)x := -D(\lambda)^{-1}x + C^T(K - \lambda M)^{-1}Cx = 0, \quad x := D(\lambda)C^TCy. \tag{19}$$

Problem (19) is of much smaller dimension than (17), and it retains the symmetry properties of (17).

Choosing a shift λ_0 close to the eigenvalues we are interested in problem (19) is further rewritten as

$$S(\lambda)x = -D(\lambda)^{-1}x + H_{\lambda_0} + (\lambda - \lambda_0)B^T(I - (\lambda - \lambda_0)A)^{-1}Bx = 0 \tag{20}$$

where $M = EE^T$ is the Cholesky factorization of M, $H_{\lambda_0} := C^T(K - \lambda_0 M)^{-1}C$, $B := E^T(K - \lambda_0 M)^{-1}C$ and $A := E^T(K - \lambda_0 M)^{-1}E$.

4 Numerical results

We consider the rational eigenvalue problem (17) where Ω is the ellipse with center $(0, 0)$ and length of semiaxes 8 and 4, and Ω_j, $j = 1, \ldots, 9$ are circles with radius 0.3 and centers $(-4, -2)$, $(0, -2)$, $(4, -2)$, $(-5, 0)$, $(0, 0)$, $(5, 0)$, $(-4, 2)$, $(0, 2)$ and $(4, 2)$. We assume that all constants in problem (17) are equal to 1.

Discretizing problem (17) by linear Lagrangian elements we obtained a rational matrix eigenvalue problem of dimension $N = 36040$

$$T(\lambda)x := -Kx + \lambda Mx + \frac{\lambda}{1 - \lambda}Cx = 0 \tag{21}$$

where C collects the contributions of all tubes, and is of rank 18.

By the methods developed in [6] it can be shown that problem (18) has 28 eigenvalues in the interval $J_1 = (0, 1)$ and a large number of eigenvalues in $(1, \infty)$, 20 of which are contained in $J_2 := (1, 3)$.

We determined these eigenvalues with $k = 4, 5, 6, 7$ block Lanczos steps. Figure 1 shows the relative errors that were obtained with shift $\lambda_0 = 1.5$.

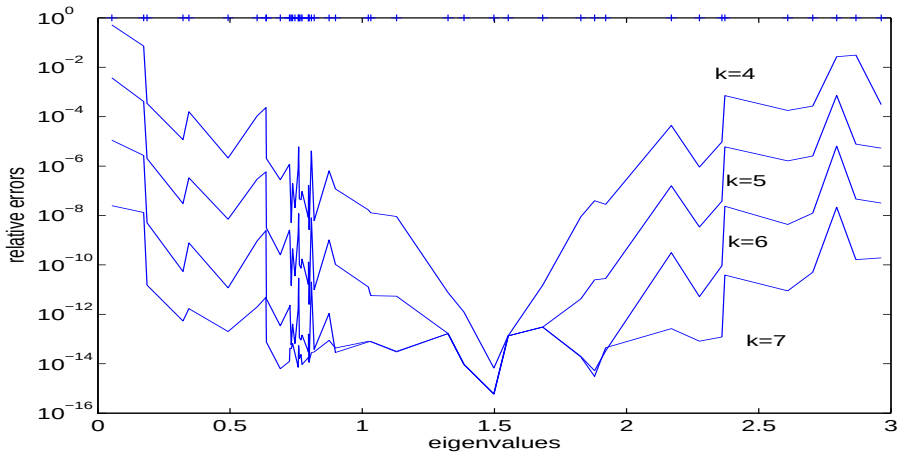

Fig. 1. Relative errors

Running the numerical experiments under MATLAB 6.5 on an Intel Centrino M processor with 1.7 GHz and 1 GB RAM the following CPU times were needed.

k	preproc.	Lanczos	eigensolver	total
4	3.99	51.65	6.07	62.12
5	3.99	63.27	5.28	72.95
6	4.01	85.73	5.80	95.96
7	3.99	92.80	13.72	111.07

For comparison we solved the rational eigenproblem (18) by the nonlinear Arnoldi method [7] which needed 83.99 seconds to obtain all 28 eigenvalues in the interval $[0, 1)$ and 86.49 seconds to compute all 20 eigenvalues in the interval $(1, 3)$. A suitable initial search space for the second interval $(1, 3)$ was determined by the method discussed in [8] and the implicitly restarted Arnoldi method in `eigs` requiring 21.27 seconds. Notice, however, that in this case we obtained the eigenvectors as well.

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