

VARIATIONAL CHARACTERIZATION OF EIGENVALUES OF NONLINEAR EIGENPROBLEMS

HEINRICH VOSS

ABSTRACT. In this paper we survey variational characterizations of eigenvalues of nonlinear eigenproblems, i.e. generalizations of Rayleigh's principle, the minmax characterization of Poincaré, and the maxmin characterization of Courant, Fischer and Weyl to eigenvalue problems containing the eigenparameter nonlinearly.

In this note we consider the nonlinear eigenvalue problem

$$(1) \quad T(\lambda)x = 0$$

where $T(\lambda)$, $\lambda \in J$, is a selfadjoint and bounded operator on a real Hilbert space H , and J is a real open interval which may be unbounded. Problems of this type appear in damped vibrations of structures, lateral buckling, and fluid-solid vibrations, e.g.

As in the linear case $T(\lambda) = \lambda I - A$ a parameter $\lambda \in J$ is called an eigenvalue of problem (1) if the equation (1) has a nontrivial solution $x \neq 0$.

For a wide class of linear selfadjoint operators $A : H \rightarrow H$ the eigenvalues of the linear eigenvalue problem (1) can be characterized by three fundamental variational principles, namely by Rayleigh's principle [11], by Poincaré's minmax characterization [10], and by the maxmin principle of Courant [2], Fischer [5] and Weyl [21]. The following theorem contains these characterizations for the simple case of a completely continuous operator. More general versions hold (cf. Dunford, Schwartz [4], p. 1543).

Theorem 1 (cf. Rektorys [12])

Let $A : H \rightarrow H$ be a selfadjoint and completely continuous operator on a real Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$, and denote by $R_A(x) := \langle Ax, x \rangle / \langle x, x \rangle$ the Rayleigh quotient of A at $x \neq 0$.

Let $\lambda_1 \geq \lambda_2 \geq \dots$ be the positive eigenvalues of A ordered by magnitude and regarding their multiplicities, and let x_1, x_2, \dots be a system of orthogonal eigenelements where x_j corresponds to λ_j . Assume that A has at least n positive eigenvalues. Then the following characterizations hold:

(i) *Rayleigh's principle*

$$\lambda_n = \max\{R_A(x) : \langle x, x_i \rangle = 0, i = 1, \dots, n-1\}.$$

(ii) *Maxmin characterization of Poincaré*

$$\lambda_n = \max_{\dim V=n} \min_{x \in V, x \neq 0} R_A(x).$$

1991 *Mathematics Subject Classification.* 49G05.

Key words and phrases. nonlinear eigenvalue problem, variational principle.

(iii) *Minmax characterization of Courant, Fischer, Weyl*

$$\lambda_n = \min_{\dim V = n-1} \max_{x \in V^\perp, x \neq 0} R_A(x).$$

Analogously the negative eigenvalues $\lambda_{-1} \leq \lambda_{-2} \leq \dots$ of A can be characterized by the three principles above if we replace min by max and vice versa.

These variational principles were generalized to the nonlinear eigenvalue problem (1) where the Rayleigh quotient $R_A(x)$ of a linear problem $Ax = \lambda x$ was replaced by the so called Rayleigh functional p .

To this end we assume that

$$f : \begin{cases} J \times H & \rightarrow \mathbb{R} \\ (\lambda, x) & \mapsto \langle T(\lambda)x, x \rangle \end{cases}$$

is continuously differentiable, and that for every fixed $x \in H$, $x \neq 0$, the real equation

$$(2) \quad f(\lambda, x) = 0$$

has at most one solution in J . Then equation (2) implicitly defines a functional p on some subset D of $H \setminus \{0\}$ which we call the Rayleigh functional, and which is exactly the Rayleigh quotient in case of a linear eigenproblem $T(\lambda) = \lambda I - A$.

We assume further that

$$(3) \quad \frac{\partial}{\partial \lambda} f(\lambda, x) \Big|_{\lambda=p(x)} > 0 \quad \text{for every } x \in D$$

which generalizes the definiteness of the operator B for the generalized linear eigenproblem $T(\lambda) := \lambda B - A$.

From the implicit function theorem it follows that D is an open set and that p is continuously differentiable on D , and differentiating the governing equation $f(p(x), x) = 0$ yields that the eigenvalues of problem (1) are the stationary vectors of the Rayleigh functional p .

If the Rayleigh functional p is defined on the entire space $H \setminus \{0\}$ then the eigenproblem (1) is called overdamped. This notation is motivated by the finite dimensional quadratic eigenvalue problem

$$(4) \quad T(\lambda)x = \lambda^2 Mx + \lambda Cx + Kx = 0$$

governing the damped free vibrations of a system where M , C and K are symmetric and positive definite matrices corresponding to the mass, the damping and the stiffness of the system, respectively.

Assume that the damping $C = \alpha \tilde{C}$ depends on a parameter $\alpha \geq 0$. Then for $\alpha = 0$ problem (4) has purely imaginary eigenvalues corresponding to harmonic vibrations of the system. Increasing α the eigenvalues move into the left half plane as conjugate complex pairs corresponding to damped vibrations. Increasing α further eigenvalue pairs reach the negative real axis as double eigenvalues where they immediately split and move into opposite directions on the negative real axis. Eventually all eigenvalues become real, and finally the ones going to the right are right of all eigenvalues moving to the left. Only for this large damping the system is called overdamped.

For overdamped systems the two solutions

$$p_{\pm}(x) = \frac{1}{2\langle Mx, x \rangle} \left(-\alpha \langle \tilde{C}x, x \rangle \pm \sqrt{\alpha^2 \langle \tilde{C}x, x \rangle^2 - 4\langle Mx, x \rangle \langle Kx, x \rangle} \right).$$

of the quadratic equation

$$(5) \quad \langle T(\lambda)x, x \rangle = \lambda^2 \langle Mx, x \rangle + \lambda \alpha \langle \tilde{C}x, x \rangle + \langle Kx, x \rangle = 0$$

are real, and they satisfy $\sup_{x \neq 0} p_-(x) < \inf_{x \neq 0} p_+(x)$. Hence, equation (5) defines two Rayleigh functionals p_- and p_+ corresponding to the intervals $J_- := (-\infty, \inf_{x \neq 0} p_+(x))$ and $J_+ := (\sup_{x \neq 0} p_-(x), \infty)$.

For general (not necessarily quadratic) overdamped problems Hadeler (in [6] for the finite dimensional case, and in [7] for $\dim H = \infty$) generalized Rayleigh's principle proving that the eigenvectors are orthogonal with respect to the generalized scalar product

$$(6) \quad [x, y] := \begin{cases} \frac{\langle (T(p(x)) - T(p(y)))x, y \rangle}{p(x) - p(y)}, & \text{if } p(x) \neq p(y) \\ \langle T'(p(x))x, y \rangle, & \text{if } p(x) = p(y) \end{cases}$$

which is symmetric, definite and homogeneous, but in general is not bilinear.

Theorem 2 (Hadeler [6], [7])

Let $T(\lambda) : H \rightarrow H$, $\lambda \in J$ be a family of selfadjoint and bounded operators. Assume that the problem (1) is overdamped and that for every $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) - \nu(\lambda)I$ is completely continuous.

Then problem (1) has at most a countable set of eigenvalues in J which we assume to be ordered by magnitude $\lambda_1 \geq \lambda_2 \geq \dots$ regarding their multiplicities. The corresponding eigenvectors x_1, x_2, \dots can be chosen orthonormally with respect to the generalized scalar product (6), and the eigenvalues can be determined recurrently by

$$(7) \quad \lambda_n = \max\{p(x) : [x, x_i] = 0, i = 1, \dots, n-1, x \neq 0\}.$$

Poincaré's maxmin characterization was first generalized by Duffin [3] to overdamped quadratic eigenproblems of finite dimension, and for more general overdamped problems of finite dimension it was proved by Rogers [13].

Infinite dimensional eigenvalue problems were studied by Turner [15], Langer [9], and Weinberger [19] who proved generalizations of both, the maxmin characterization of Poincaré and of the minmax characterization of Courant, Fischer and Weyl for quadratic (and in [16] for polynomial) overdamped problems.

The corresponding generalizations for general overdamped problems of infinite dimension contained in Theorem 3 were derived by Hadeler [7]. Similar results weakening the compactness requirements are contained in Rogers [14], Werner [20] and Hadeler [8].

Theorem 3 (Hadeler [7])

Assume that the general conditions of Theorem 2 are satisfied, and let the eigenvalues λ_j be numbered in nonincreasing order according to multiplicities.

Then the eigenvalues of problem (1) in J can be characterized by the following two variational principles

$$\begin{aligned}\lambda_n &= \max_{\dim V=n} \min_{x \in V, x \neq 0} p(x) \\ &= \min_{\dim V=n-1} \max_{x \in V^\perp, x \neq 0} p(x).\end{aligned}$$

Barston [1] characterized some of the extreme eigenvalues of a nonoverdamped quadratic eigenproblem of finite dimension, and Werner and the author studied the general nonoverdamped case. The following maxmin characterization was proved in [18] and the corresponding minmax characterization in [17].

Clearly in this case the natural ordering to call the largest eigenvalue the first one, the second largest the second one, etc., is not appropriate, as it is seen if we make a linear eigenvalue $(\lambda I - A)x = 0$ nonlinear by restricting it to an interval J which does not contain the largest eigenvalue of A . Then, obviously, all conditions are satisfied, p is the restriction of the Rayleigh quotient R_A to $D := \{x \neq 0 : R_A(x) \in J\}$, and $\sup_{x \in D} p(x)$ will not be an eigenvalue.

Obviously, $\lambda \in J$ is an eigenvalue of $T(\cdot)$ if and only if $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$. The key idea is to orientate the number of λ on the location on the eigenvalue $\mu = 0$ in the spectrum of the linear operator $T(\lambda)$.

To this end we assume that for every $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that the linear operator $S := T(\lambda) - \nu(\lambda)I$ is completely continuous. If $\lambda \in J$ is an eigenvalue of the nonlinear problem (1) then $\mu = 0$ is an eigenvalue of $T(\lambda)$, and $-\nu(\lambda)$ is a negative eigenvalue of S . By Theorem 1 there exists $n \in \mathbb{N}$ such that

$$-\nu(\lambda) = \min_{\dim V=n} \max_{x \in V, x \neq 0} R_S(x), \quad \text{i.e. } 0 = \min_{\dim V=n} \max_{x \in V, x \neq 0} R_{T(\lambda)}(x).$$

In this case we assign n to the eigenvalue λ of problem (1) as its number.

With this enumeration the following minmax and maxmin characterizations hold for nonlinear eigenvalue problems, where again the compactness condition can be relaxed.

Theorem 4 (Voss, Werner [18], [17])

Assume that for every $x \in H$, $x \neq 0$ the real equation $f(\lambda, x) = 0$ has at most one solution $\lambda =: p(x) \in J$, and that condition (3) holds, and suppose that for every $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) - \nu(\lambda)I$ is completely continuous.

Then the nonlinear eigenvalue problem (1) has at most a countable number of eigenvalues. Enumerating them as above the following characterizations hold:

(i) If $\lambda_n \in J$ is an n -th eigenvalue then

$$(8) \quad \lambda_n = \max_{\dim V=n, V \cap D \neq \emptyset} \inf_{x \in D \cap V} p(x).$$

If conversely

$$\lambda_n = \sup_{\dim V=n, V \cap D \neq \emptyset} \inf_{x \in D \cap V} p(x) \in J$$

then λ_n is an n -th eigenvalue of (1) and (8) holds.

(ii) If problem (1) has an n -th eigenvalue λ_n then

$$\lambda_n = \min_{\dim V=n-1, V^\perp \cap D \neq \emptyset} \sup_{x \in D \cap V^\perp} p(x).$$

REFERENCES

1. E.M. Barston, *A minimax principle for nonoverdamped systems*, Int.J.Engng.Sci. **12** (1974), 413 – 421.
2. R. Courant, *Über die Eigenwerte bei den Differentialgleichungen der mathematischen Physik*, Math.Z. **7** (1920), 1 – 57.
3. R.J. Duffin, *A minmax theory for overdamped networks*, J.Rat.Mech.Anal. **4** (1955), 221 – 233.
4. N. Dunford and J.T. Schwartz, *Linear operators. part II*, Wiley, New York, London, 1963.
5. E. Fischer, *Über quadratische Formen mit reellen Koeffizienten*, Monatshefte für Math. und Phys. **16** (1905), 234 – 249.
6. K. P. Hadeler, *Mehrparametrische und nichtlineare Eigenwertaufgaben*, Arch. Rat. Mech. Anal. **27** (1967), 306–328.
7. ———, *Variationsprinzipien bei nichtlinearen Eigenwertaufgaben*, Arch.Rat.Mech.Anal. **30** (1968), 297 – 307.
8. ———, *Nonlinear eigenvalue problems*, Numerische Behandlung von Differentialgleichungen (R. Ansorge, L. Collatz, G. Hämmerlin, and W. Törnig, eds.), ISNM 27, Birkhäuser, Stuttgart, 1975, pp. 111–129.
9. H. Langer, *Über stark gedämpfte Scharen im Hilbertraum*, J.Math.Mech. **17** (1968), 685 – 705.
10. H. Poincaré, *Sur les équations aux dérivées partielles de la physique mathématique*, Amer.J.Math. **12** (1890), 211 – 294.
11. Lord Rayleigh, *Some general theorems relating to vibrations*, Proc. London Math.Soc. **4** (1873), 357–368.
12. K. Rektorys, *Variationsmethoden in Mathematik, Physik und Technik*, Carl Hanser Verlag, München, 1984.
13. E.H. Rogers, *A minmax theory for overdamped systems*, Arch.Rat.Mech.Anal. **16** (1964), 89 – 96.
14. ———, *Variational properties of nonlinear spectra*, J.Math.Mech. **18** (1968), 479 – 490.
15. R.E.L. Turner, *Some variational principles for nonlinear eigenvalue problems*, J.Math.Anal.Appl. **17** (1967), 151 – 160.
16. ———, *A class of nonlinear eigenvalue problems*, J.Func.Anal. **7** (1968), 297 – 322.
17. H. Voss, *A maxmin principle for nonlinear eigenvalue problems with application to a rational spectral problem in fluid–solid vibration*, Tech. Report 48, Dept.Math., TU Hamburg-Harburg, 2002.
18. H. Voss and B. Werner, *A minimax principle for nonlinear eigenvalue problems with application to nonoverdamped systems*, Math.Meth.Appl.Sci. **4** (1982), 415 – 424.
19. H.F. Weinberger, *On a nonlinear eigenvalue problem*, J.Math.Anal.Appl. **21** (1968), 506–509.
20. B. Werner, *Das Spektrum von Operatorenscharen mit verallgemeinerten Rayleighquotienten*, Arch.Rat.Mech.Anal. **42** (1971), 223 – 238.
21. H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math.Ann. **71** (1912), 441 – 479.

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY OF HAMBURG – HARBURG, D–21071 HAMBURG, GERMANY

E-mail address: voss@tu-harburg.de

URL: <http://www.tu-harburg.de/mat/hp/voss>