Eigenvalue Reanalysis and Condensation with General Masters

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Abstract: In design optimization very often several changes of a structure are necessary to reach predetermined demands, and in each step very large eigenvalue problems have to be solved which are small modifications of each other. For large eigenvalue problems condensation methods are used to reduce the number of degrees of freedom to manageable size. In this note we take advantage of preceding computations by implementing eigenvectors of previous models as general masters into condensation.

Keywords: eigenvalue problem, condensation, generalized masters, reanalysis

1 Introduction

One purpose of a design process of a structure is to satisfy predetermined demands, such as given natural frequencies or dynamic responses. During this process the structure is modified a couple of times, and several eigenvalue problems appear which are small modifications of each other and which have similar eigenforms. Hence, reanalytical methods are welcome which take advantage of preceding calculations. A common way in reanalysis is to use Taylor expansions or Rayleigh quotient approximations or assumed mode approaches in a Rayleigh – Ritz method. In this note we consider condensation methods in eigenvalue reanalysis in two ways which are usually applied to reduce the number of degrees of freedom of large eigenvalue problems.

First we demonstrate that the algebraic cost of condensation can be reduced substantially if the masters are chosen as interface degrees of freedom in a substructure decomposition of the underlying structure and only a few substructures are modified. Secondly the approximation properties of condensation are enhanced considerably if eigenmodes of previous models are used as general masters.

2 Substructuring and condensation

We consider the general eigenvalue problem

$$Kx = \lambda Mx \tag{1}$$

where $K \in \mathbb{R}^{(n,n)}$ and $M \in \mathbb{R}^{(n,n)}$ are symmetric and positive definite matrices which are usually the stiffness and mass matrix of a finite element model of a structure, respectively. To reduce the number of unknowns Irons and Guyan independently proposed nodal condensation, i.e. to choose a small number of degrees of freedom (called masters) which seem to be representative for the dynamic behaviour of the entire structure, and to eliminate the remaining unknowns (called slaves) neglecting inertia terms in some of the equations of (1).

It is well known that substructuring leads to data structures such that the proportions of the reduced matrices belonging to different substructures can be computed independently. To this end assume that the structure under consideration has been decomposed into r substructures and let the masters be chosen as interface degrees of freedom such that the substructures connect to each other through the master variables only. If the slave variables are numbered appropriately, then the stiffness matrix is given by

$$K = \begin{pmatrix} K_{mm} & K_{ms1} & K_{ms2} & \dots & K_{msr} \\ K_{sm1} & K_{ss1} & 0 & \dots & 0 \\ K_{sm2} & 0 & K_{ss2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{smr} & 0 & 0 & \dots & K_{ssr} \end{pmatrix}, \tag{2}$$

and the mass matrix M has the same block form. In this case it is easily seen that the condensed eigenvalue problem obtains the form

$$K_0 x_m = \lambda M_0 x_m \tag{3}$$

where the reduced stiffness and mass matrices K_0 and M_0, respectively, are given by

$$K_0 = K_{mm} - \sum_{j=1}^{r} K_{mmj} := K_{mm} - \sum_{j=1}^{r} K_{msj} K_{ssj}^{-1} K_{smj}$$

and

$$M_0 = M_{mm} - \sum_{j=1}^{r} M_{mmj},$$

$$M_{mmj} := K_{msj} K_{ssj}^{-1} M_{smj} + M_{msj} K_{ssj}^{-1} K_{smj} - K_{msj} K_{ssj}^{-1} M_{ssj} K_{ssj}^{-1} K_{smj}.$$

Obviously, these matrices can be updated at low cost if only a small number of substructures is modified in a reanalysis step since each of the terms in the sums depends only on data of a single substructure. Notice that this concept can be generalized to condensation methods in the presence of general masters. Details are given in [1].

3 Condensation with general masters

Nodal condensation has the disadvantage that it produces accurate results only for a small part of the lower end of the spectrum. The approximation properties can be improved substantially if general masters are considered (cf. [2]). Let $z_1, \ldots, z_m \in \mathbb{R}^n$ be independent vectors, and define $Z := (z_1, \ldots, z_m) \in \mathbb{R}^{(n,m)}$. Then the projected eigenvalue problem

$$K_0 x_m := P^T K P x_m = \lambda P^T M P x_m =: \lambda M_0 x_m, \tag{4}$$

with

$$P = K^{-1} Z (Z^T K^{-1} Z)^{-1} Z^T Z$$

is called condensed eigenvalue problem with general masters $z_1, \ldots, z_m$. It is easily seen that this is exactly the reduced problem of nodal condensation if we choose $z_1, \ldots, z_m$ as unit vectors.

Since $(Z^T K^{-1} Z)^{-1} Z^T Z$ is a nonsingular matrix the condensed problem (4) is equivalent to the projection of problem (1) to the space spanned by the columns of $K^{-1}Z$. Hence, condensation is nothing else but one step of simultaneous inverse iteration with initial guess $X = M^{-1}Z \in \mathbb{R}^{(n,m)}$. Therefore, we can expect good approximation properties of condensation if we include general masters $z_j = M x_j$ where x_j are approximate eigenvectors of problem (1) corresponding to the desired eigenvalues. Hence, choosing approximate eigenvectors from previous design steps as general masters should improve the approximation properties.

4 A numerical example

To demonstrate the gain of accuracy by general masters we consider the free vibrations of a uniform thin clamped plate covering the rectangular region $\Omega := (0,5) \times (0,3)$. We discretized this problem by Bogner-Fox-Schmidt elements on a quadratic mesh with meshsize $h = 0.1$ and obtained a matrix eigenvalue problem of dimension $n = 5684$. Dividing Ω into 15 identical substructures each of them being a square of sidelength 1 and choosing all interface degrees of freedom as masters one obtains a reduced problem of dimension $m = 824$.

We modified the problem by doubling the mass in the rectangles $(2,3) \times (0,3)$ and $(4,5) \times (0,3)$, respectively. The following table contains the relative errors of the 12 smallest eigenvalues using nodal condensation (column 2 and 4) and of condensation if we complement the interface masters by 12 eigenvectors of the uniform plate as general masters (columns 3 and 5).

	$(2,3) \times (0,2)$		$(4,5) \times (0,2)$	
1	3.54e-3	1.19e-9	3.04e-3	2.63e-9
2	5.41e-3	3.15e-7	6.15e-3	7.99e-8
3	1.50e-2	1.70e-7	1.38e-2	1.67e-6
4	1.63e-2	1.19e-5	1.41e-2	1.84e-5
5	1.57e-2	1.18e-5	1.86e-2	5.58e-5
6	1.51e-2	9.46e-5	2.68e-2	2.76e-5
7	2.87e-2	9.78e-5	2.74e-2	2.30e-4
8	1.25e-1	3.85e-4	5.71e-2	5.26e-4
9	2.34e-2	3.93e-4	7.03e-2	1.63e-4
10	8.14e-2	4.23e-5	1.19e-1	6.18e-3
11	9.20e-2	1.62e-3	1.11e-1	2.09e-3
12	1.29e-1	8.48e-4	1.27e-1	7.93e-3

References

[1] B. Hofferek. Anwendung der Kondensation mit verallgemeinerten Mastern in der Eigenwert Reanalysis. Technical report.

[2] W. Mackens and H. Voss. Nonnodal condensation of eigenvalue problems. *ZAMM*, 79:243 – 255, 1999.