Variational characterization of real eigenvalues in linear viscoelastic oscillators

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Abstract
This paper proposes a new approach for computing the real eigenvalues of a multiple-degrees-of-freedom viscoelastic system in which we assume an exponentially decaying damping. The free-motion equations lead to a nonlinear eigenvalue problem. If the system matrices are symmetric, the eigenvalues allow for a variational characterization of maxmin type, and the eigenvalues and eigenvectors can be determined very efficiently by the safeguarded iteration, which converges quadratically and, for extreme eigenvalues, monotonically. Numerical methods demonstrate the performance and the reliability of the approach. The method succeeds where some current approaches, with restrictive physical assumptions, fail.

Keywords
Variational characterization, viscoelastic system, safeguarded iteration, nonviscous frequency, exponential damping

1. Introduction
It is essential to understand the vibration properties of complex engineering structures, such as automobile, civil, and aeronautics structures for mitigating fatigue and developing vibration suppression techniques. In view of their favorable damping characteristics, polymers and composite materials are of great importance in vibration control applications. These materials show a combination of elastic and viscoelastic responses and are usually characterized as viscoelastic materials. These materials exhibit dependency on frequency, load, and temperature while deformation is undergoing. These characteristics are often applied for a wide range of engineering applications, such as vibration control in almost all areas of engineering: mechanical, architecture, civil, industrial, aeronautics, and automotive. For instance, viscoelastic materials are used to mitigate earthquake effects in buildings or for vibration isolation. However, there are challenges in developing mathematical models for investigating the deformation and dynamic behavior of structures composed of viscoelastic materials.

Recently, several material models have been developed to determine various aspects of deformation, stiffness, and damping in materials to study the dynamic behavior of structures composed of viscoelastic materials [1].

From the mathematical point of view, any causal model that makes the energy dissipation functional non-negative is a candidate for a nonviscous damping model. Using the physics of the problem or selecting a model...
a priori and fitting its parameters from experiments, obviously there is a lot of choice for the model, for example, rational polynomials for Biot [2], Golla–Hughes–McTavish [3, 4] and inelastic displacement field models [5], and fractional polynomials [6]. Some representative damping functions, including complex transcendental functions, are tabulated in Wagner and Adhikari [7].

In the most general case, the structures that include viscoelastic materials are determined by hereditary energy dissipation mechanisms; this means that the damping forces depend on the history of the velocity response. Mathematically, this fact is represented by convolution integrals that involve the velocities of the degrees of freedom over certain kernel functions.

To investigate nonviscous damping of structures composed of viscoelastic materials, in this study we consider an exponential damping model was proposed by Biot [2], which assumes that the dissipative forces depend on the history of motion via the convolution integral over an exponentially decreasing kernel function. The exponential damping model is physically the most meaningful, and it has been widely used in recent studies [1].

The equation of motion of an N-degrees-of-freedom linear system of this type reads

\[ M\ddot{u}(t) + \int_{-\infty}^{t} G(t - \tau)\dot{u}(\tau)d\tau + Ku(t) = f(t), \quad (1) \]

together with initial conditions \( u(0) = u_0, \dot{u}(0) = \dot{u}_0, \) where \( u \in \mathbb{R}^N \) is the displacement vector, \( f \in \mathbb{R}^N \) is the forcing vector, \( M \in \mathbb{R}^{N \times N} \) and \( K \in \mathbb{R}^{N \times N} \) are the mass and stiffness matrix, respectively, and \( G \in \mathbb{R}^{N \times N} \) is the kernel function of damping.

For exponential damping, the kernel function has the form

\[ G(t) = \sum_{j=1}^{n} \mu_j \exp(-\mu_j t)C_j, \quad (2) \]

where, for \( j = 1, \ldots, n, \) the constants \( \mu_j \in \mathbb{R}_+ \) are the relaxation parameters and \( C_j \in \mathbb{R}^{N \times N} \) are damping coefficient matrices.

Here, \( M \) and \( K \) are assumed both symmetric and positive definite. The modes of the system can be determined as nontrivial solutions of a free-motion problem obtained by setting \( f = u_0 = \dot{u}_0 = 0 \) in equation (1). Therefore, considering functions of the form \( u(t) = uc^s, \) we have

\[ T(s) := (s^2M + sG(s) + K)u = 0, \quad (3) \]

where \( G \) is the Laplace transform of the kernel function \( G(t). \) This is a nonlinear eigenvalue problem, which depends nonlinearly on the eigenparameter \( s. \) Recall that \( s \) is an eigenvalue of equation (3) if equation (3) has a nontrivial solution \( u \neq 0. \)

Considering that the damping model is strictly dissipative (conditions are provided in Golla and Hughes [3]), it can be shown [8, 9] that equation (3) has \( 2N + r \) eigenvalues, \( r \geq 0, \) where \( 2N \) roots are in complex conjugate pairs. The remaining eigenvalues are negative real numbers, since positive eigenvalues and dissipative behavior cannot coexist. These eigenvalues, which are named nonviscous eigenvalues, are presented as an intrinsic property of viscoelastic models. The number of nonviscous eigenvalues depends on the nature of the damping function and they are associated with nonoscillatory modes.

Several methods have been developed to compute the nonviscous eigenvalues of viscoelastic systems. The most prominent approach to estimating these eigenvalues deals in converting the integro-differential equation (equation (1)) to an associated state-space model [3–11], by introducing internal variables and computing the resulting eigenvalues and eigenvectors. Such a realization in state-space form introduces a large number of fictitious internal variables, leading to computationally expansive large matrices for eigenvalue computations. To overcome this challenge, non-state-space methods have been investigated by several researchers. Approximated closed-form expressions for the complex and real eigenvalues of the viscoelastic system were obtained by Adhikari and Pascual [12] for three mathematically different cases based on the number of kernel functions. The approximations utilize Taylor series expansion in the complex domain and are based on certain simplifying physical assumptions. The approach is valid only for systems with small viscoelasticity. Adhikari and Pascual [13] also give five different iterative algorithms for estimating the real and complex eigenvalues associated with single- and multiple-degrees-of-freedom systems. The iterative method for the complex eigenvalue is applicable
to any general viscoelastic kernel, while the iterative method for real eigenvalues is specific to exponential damping. A fixed-point iteration method has been proposed [14] toward the computation of eigenvalues of a viscoelastic system. Although, under certain conditions, the method may also be used to obtain nonviscous eigenvalues, it can only be implemented for systems with a proportional, or lightly nonproportional, damping matrix.

Introducing the novel concept of the “nonviscous set,” a new mathematical characterization of the nonviscous eigenvalues has been proposed [15]. Using the nonviscous set, a closed-form expression that estimates each real eigenvalue has been obtained for systems with exponential kernels. It is proved that for lightly or moderately damped systems, the set of real eigenvalues can be obtained by solving as many linear eigenvalue problems as exponential kernels [16]. A numerical method for computing the eigenvalues and respective left and right eigenvectors has been presented [17], which is based on Newton’s eigenvalue iteration method.

In this paper, we will develop a non-state-step method invoking the variational or maxmin characterization of real eigenvalues; this has not, to our knowledge, been derived for viscoelastic systems in previous studies. Variational characterizations of eigenvalues are very powerful tools when studying nonlinear eigenvalue problems. Usually, non-state-step methods are based on some restrictive physical assumptions; they fail to obtain nonviscous eigenvalues efficiently when the physical assumptions are not valid. It is shown by several realistic physical examples that our method successfully determines eigenvalues while current methods fail, as a result of their restrictive physical assumptions.

2. Variational characterization for nonlinear eigenvalue problems


We consider the nonlinear eigenvalue problem

$$ T(\lambda)x = 0, \quad (4) $$

where \( T(\lambda) \in \mathbb{C}^{n \times n}, \lambda \in J \), is a family of Hermitian matrices depending continuously on the parameter \( \lambda \in J \), and \( J \) is a real open interval, which may be unbounded.

To generalize the variational characterization of eigenvalues, we need a generalization of the Rayleigh quotient. To this end, we make an assumption.

**Assumption 1.** For every fixed \( x \in \mathbb{C}^{n}, x \neq 0 \), the scalar real equation,

$$ f(\lambda; x) := x^H T(\lambda)x = 0, \quad (5) $$

has at most one solution \( p(x) \in J \).

Then \( f(\lambda; x) = 0 \) implicitly defines a functional \( p \) on some subset \( D \subset \mathbb{C}^{n} \), which is called the Rayleigh functional of equation (4), and which is exactly the Rayleigh quotient in the case of a monic linear matrix function \( T(\lambda) = \lambda I - A \).

Generalizing the definiteness requirement for linear pencils, \( T(\lambda) = \lambda B - A \), we make a further assumption.

**Assumption 2.** For every \( x \in D \) and every \( \lambda \in J \) with \( \lambda \neq p(x) \), it holds that

$$ (\lambda - p(x))f'(\lambda; x) > 0. \quad (6) $$

If \( p \) is defined on \( D = \mathbb{C}^{n} \setminus \{0\} \) then the problem \( T(\lambda)x = 0 \) is called overdamped. This notation is motivated by the finite-dimensional quadratic eigenvalue problem

$$ T(\lambda)x = \lambda^2 Mx + \lambda Cx + Kx = 0, \quad (7) $$

where \( M, C, \) and \( K \) are Hermitian and positive definite matrices. If \( C \) is large enough that \( d(x) := (x^H Cx)^2 - 4(x^H Kx)(x^H Mx) > 0 \) for every \( x \neq 0 \), then \( T(\cdot) \) is overdamped. Generalizations of the minmax and maxmin characterizations of eigenvalues were proved by Duffin [22] for the quadratic case and by Rogers [23] for general overdamped problems.
It is worth mentioning that for quadratic eigenproblems and, more generally, nonviscously damped systems, Phani and Adhikari [24] discussed Rayleigh quotients and their stationarity properties. Here, we take advantage of the approximation properties of Rayleigh functionals [18, 22, 23] which apply to a much wider class of nonlinear eigenvalue problems and which additionally allows for variational characterization of eigenvalues.

For nonoverdamped eigenproblems, the natural ordering, in which the largest eigenvalue is labeled the first one, the second largest the second one, etc., is not appropriate. This is obvious if we make a linear eigenvalue problem $T(\lambda)x := (\lambda I - A)x = 0$ nonlinear by restricting it to an interval $J$ that does not contain the largest eigenvalue of $A$. Then Assumptions 1 and 2 are satisfied, $p$ is the restriction of the Rayleigh quotient $R_J$ to

$$D := \{x \neq 0 : R_J(x) \in J\},$$

and $\sup_{x \in D} p(x)$ will, in general, not be an eigenvalue.

If $\lambda \in J$ is an eigenvalue of $T(\cdot)$, then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $\ell \in \mathbb{N}$, such that

$$0 = \min_{V \in H_\ell} \max_{v \in V \setminus \{0\}} \frac{\langle v, T(\lambda)v \rangle}{\|v\|^2},$$

where $H_\ell$ denotes the set of all $\ell$-dimensional subspaces of $C^n$. In this case, $\lambda$ is called an $\ell$th eigenvalue of $T(\cdot)$.

With this enumeration, the following maxmin characterization for eigenvalues was proved [25, 26].

**Theorem 1.** Let $J$ be an open interval in $\mathbb{R}$, and let $T(\lambda) \in C^{n \times n}$, $\lambda \in J$, be a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$, such that Assumptions 1 and 2 are satisfied. Then the following statements hold.

1. For every $\ell \in \mathbb{N}$, there is at most one $\ell$th eigenvalue of $T(\cdot)$, which can be characterized by

$$\lambda_\ell = \max_{V \in H_\ell} \inf_{v \in V \cap D} p(v). \tag{8}$$

2. If

$$\lambda_\ell := \sup_{V \in H_\ell} \inf_{v \in V \cap D} p(v) \in J$$

for some $\ell \in \mathbb{N}$, then $\lambda_\ell$ is the $\ell$th eigenvalue of $T(\cdot)$ in $J$, and equation (8) holds.

3. If there exist the $k$th and the $\ell$th eigenvalue $\lambda_k$ and $\lambda_\ell$ in $J$ ($k > \ell$), then $J$ contains the $j$th eigenvalue $\lambda_j$ ($k \geq j \geq \ell$) as well, and $\lambda_k \leq \lambda_j \leq \lambda_\ell$.

4. Let $\lambda_1 = \sup_{x \in D} p(x) \in J$ and $\lambda_\ell \in J$. If the maximum in equation (8) is attained for an $\ell$-dimensional subspace $V$, then $V \subset D \cup \{0\}$, and equation (8) can be replaced with

$$\lambda_\ell = \max_{V \in H_\ell, V \subset D \cup \{0\}} \inf_{v \neq 0} p(v). \tag{9}$$

5. $\bar{\lambda}$ is an $\ell$th eigenvalue if and only if $\mu = 0$ is the $\ell$th smallest eigenvalue of the linear eigenproblem $T(\lambda)x = \mu x$.

6. The maximum in equation (8) is attained for the invariant subspace of $T(\lambda_\ell)$ corresponding to its $\ell$th smallest eigenvalues.

The number of eigenvalues can be determined from the following generalization of Sylvester’s law of inertia for nonlinear eigenvalue problems, which was proved by Kostic and Voss [27].

**Theorem 2.** Let $T : J \to C^{n \times n}$ satisfy the conditions of the maxmin characterization and let $(n_+, n_-, n_0)$ be the inertia of $T(\sigma)$ for some $\sigma \in J$.

1. If $\sup_{x \in D} p(x) \in J$, then the nonlinear eigenvalue problem $T(\lambda)x = 0$ has exactly $n_-$ eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_-}$ in $J$ not exceeding $\sigma$.

2. Let $\sigma, \tau \in J$, $\sigma < \tau$, and let $(n_+(\sigma), n_-(\sigma), n_0(\sigma))$ and $(n_+(\tau), n_-(\tau), n_0(\tau))$ be the inerties of $T(\sigma)$ and $T(\tau)$, respectively. Then the inequality $n_-(\sigma) \geq n_-(\tau)$ holds, and the nonlinear eigenvalue problem $T(\lambda)x = 0$ has exactly $n_-(\tau)$ eigenvalues $\lambda_{n_-(\tau)+1} \geq \cdots \geq \lambda_{n_-(\sigma)}$ in the interval $(\sigma, \tau)$. 
3. Real eigenvalues of viscoelastic systems

In this section, we shall derive the variational characterization of real eigenvalues of equation (3). We consider the exponentially damped linear system of equations (1) and (2), which, in the frequency domain, is of the form

\[ T(s)u := (s^2M + sG(s) + K)u = 0, \]  

where

\[ G(s) = \sum_{j=1}^{n} \frac{\mu_j}{s + \mu_j}C_j \]  

is the Laplace transform of the kernel function \( G(t) \).

We assume that the mass matrix \( M \) and the stiffness matrix \( K \) are positive definite, that the damping coefficient matrices \( C_j \neq 0 \), \( j = 1, \ldots, n \) are positive semi-definite, and that the relaxation parameters are ordered by magnitude \( 0 < \mu_1 < \cdots < \mu_n \). We denote by \( r_j \) the rank of the matrix \( C_j \) and the \( j \)th interval by \( I_j := (-\mu_j, -\mu_{j-1}) \), \( j = 1, \ldots, n \) where \( \mu_0 := 0 \).

For every \( u \in \mathbb{R}^N \setminus \{0\} \) we consider the projection of equation (10) to \( \text{span}\{u\} \)

\[ f(s; u) := s^2u^TMu + s\sum_{j=1}^{n} \frac{\mu_j}{s + \mu_j}u^TC_ju + u^TKu \]

\[ =: ms^2 + s\sum_{j=1}^{n} \frac{\mu_j}{s + \mu_j}c_j + k = 0, \]

where constants \( m, c_j, \) and \( k \) are projections of matrices \( M, C_j, \) and \( K \) to the one-dimensional subspace \( \text{span}\{u\} \). Equation (12) is the characteristic equation of a single-degree-of-freedom system. We assume that each of these systems is underdamped, i.e., two roots are in a complex conjugate pair, such that this single-degree-of-freedom system allows for oscillatory motions. Conditions for this behavior of a single-degree-of-freedom system with one exponential term only are discussed by Adhikari [8].

3.1. Case 1: All \( C_j \) have full rank \( N \)

Multiplying equation (12) with \( \prod_{j=1}^{n}(s + \mu_j) \) yields a polynomial of degree \( 2 + n \) (since all \( c_j \) are different from zero), which has \( 2 + n \) roots. For every \( j = 1, \ldots, n \) the function \( f(\cdot; u) \) is continuous in \( I_j := (-\mu_j, -\mu_{j-1}) \) and it holds that

\[ \lim_{s \to -\mu_j + 0} \frac{\mu_j c_j}{s + \mu_j} = -\infty, \]

and therefore \( \lim_{s \to -\mu_j + 0} f(s; u) = -\infty \). For \( j = 1 \), we have \( f(0; u) = u^TKu > 0 \), and \( f(\cdot; u) \) has a root in \( (-\mu_1, 0) \), and for \( j > 1 \)

\[ \lim_{s \to -\mu_{j-1} - 0} \frac{s \mu_{j-1} c_{j-1}}{s + \mu_{j-1}} = \infty, \]

and thus \( \lim_{s \to -\mu_{j-1} - 0} f(s; u) = \infty \), and \( f(\cdot; u) \) also has a root in \( I_j \).

Hence, owing to the underdamping condition, \( f(\cdot; u) \) has a single real root \( p_j(u) \in (-\mu_j, -\mu_{j-1}) \) in each of the intervals \( I_j \), \( f(s, u) < 0 \) for \( -\mu_j < s < p_j(u) \), and \( f(s, u) > 0 \) for \( p_j(u) < s < -\mu_{j-1} \). Therefore, the nonlinear eigenvalue problem of equations (10) and (11) satisfies Assumptions 1 and 2 of the maxmin characterization for each of the intervals \( I_j \); since \( p_j \) is defined on \( \mathbb{R}^N \setminus \{0\} \), each of them is overdamped (in the sense of Section 2). Thus, we obtain the following result.

**Theorem 3.** Assume that the general conditions on \( M, K, \) and \( \mu_j, j = 1, \ldots, n \) are satisfied, and assume that all matrices \( C_j \) are positive definite. Then the exponentially damped linear system of equations (10) and (11) has \( N \) conjugate complex eigenvalue pairs, and \( Nn \) real eigenvalues.

In each interval \( I_j = (-\mu_j, -\mu_{j-1}) \), \( j = 1, \ldots, n \) there are \( N \) real eigenvalues \( \lambda_{(j)}^{(\ell)} \geq \cdots \geq \lambda_{(j)}^{(N)} \), which can be characterized as

\[ \lambda_{(j)}^{(\ell)} = \max_{\nu \in \mathbb{R}^N, \dim \nu = \ell} \inf_{\nu \in \ell} p_j(\nu), \quad \ell = 1, \ldots, N. \]  

(13)
3.2. Case 2: Some or all matrices $C_j$ are rank deficient

With $r_j := \text{rank}(C_j)$, it was shown by Wagner and Adhikari [7] that the eigenvalue problem of equations (10) and (11) has $m = 2N + \sum_{j=1}^{n} r_j$ eigenvalues,

$$r := \sum_{j=1}^{n} r_j,$$

which are real, owing to the underdamping assumption.

We first consider the interval $I_1 = (-\mu_1, 0)$, and show that the real eigenvalues in $I_1$ satisfy a maxmin characterization.

That $f(s; u) = 0$ has at most one root $p_1(u) \in (-\mu_1, 0)$ and that the condition $(A_2)$ is satisfied, and hence that the eigenvalues in $(-\mu_1, 0)$ are maxmin values of the Rayleigh functional $p_1$ follows from the underdamping assumption.

**Assumption 3.** Assume that $u^T C_j u \neq 0$ for $j = j_1 \leq j_2 \leq \cdots \leq j_\ell$, $\ell \leq n$ and consider the one-dimensional system

$$f(s; u) = s^2 m + s \sum_{j=1}^{n} \frac{\mu_j}{s + \mu_j} + k = s^2 m + s \sum_{i=1}^{\ell} \frac{\mu_i}{s + \mu_i} + k.$$

Then $f(\cdot; u)$ has $\ell$ poles $\mu_{j_1}, \ldots, \mu_{j_\ell}$ each of order 1, and since $f(s; u) = 0$ has $\ell + 2$ roots, two of which are nonreal, it has exactly one root in each of the intervals $(-\mu_{j_i}, -\mu_{j_{i-1}})$ for $i = 1, \ldots, \ell$, where $j_0 = 0$.

By the definiteness of $K$ it holds that $u^T T(0) u = u^T Ku > 0$, and therefore, owing to the underdamping assumption, the domain of definition $D_1$ of the Rayleigh functional $p_1$ consists of all $u \in \mathbb{R}^N$, such that $\lim_{s \to -\mu_1^+} u^T T(s) u < 0$.

For $u^T C_j u > 0$,

$$\lim_{s \to -\mu_1^+} \frac{s}{s + \mu_1} u^T C_j u = -\infty, \quad \text{hence} \quad \lim_{s \to -\mu_1^+} f(s; u) = -\infty,$$

holds; therefore, $u \in D_1$.

For $u^T C_j u = 0$, choose the minimal $j \in \{2, \ldots, n\}$, such that $u^T C_j u > 0$. Then $f(s; u) = u^T T(s) u$ is continuous in $(-\mu_1, 0)$ and, as for the case $j = 1$,

$$\lim_{s \to -\mu_j^+} \frac{s}{s + \mu_j} u^T C_j u = -\infty, \quad \text{hence} \quad \lim_{s \to -\mu_j^+} f(s; u) = -\infty,$$

and $u \in D_1$ if and only if $f(-\mu_1; u) < 0$.

From

$$\frac{\partial}{\partial s} f(s; u) \bigg|_{s=0} = \sum_{j=1}^{n} u^T C_j u > 0,$$

it follows that

$$\sup_{u \in D_1} p_1(u) < 0.$$

Moreover, $D_1$ contains an $r_1$-dimensional subspace of $\mathbb{R}^N$; therefore, $I_1$ contains at least $r_1$ real eigenvalues $\lambda_j^{(1)}$, allowing for a variational characterization (equation (9)).

The exact number of eigenvalues in $I_1$ is obtained from the generalization of Sylvester’s law in Theorem 2.

If $(n_{+}^{(1)}, n_{-}^{(1)}, n_{0}^{(1)})$ is the inertia of $T(-\mu_1 + \varepsilon)$ for sufficiently small $\varepsilon > 0$, then $n_{+}^{(1)}$ is the number of eigenvalues in $I_1$. The inertia can be obtained easily from an $LDL^T$ factorization of $T(-\mu_1 + \varepsilon)$.

From Theorem 1, we obtain the following theorem.
Theorem 4. Assume that the general conditions on $M$, $K$, and $\mu$, $j = 1, \ldots, n$ are satisfied, and assume that all matrices $C_j$ are positive semi-definite, and let $(n_+^{(1)}, n_-^{(1)}, n_0^{(1)})$ be the inertia of $T(-\mu_1 + \varepsilon)$ for sufficiently small $\varepsilon > 0$.

Then $I_1$ contains at least $n_-^{(1)}$ eigenvalues, and all eigenvalues $\lambda_j^{(1)} \geq \lambda_2^{(1)} \geq \cdots$ in the interval $I_1 = (-\mu_1, 0)$ can be characterized as

$$
\lambda_j^{(1)} = \max_{V \subset D_1, \dim V = \ell} \inf_{u \in V} p_1(u),
$$

(14)

The following example demonstrates that $I_1$ may contain more than $r_1$ eigenvalues.

Example 1. Let

$$
M = \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\
0 & k \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\
0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\
0 & 1 \end{bmatrix}
$$

and $\mu := [1, 2]$.

Then, for $u := [1, 0]^T$,

$$
f_1(s) := u^T T(s) u = s^2 + \frac{1}{1 + s} + 1
$$

has the roots $s_1^{(1)} = -0.5698$ and $s_2^{(1)} = -0.2151 \pm 1.3071 i$, and for $v := [0, 1]^T$,

$$
f_2(s) := v^T T(s) v = s^2 + \frac{2s}{2 + s} + k
$$

has the roots $s_1^{(2)} = -0.9459$, $s_2^{(2)} = -0.5270 \pm 1.2748 i$ for $k = 0.9$, and $s_1 = -0.4785$, $s_2^{(2)} = -0.7608 \pm 1.0456 i$ for $k = 0.4$. Since $T(s)u = f_1(s)u$ and $T(s)v = f_2(s)v$ holds, $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ are the real eigenvalues of $T$, demonstrating that the interval $I_1$ contains more than $r_1$ eigenvalues and that there are no eigenvalues in $I_2$.

By Theorem 4, the eigenvalues corresponding to $u^T C_1 u > 0$ are guaranteed to be in $I_1$. For $k = 0.4$, this example demonstrates that additional eigenvalues (corresponding to the denominator $\mu + s$ for $j \geq 2$) can congregate with these eigenvalues.

We now consider an interval $I_j$ for some $j > 2$. If $u^T D_j u = 0$ for $i = 1, \ldots, n$ then $f(s; u) = s^2 u^T M u + u^T K u > 0$ for $s < 0$, and $u \notin D_j$.

Assume that $u \in D_j$, and let $i_1 := \max\{i \leq j - 1 : u^T C_i u > 0\}$ where $C_0 := K$. Then it follows from the continuity of $f(s; u)$ in $(-\mu_j, -\mu_i)$ that $\lim_{s \rightarrow -\mu_i} f(s; u) > 0$, and similarly to the first interval, the underdamping condition implies that Assumption 2 is satisfied. Hence, all eigenvalues in $I_j$ can be characterized as maximin values of the Rayleigh functional $p_j$.

The number of eigenvalues and the enumeration according to the maximin characterization is obtained from Theorem 2, as follows.

Remark 1. If $(n_+(\sigma), n_-(\sigma), n_0(\sigma))$ and $(n_+(\tau), n_-(\tau), n_0(\tau))$ are the inertias of $T(\sigma)$ and $T(\tau)$, respectively, where $\mu_j < \sigma < \tau < \mu_{j-1}$ then $T(\cdot)$ has $n_-(\sigma) - n_-(\tau)$ eigenvalues $\lambda_{n_-(\sigma)+1} \geq \cdots \geq \lambda_{n_-(\tau)}$ in the interval $(\sigma, \tau)$.

4. Numerical examples

The proof of Theorem 1 reveals that the subspace for which the maximum in equation (8) is attained is the invariant subspace of $T(\lambda)$, which is spanned by the eigenvectors of the matrix $T(\lambda)$ corresponding to its $j$th smallest eigenvalue, and that the minimum is attained for every eigenvector of $T(\lambda) u = \kappa u$ corresponding to its eigenvalue $\kappa = 0$. This suggests the method of Algorithm 1, called safeguarded iteration, for computing the $j$th eigenvalue of $T(\cdot)$.

The following theorem contains the convergence properties of the safeguarded iteration (cf. Niendorf and Voss [28]).

Theorem 5. Let $J \subset \mathbb{R}$ be an open interval and let $T(\lambda) \in \mathbb{C}^{n \times n}$, $\lambda \in J$, be a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$, such that Assumptions 1 and 2 are satisfied.
be determined easily, and its unique root inators, one obtains a polynomial of degree at most 2 applied easily to f to be in proximities of the poles µ.

\[ \text{Algorithm 1} \text{ Safeguarded iteration.} \]

**Require:** initial vector \( x_0 \in D \)

1. compute \( \sigma_0 = p(x_0) \)
2. for \( k = 1, 2, \ldots \) until convergence do
3. determine an eigenvector \( x_k \) corresponding to the \( j \) smallest eigenvalue of \( T(\sigma_{k-1}) \)
4. determine Rayleigh functional \( \sigma_k := p(x_k) \), i.e. solve \( x_k^T T(\sigma_k) x_k = 0 \) for \( \sigma_k \)
5. end for

1. If \( \lambda_1 := \sup_{x \in D} p(x) \in J \) and \( x_0 \in D \), then the safeguarded iteration converges globally and is monotonically increasing to \( \lambda_1 \).
2. If \( T(\lambda) \) is holomorphic on a neighborhood \( U \subset \mathbb{C} \) of a \( j \)th eigenvalue of \( T(\cdot) \) and \( \lambda_j \) is a simple eigenvalue, then the safeguarded iteration converges locally and quadratically to \( \lambda_j \).

A globally convergent variant of the safeguarded iteration is presented by Voss [29].

To compute real eigenvalues and corresponding eigenvectors of the viscoelastic eigenvalue problem, the safeguarded iteration requires the following tasks. Assume that we are given an initial or the current vector \( u \) of the eigenvector when computing the \( i \)th largest eigenvalue \( \lambda_i^{(j)} \) in the interval \( I_j \). First, we evaluate the Rayleigh functional \( p(u) \), solving the real equation \( f(\lambda; u) := u^T T(\lambda) u = 0 \). Multiplying \( f(\lambda; u) \) by all appearing denominators, one obtains a polynomial of degree at most \( 2 + n \), and since \( n \) is usually quite small, all its roots can be determined easily, and its unique root \( p(u) \) in \( I_j \) can be determined (alternatively, Newton’s method can be applied easily to \( f(\lambda; u) = 0 \). However, if the wanted root \( p(u) \) is close to the boundary of \( I_j \) it may happen that the initial value is not in the zone of attraction of Newton’s method. In this case, we can make use of Theorem 2 and bisection to improve the initial guess and enforce convergence). Once the approximation \( p(u) \) to the wanted eigenvalue has been found the linear eigenvalue problem \( T(p(u)) v = \kappa v \) is solved, and the eigenvector corresponding to the \( j \)th smallest eigenvalue \( \kappa \) is chosen as the new approximation \( u \) to the wanted eigenvector.

Usually, non-state-space approaches for viscoelastic problems are based on some restrictive physical assumptions. For instance, the approach that has been developed in Lázaro [16] is based on linearization of \( T(\lambda) \) at some \( \mu_j \), and is only applicable for lightly damped systems where the real eigenvalues can be shown to be in proximities of the poles \( \mu_j \). Adhikari and Pascual [12, 13] transform \( T(\cdot) \) to modal coordinates of the undamped system \( KX = MX \Omega^2 \), i.e.

\[ \tilde{T} = I + \lambda \sum_{j=1}^{m} \frac{\mu_j}{\lambda + \mu_j} X^T C_j X + \Omega^2, \]

and then they neglect the off-diagonal entries of \( X^T C_j X \). It is assumed that the nonproportionality is small, i.e. that the off-diagonal entries of \( \tilde{T} \) are small compared with its diagonal entries. Example 3 reveals that the safeguarded iteration also succeeds in cases when such methods fail regarding their restrictive physical assumptions. This is because the safeguarded iteration mainly relies on the mathematical form of the function \( T(\lambda) \). Regardless of restrictive physical assumptions, such as nonproportionality of the damping, this method is successful while the function \( T(\lambda) \) satisfies the assumptions of Theorem 5 and the initial vector \( x_0 \) is chosen properly.

To show the reliability and efficiency of our method, we implement it for some realistic physical examples. For the following examples, our stopping criterion in the safeguarded iteration is that the relative error between successive iterations should be less than \( 1 \times 10^{-15} \). The relative error is calculated using a basis of one.

**Example 2.** The following example is taken from Adhikari and Wagner [11]. Let \( \mu = [1, 5] \), and

\[
M = 3I, \quad K = 2 \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{bmatrix}, \quad C_1 = 0.6 \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad C_2 = 0.2 \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & \lambda \\
0 & -1 & 1
\end{bmatrix}.
\]

Then \( r_1 := \text{rank}(C_1) = 2 \) implies that there are at least two real eigenvalues in \( I_1 = (-1, 0) \), and Sylvester’s law implies that \( I_1 \) contains exactly two eigenvalues in \( I_1 \) and another in \( I_2 = (-5, -1) \).
Table 1. Convergence behavior for first eigenvalue in $I_1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda_k$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-0.8850009423012485$</td>
<td>$2.33 \times 10^{-2}$</td>
</tr>
<tr>
<td>1</td>
<td>$-0.86501639393549$</td>
<td>$1.71 \times 10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.864853263454781$</td>
<td>$8.53 \times 10^{-9}$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.864853512652167$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Convergence behavior for second eigenvalue in $I_1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda_k$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-0.9353368996719444$</td>
<td>$3.11 \times 10^{-3}$</td>
</tr>
<tr>
<td>1</td>
<td>$-0.9324383219856075$</td>
<td>$5.37 \times 10^{-6}$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.9324333103801842$</td>
<td>$1.63 \times 10^{-11}$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.9324333103649718$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Convergence behavior for first eigenvalue in $I_2$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda_k$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-4.890528934075012$</td>
<td>$3.31 \times 10^{-3}$</td>
</tr>
<tr>
<td>1</td>
<td>$-4.874418912730874$</td>
<td>$1.67 \times 10^{-7}$</td>
</tr>
<tr>
<td>2</td>
<td>$-4.874418097646541$</td>
<td>$5.46 \times 10^{-16}$</td>
</tr>
<tr>
<td>3</td>
<td>$-4.874418097646538$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Convergence behavior for first eigenvalue in $I_1$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda_k$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-0.5745285588902163$</td>
<td>$1.28 \times 10^{-01}$</td>
</tr>
<tr>
<td>1</td>
<td>$-0.5120406384543340$</td>
<td>$5.66 \times 10^{-03}$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.5091675848745895$</td>
<td>$1.78 \times 10^{-05}$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.5091585213016412$</td>
<td>$1.80 \times 10^{-10}$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.5091585212098354$</td>
<td>$1.09 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

Determining initial values for the eigenvalues by the first-order approximation in Lázaro [16], one obtains the eigenvalue approximations in Tables 1 and 2 by the safeguarded iteration in $I_1$, where the relative errors indicate the quadratic convergence.

Notice that we did not initialize the iterations by approximations to eigenvectors but to the eigenvalues. Therefore, the monotone convergence to $\lambda_1^{(1)}$ in Table 1 holds only after the first iteration, $\lambda_1$. For the eigenvalue in $I_2$ we obtain the results given in Table 3.

Example 3. This example demonstrates that the safeguarded iteration also succeeds in cases when some current methods fail, owing to their restrictive physical assumptions.

Again we consider the matrices $M$, $K$, and $C_1$ and $\mu$ of Example 2 but we replace $C_2$ with

$$C_2 = 5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$ 

The safeguarded iteration converges globally to the largest eigenvalue in $I_1 = (-1, 0)$. Choosing the initial vector at random, we obtain the iteration history reported in Table 4.

Continuing with the eigenvector $u$ corresponding to the second eigenvalue of $T(\lambda_1)$, $\lambda_1 = -0.5091585212098354$, and aiming at a second eigenvalue in $I_1 = (-1, 0)$, one gets the results in Table 5.
Table 5. Convergence behavior for second eigenvalue in \( I_1 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \lambda_k )</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-0.9713917624952569)</td>
<td>(5.81 \times 10^{-02})</td>
</tr>
<tr>
<td>1</td>
<td>(-0.9214037635827200)</td>
<td>(3.60 \times 10^{-03})</td>
</tr>
<tr>
<td>2</td>
<td>(-0.9181304185737342)</td>
<td>(3.77 \times 10^{-05})</td>
</tr>
<tr>
<td>3</td>
<td>(-0.9180957791399321)</td>
<td>(4.40 \times 10^{-09})</td>
</tr>
<tr>
<td>4</td>
<td>(-0.9180957751045256)</td>
<td>(7.26 \times 10^{-16})</td>
</tr>
</tbody>
</table>

Table 6. Convergence behavior for first eigenvalue in \( I_2 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \lambda_k )</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-4.697038664163345)</td>
<td>(3.59 \times 10^{+01})</td>
</tr>
<tr>
<td>1</td>
<td>(-1.163682728784905)</td>
<td>(1.38 \times 10^{-01})</td>
</tr>
<tr>
<td>2</td>
<td>(-1.054801858216535)</td>
<td>(3.14 \times 10^{-02})</td>
</tr>
<tr>
<td>3</td>
<td>(-1.032346569236416)</td>
<td>(9.45 \times 10^{-03})</td>
</tr>
<tr>
<td>4</td>
<td>(-1.024628584720732)</td>
<td>(1.90 \times 10^{-03})</td>
</tr>
<tr>
<td>5</td>
<td>(-1.022809231521676)</td>
<td>(1.21 \times 10^{-04})</td>
</tr>
<tr>
<td>6</td>
<td>(-1.022685713203887)</td>
<td>(5.69 \times 10^{-07})</td>
</tr>
<tr>
<td>7</td>
<td>(-1.022685131138450)</td>
<td>(1.26 \times 10^{-11})</td>
</tr>
<tr>
<td>8</td>
<td>(-1.022685131125515)</td>
<td></td>
</tr>
</tbody>
</table>
Table 7. Real eigenvalues of the system in Example 4.

<table>
<thead>
<tr>
<th>$\lambda^{(1)}$</th>
<th>$\lambda^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1^{(1)}$</td>
<td>$-49.1401549067203$</td>
</tr>
<tr>
<td>$\lambda_2^{(1)}$</td>
<td>$-49.5396268709059$</td>
</tr>
<tr>
<td>$\lambda_3^{(1)}$</td>
<td>$-49.7452598611765$</td>
</tr>
<tr>
<td>$\lambda_4^{(1)}$</td>
<td>$-49.9286161974936$</td>
</tr>
</tbody>
</table>

Table 8. Approximate eigenvalues in $I_1$ and $I_2$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\lambda_j^{(1)}$</th>
<th>Relative error</th>
<th>$\lambda_j^{(2)}$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-49.14015490672037$</td>
<td>$1.45 \times 10^{-15}$</td>
<td>$-69.06598684153966$</td>
<td>$2.26 \times 10^{-15}$</td>
</tr>
<tr>
<td>2</td>
<td>$-49.53962687090588$</td>
<td>$4.30 \times 10^{-16}$</td>
<td>$-69.51999213879260$</td>
<td>$1.43 \times 10^{-15}$</td>
</tr>
<tr>
<td>3</td>
<td>$-49.74525986117655$</td>
<td>$1.00 \times 10^{-16}$</td>
<td>$-69.73949030269418$</td>
<td>$2.03 \times 10^{-16}$</td>
</tr>
<tr>
<td>4</td>
<td>$-49.92861619749364$</td>
<td>$8.53 \times 10^{-16}$</td>
<td>$-69.92817947042275$</td>
<td>$2.24 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

Table 9. Convergence behavior of largest eigenvalues in $I_1$ and $I_2$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\lambda_j^{(1)}$</th>
<th>Relative error</th>
<th>$\lambda_j^{(2)}$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-49.63970412945238$</td>
<td>$6.19 \times 10^{-03}$</td>
<td>$-69.73955417360676$</td>
<td>$2.70 \times 10^{-03}$</td>
</tr>
<tr>
<td>1</td>
<td>$-49.21342383480226$</td>
<td>$6.11 \times 10^{-04}$</td>
<td>$-69.91403489875896$</td>
<td>$2.02 \times 10^{-04}$</td>
</tr>
<tr>
<td>2</td>
<td>$-49.14111828211603$</td>
<td>$1.04 \times 10^{-05}$</td>
<td>$-69.92806651050697$</td>
<td>$1.62 \times 10^{-06}$</td>
</tr>
<tr>
<td>3</td>
<td>$-49.14015490672026$</td>
<td>$3.16 \times 10^{-09}$</td>
<td>$-69.92817946305229$</td>
<td>$1.05 \times 10^{-10}$</td>
</tr>
<tr>
<td>4</td>
<td>$-49.14015490672037$</td>
<td>$1.45 \times 10^{-15}$</td>
<td>$-69.92817947042275$</td>
<td>$2.26 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

which contains all eigenvalues, and the relative length $\lambda_1^{(i)} - \lambda_4^{(i)}$ of which is only $1.57 \times 10^{-02}$ for $i = 1$ and $1.23 \times 10^{-02}$ for $i = 2$.

Singh [17] solved the problem iteratively, approximating each iteration $T(\lambda)$ by a third-order Taylor expansion and solving the produced polynomial eigenvalue problem of degree three by linearization. He reports the following approximations to the real eigenvalues:

$-49.1402, -49.7453, -49.9286, -53.7052, -65.4264, -69.0660, -69.7395, -69.9282,$

requiring 121, 10, 8, 31, 27, 13, 15, and 462 iterations, respectively. Notice that in the interval $I_1$ only three approximate eigenvalues were found, while Theorem 3 guarantees four eigenvalues in $I_1$.

It is noteworthy that, in these examples, the safeguarded iteration determines the next eigenvalue iteration from the Rayleigh functional of an eigenvector approximation, which is obtained from an inverse iteration step. Thus, since the the eigenvalue approximations converge quadratically, the same must hold for the eigenvector approximations.

5. Conclusions

The computation of eigenvalues of viscoelastic systems with exponential damping model requires the solution of a nonlinear eigenvalue problem. If the corresponding matrices are symmetric, then the eigenvalues of the problem allow for a variational characterization, and the safeguarded iteration enables the eigenvalues to be computed very efficiently, even if the problem is strongly viscoelastic or strongly nonproportional. The safeguarded iteration converges quadratically and its convergence is monotone for extreme eigenvalues. Numerical examples demonstrate the efficiency of the approach.

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References


