VARIATIONAL CHARACTERIZATION OF REAL EIGENVALUES IN LINEAR VISCOELASTIC OSCILLATORS

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Abstract. This paper proposes a new approach for computing the real eigenvalues of a multiple–degree–of–freedom viscoelastic system where we assume an exponentially decaying damping. The free–motion equations lead to a nonlinear eigenvalue problem. If the system matrices are symmetric then the eigenvalues allow for a variational characterization of maxmin type, and the eigenvalues and eigenvectors can be determined very efficiently by the safeguarded iteration which converges quadratically and for extreme eigenvalues monotonically. Numerical methods demonstrate the performance and the reliability of the approach. The method succeeds when some of current approaches fail regarding their restrictive physical assumptions.

1. Introduction. It is essential to understand the vibration properties of complex engineering structures, such as automobile, civil and aeronautics structures for mitigating fatigue and developing vibration suppression techniques. In view of favorable damping characteristics, polymers and composite materials are of great importance in vibration control applications. These materials show a combination of elastic and viscoelastic response and usually characterized as viscoelastic materials. These materials are shown dependency in frequency, load and temperature while deformation is undergoing. These characteristics are often applied for a wide range of engineering applications such as vibrating control in almost all areas of engineering: mechanical, architecture, civil, industrial, aeronautics, and automotive. For instance, viscoelastic materials are used for mitigating earthquake effects in buildings or vibration isolation. However, there are challenges in developing mathematical models for investigating the deformation and dynamic behavior of structures composed of viscoelastic materials

Recently, several material models have been developed to determine various aspects of deformation, stiffness and damping in the materials to study the dynamic behavior of structures composed of viscoelastic materials [12].

From the mathematical point of view, any causal model that makes the energy dissipation functional nonnegative is a candidate for a nonviscous damping model. Using the physics of the problem or selecting a model a priori and fitting its parameters from experiments, obviously there are a lot of choice for the model. For example, they can be rational polynomials for Biot [7], Golla-Hughes-McTavish [11, 19] and inelastic displacement field models [17], and fractional polynomials [6]. Some representative damping functions, including complex transcendental functions, are tabulated in [29].

In the most general case, the structures that include viscoelastic materials are determined by hereditary energy dissipation mechanisms which means that the damping forces depend on the history of the velocity response. Mathematically, this fact is represented by convolution integrals that involve the velocities of the degrees–of–freedom over certain kernel functions.

In order to investigate nonviscous damping of structures composed of viscoelastic materials, in this paper we consider an exponential damping model which was proposed by Biot [7] and which assumes that the dissipative forces depend on the

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history of motion via the convolution integral over an exponentially decreasing kernel function. The exponential damping model is physically the most meaningful one, and it has been widely used in recent studies [12].

The equation of motion of an N–degree–of–freedom linear system of this type reads

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \int_{-\infty}^{t} \mathcal{G}(t-\tau)\dot{\mathbf{u}}(\tau)d\tau + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t), \tag{1.1}$$

together with initial conditions $\mathbf{u}(0) = \mathbf{u_0}$, $\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$, where $\mathbf{u} \in \mathbb{R}^N$ is the displacement vector, $\mathbf{f} \in \mathbb{R}^N$ is the forcing vector, $\mathbf{M} \in \mathbb{R}^{N\times N}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass and stiffness matrix, respectively, and $\mathcal{G} \in \mathbb{R}^{N\times N}$ is the kernel function of damping.

For exponential damping the kernel function has the following form

$$\mathcal{G}(t) = \sum_{j=1}^{n} \mu_j exp(-\mu_j t)\mathbf{C}_j, \tag{1.2}$$

where for $j = 1, \ldots, n$ the constants $\mu_j \in \mathbb{R}_+$ are the relaxation parameters and $\mathbf{C}_j \in \mathbb{R}^{N\times N}$ are damping coefficient matrices.

Here, $\mathbf{M}$ and $\mathbf{K}$ are assumed both symmetric and positive definite. The modes of the system can be determined as a non-trivial solutions of free-motion problem obtained by setting $\mathbf{f} = \mathbf{u}_0 = \dot{\mathbf{u}}_0 = 0$ in equation (1.1). Therefore, considering functions of the form $\mathbf{u}(t) = \mathbf{u}e^{st}$ we have

$$\mathbf{T}(s) := (s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K})\mathbf{u} = 0, \tag{1.3}$$

where $\mathbf{G}$ is the Laplace transform of the kernel function $\mathcal{G}(t)$. This is a nonlinear eigenvalue problem which depends nonlinearly on the eigenparameter s. Recall that s is an eigenvalue of (1.3) if equation (1.3) has a nontrivial solution $\mathbf{u} \neq 0$.

Considering that the damping model is strictly dissipative (conditions are provided in [11]), it can be shown [1, 20] that equation (1.3) has $2N + r$ eigenvalues, $r \geq 0$, where $2N$ roots are in complex conjugate pairs. The rest eigenvalues are negative real numbers since positive eigenvalues and dissipative behavior can not coexist. These eigenvalues which are named non-viscous eigenvalues are presented as an intrinsic property of viscoelastic models. The number of non-viscous eigenvalues depends on the nature of the damping function and they are associated with non-oscillatory modes.

Several methods have been developed to compute the non-viscous eigenvalues of viscoelastic systems. The most prominent approach to estimate these eigenvalues deals in converting the integro-differential equation (1.1) to an associated state-space model [1, 4, 5, 6, 11, 19, 17, 20, 29], by introducing internal variables and computing the resulting eigenvalues and eigenvectors. Such a realization in state-space form introduces a large number of fictitious internal variables leading to computationally expansive large size matrices for eigenvalue computations. To overcome this challenge, non-state space methods are investigated by several researchers. Approximated closed-form expressions for the complex and real eigenvalues of the viscoelastic system are obtained in [2] for three mathematically different cases based on the number of kernel functions. The approximations utilize Taylor series expansion in the complex domain and are based on certain simplifying physical assumptions. The approach is

valid only for systems with small viscoelasticity. In [3], five different iterative algorithms are also given for estimating the real and complex eigenvalues associated with single and multiple degree-of-freedom systems. The iterative method for the complex eigenvalue is applicable to any general viscoelastic kernel while the iterative method for real eigenvalues is specific to exponential damping. A fixed point iteration method is proposed in [16] towards the computation of eigenvalues of a viscoelastic system. Although under certain conditions the method may also be used to obtain non–viscous eigenvalues, the method can only be implemented for systems with a proportional, or lightly non-proportional, damping matrix.

Introducing the novel concept of "non-viscous set", a new mathematical characterization of the non-viscous eigenvalues has been proposed in [15]. Using the non-viscous set, a closed form expression that estimates each real eigenvalues has been obtained for systems with exponential kernels. It is proved that for lightly or moderately damped systems, the set of real eigenvalues can be obtained solving as many linear eigenvalue problems as exponential kernels [14]. A numerical method for computing the eigenvalues and respective left and right eigenvectors has been presented in [24], which is based on Newton's eigenvalue iteration method.

In this paper we will develop a nonstate-step method invoking the variational or maxmin characterization of real eigenvalues which has not been derived for viscoelastic systems in the previous studies. Variational characterizations of eigenvalues are very powerful tools when studying nonlinear eigenvalue problems. Usually, nonstate-step methods are based upon some restrictive physical assumptions and they fail to obtain vonviscous eigenvalues efficiently when the physical assumptins are not valid. It is shown by several realistic physical examples that our method succesfully determines eigenvalues while current methods fail regarding their restrictive physical assumptions.

2. Variational characterization for nonlinear eigenvalue problems. Our main tools in this paper are variational characterizations of eigenvalues of nonlinear eigenvalue problems [27] generalizing the well known minmax characterization of Poincaré [22] or Courant [8] and Fischer [10] for linear eigenvalue problems.

We consider the nonlinear eigenvalue problem

$$\mathbf{T}(\lambda)\mathbf{x} = 0, \tag{2.1}$$

where $\mathbf{T}(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, is a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$, and J is a real open interval which may be unbounded.

To generalize the variational characterization of eigenvalues we need a generalization of the Rayleigh quotient. To this end we assume that

(A_1) for every fixed $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq 0$ the scalar real equation

$$f(\lambda; \mathbf{x}) := \mathbf{x}^H \mathbf{T}(\lambda)\mathbf{x} = 0 \tag{2.2}$$

has at most one solution $p(\mathbf{x}) \in J$.

Then $f(\lambda; \mathbf{x}) = 0$ implicitly defines a functional p on some subset $\mathcal{D} \subset \mathbb{C}^n$ which is called the Rayleigh functional of (2.1), and which is exactly the Rayleigh quotient in case of a monic linear matrix function $\mathbf{T}(\lambda) = \lambda\mathbf{I} - \mathbf{A}$.

Generalizing the definiteness requirement for linear pencils $\mathbf{T}(\lambda) = \lambda\mathbf{B} - \mathbf{A}$ we further assume that

(A_2) for every $\mathbf{x} \in \mathcal{D}$ and every $\lambda \in J$ with $\lambda \neq p(\mathbf{x})$ it holds that

$$(\lambda - p(\mathbf{x}))f(\lambda; \mathbf{x}) > 0. \tag{2.3}$$

If p is defined on $\mathcal{D} = \mathbb{C}^n \setminus \{0\}$ then the problem $\mathbf{T}(\lambda)\mathbf{x} = 0$ is called overdamped. This notation is motivated by the finite dimensional quadratic eigenvalue problem

$$\mathbf{T}(\lambda)\mathbf{x} = \lambda^2\mathbf{M}\mathbf{x} + \lambda\mathbf{C}\mathbf{x} + \mathbf{K}\mathbf{x} = 0, \tag{2.4}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are Hermitian and positive definite matrices. If $\mathbf{C}$ is big enough such that $d(\mathbf{x}) := (\mathbf{x}^H\mathbf{C}\mathbf{x})^2 - 4(\mathbf{x}^H\mathbf{K}\mathbf{x})(\mathbf{x}^H\mathbf{M}\mathbf{x}) > 0$ for every $\mathbf{x} \neq 0$ then $\mathbf{T}(\cdot)$ is overdamped. Generalizations of the minmax and maxmin characterizations of eigenvalues were proved by Duffin [9] for the quadratic case and by Rogers [23] for general overdamped problems.

For nonoverdamped eigenproblems the natural ordering to call the largest eigenvalue the first one, the second largest the second one, etc., is not appropriate. This is obvious if we make a linear eigenvalue problem $\mathbf{T}(\lambda)\mathbf{x} := (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = 0$ nonlinear by restricting it to an interval J which does not contain the largest eigenvalue of $\mathbf{A}$. Then the conditions (A_1) and (A_2) are satisfied, p is the restriction of the Rayleigh quotient R_A to

$$\mathcal{D} := \{\mathbf{x} \neq 0 \,:\, R_A(\mathbf{x}) \in J\},$$

and $\sup_{\mathbf{x}\in\mathcal{D}} p(\mathbf{x})$ will in general not be an eigenvalue.

If $\lambda \in J$ is an eigenvalue of $\mathbf{T}(\cdot)$ then $\mu = 0$ is an eigenvalue of the linear problem $\mathbf{T}(\lambda)\mathbf{y} = \mu\mathbf{y}$, and therefore there exists $\ell \in \mathbb{N}$ such that

$$0 = \min_{V\in H_\ell} \max_{\mathbf{v}\in V\setminus\{0\}} \frac{\mathbf{v}^H\mathbf{T}(\lambda)\mathbf{v}}{\|\mathbf{v}\|^2},$$

where H_ℓ denotes the set of all ℓ–dimensional subspaces of $\mathbb{C}^n$. In this case λ is called an ℓth eigenvalue of $\mathbf{T}(\cdot)$.

With this enumeration the following maxmin characterization for eigenvalues was proved in [26, 28].

THEOREM 2.1. *Let J be an open interval in $\mathbb{R}$, and let $\mathbf{T}(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, be a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$ such that the conditions (A_1) and (A_2) are satisfied. Then the following statements hold.*

(i) For every $\ell \in \mathbb{N}$ there is at most one ℓth eigenvalue of $\mathbf{T}(\cdot)$ which can be characterized by

$$\lambda_\ell = \max_{V\in H_\ell,\ V\cap\mathcal{D}\neq\emptyset} \inf_{\mathbf{v}\in V\cap\mathcal{D}} p(\mathbf{v}). \tag{2.5}$$

(ii) If

$$\lambda_\ell := \sup_{V\in H_\ell,\ V\cap\mathcal{D}\neq\emptyset} \inf_{\mathbf{v}\in V\cap\mathcal{D}} p(\mathbf{v}) \in J$$

for some $\ell \in \mathbb{N}$ then λ_ℓ is the ℓth eigenvalue of $\mathbf{T}(\cdot)$ in J, and (2.5) holds.

(iii) If there exist the kth and the ℓth eigenvalue λ_k and λ_ℓ in J $(k > \ell)$, then J contains the jth eigenvalue λ_j $(k \geq j \geq \ell)$ as well, and $\lambda_k \leq \lambda_j \leq \lambda_\ell$.

(iv) Let $\lambda_1 = \sup_{x\in\mathcal{D}} p(\mathbf{x}) \in J$ and $\lambda_\ell \in J$. If the maximum in (2.5) is attained for an ℓ dimensional subspace V, then $V \subset \mathcal{D}\cup\{0\}$, and (2.5) can be replaced with

$$\lambda_\ell = \max_{V\in H_\ell,\ V\subset\mathcal{D}\cup\{0\}} \inf_{\mathbf{v}\in V,\ \mathbf{v}\neq 0} p(\mathbf{v}). \tag{2.6}$$

(v) $\tilde{\lambda}$ *is an* ℓ*th eigenvalue if and only if* $\mu = 0$ *is the* ℓ*th smallest eigenvalue of the linear eigenproblem* $\mathbf{T}(\tilde{\lambda})\mathbf{x} = \mu\mathbf{x}$.

(vi) The maximum in (2.5) *is attained for the invariant subspace of* $\mathbf{T}(\lambda_\ell)$ *corresponding to its* ℓ*th smallest eigenvalues.*

The number of eigenvalues can be determined from the following generalization of Sylvester's law of inertia for nonlinear eigenvalue problems which was proved in [13].

THEOREM 2.2. *Let* $\mathbf{T} : J \to \mathbb{C}^{n\times n}$ *satisfy the conditions of the maxmin characterization and let* (n_+, n_-, n_0) *be the inertia of* $\mathbf{T}(\sigma)$ *for some* $\sigma \in J$.

(i) If $\sup_{\mathbf{x}\in\mathcal{D}} p(\mathbf{x}) \in J$ *then the nonlinear eigenvalue problem* $\mathbf{T}(\lambda)\mathbf{x} = 0$ *has exactly* n_- *eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_-}$ *in* J *not exceeding* σ.

(ii) Let $\sigma, \tau \in J$, $\sigma < \tau$, *and let* $(n_+(\sigma)n_-(\sigma), n_0(\sigma))$ *and* $(n_+(\tau), n_-(\tau), n_0(\tau))$ *be the inertias of* $\mathbf{T}(\sigma)$ *and* $\mathbf{T}(\tau)$, *respectively. Then the inequality* $n_-(\sigma) \geq n_-(\tau)$ *holds , and the nonlinear eigenvalue problem* $\mathbf{T}(\lambda)\mathbf{x} = 0$ *has exactly* $n_-(\sigma) - n_-(\tau)$ *eigenvalues* $\lambda_{n_-(\sigma)+1} \geq \cdots \geq \lambda_{n_-(\tau)}$ *in the interval* (σ, τ).

3. Real eigenvalues of viscoelastic systems. In this section we shall derive the variational characterization of real eigenvalues of (1.3) viscoelastic systems. We consider the exponentially damped linear system (1.1), (1.2) which in frequency domain receives the form

$$\mathbf{T}(s)\mathbf{u} := (s^2\mathbf{M} + s\mathbf{G}(s) + \mathbf{K})\mathbf{u} = 0, \tag{3.1}$$

where

$$\mathbf{G}(s) = \sum_{j=1}^{n} \frac{\mu_j}{s+\mu_j}\mathbf{C}_j, \tag{3.2}$$

is the Laplace transform of the kernel function $\mathcal{G}(t)$.

We assume that the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are positive definite, that the damping coefficient matrices $\mathbf{C}_j \neq 0$, $j = 1, \ldots, n$ are positive semi–definite, and that the relaxation parameters are ordered by magnitude $0 < \mu_1 < \cdots < \mu_n$. We denote by r_j the rank of the matrix $\mathbf{C}_j$ and the jth interval by $I_j := (-\mu_j, -\mu_{j-1})$, $j = 1, \ldots, n$ where $\mu_0 := 0$.

For every $\mathbf{u} \in \mathbb{R}^N \setminus \{0\}$ we consider the projection of problem (3.1) to span$\{\mathbf{u}\}$

$$\begin{aligned} f(s;\mathbf{u}) &:= s^2\mathbf{u}^T\mathbf{M}\mathbf{u} + s\sum_{j=1}^{n}\frac{\mu_j}{s+\mu_j}\mathbf{u}^T\mathbf{C}_j\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} \\ &=: ms^2 + s\sum_{j=1}^{n}\frac{\mu_j}{s+\mu_j}c_j + k = 0, \end{aligned} \tag{3.3}$$

which is the characteristic equation of a single–degree–of–freedom system. We assume that each of these systems is underdamped, i.e. two roots are in a complex conjugate pair, such that this single–degree–of–freedom system allows for oscillatory motions. Conditions for this behavior of a single–degree–of–freedom system with one exponential term only is discussed by Adhikari [1].

3.1. Case 1: All C_j have full rank N. Multiplying equation (3.3) with $\prod_{j=1}^{n}(s+\mu_j)$ yields a polynomial of degree $2+n$ (since all c_j are different from

zero) which has $2+n$ roots. For every $j=1,\dots,n$ the function $f(\cdot;\mathbf{u})$ is continuous in $I_j := (-\mu_j, -\mu_{j-1})$ and it holds that

$$\lim_{s\to-\mu_j+0} \frac{s\mu_j c_j}{s+\mu_j} = -\infty,$$

and therefore $\lim_{s\to-\mu_j+0} f(s;\mathbf{u}) = -\infty$. For $j=1$ we have $f(0;\mathbf{u}) = \mathbf{u}^T\mathbf{K}\mathbf{u} > 0$, and $f(\cdot;\mathbf{u})$ has a root in $(-\mu_1, 0)$, and for $j>1$

$$\lim_{s\to-\mu_{j-1}-0} \frac{s\mu_{j-1}c_{j-1}}{s+\mu_{j-1}} = \infty, \quad \text{and thus } \lim_{s\to-\mu_{j-1}-0} f(s;\mathbf{u}) = \infty,$$

and $f(\cdot;\mathbf{u})$ has also a root in I_j.

Hence, due to the underdamping condition, $f(\cdot;\mathbf{u})$ has a single real root $p_j(\mathbf{u}) \in (-\mu_j, -\mu_{j-1})$ in each of the intervals I_j, and $f(s,\mathbf{u}) < 0$ for $-\mu_j < s < p_j(\mathbf{u})$ and $f(s,\mathbf{u}) > 0$ for $p_j(\mathbf{u}) < s < -\mu_{j-1}$. Therefore, the nonlinear eigenvalue problem (3.1), (3.2) satisfies the conditions (A_1) and (A_2) of the maxmin characterization for each of the intervals I_j, and since p_j is defined on $\mathbb{R}^N \setminus \{0\}$ each of them is overdamped (in the sense of Section 2). Thus, we obtain the following result:

THEOREM 3.1. *Assume that the general conditions on* $\mathbf{M}$, $\mathbf{K}$, *and* μ_j, $j = 1..n$ *are satisfied, and assume that all matrices* $\mathbf{C_j}$ *are positive definite. Then the exponentially damped linear system* (3.1), (3.2) *has* N *conjugate complex eigenvalue pairs, and* Nn *real eigenvalues.*

In each of the interval $I_j = (-\mu_j, -\mu_{j-1})$, $j = 1,\dots,n$ *it has* N *real eigenvalues* $\lambda_1^{(j)} \ge \cdots \ge \lambda_N^{(j)}$ *which can be characterized as*

$$\lambda_\ell^{(j)} = \max_{V\subset\mathbb{R}^N, \dim V=\ell} \quad \inf_{\mathbf{v}\in V} \; p_j(\mathbf{v}), \;\; \ell = 1,\cdots,N. \tag{3.4}$$

3.2. Case 2: Some or all matrices $\mathbf{C}_j$ are rank deficient. With $r_j := \operatorname{rank}(\mathbf{C}_j)$ it was shown by Wagner and Adhikari [29] that the eigenvalue problem (3.1), (3.2) has $m = 2N + \sum_{j=1}^n r_j$ eigenvalues,

$$r := \sum_{j=1}^n r_j$$

of which are real due to the underdamping assumption.

We first consider the interval $I_1 = (-\mu_1, 0)$, and show that the real eigenvalues in I_1 satisfy a maxmin characterization.

That $f(s;\mathbf{u}) = 0$ has at most one root $p_1(\mathbf{u}) \in (-\mu_1, 0)$ and that the condition (A_2) is satisfied, and hence that the eigenvalues in $(-\mu_1, 0)$ are maxmin values of the Rayleigh functional p_1 follows from the underdamping assumption: Assume that $\mathbf{u}^T\mathbf{C}_j\mathbf{u} \neq 0$ for $j = j_1 \le j_2 \le \cdots \le j_\ell$, $\ell \le n$ and consider the one–dimensional system

$$f(s;\mathbf{u}) = s^2 m + s\sum_{j=1}^n \frac{\mu_j}{s+\mu_j} c_j + k = s^2 m + s\sum_{i=1}^\ell \frac{\mu_{j_i}}{s+\mu_{j_i}} c_{j_i} + k.$$

Then $f(\cdot;\mathbf{u})$ has ℓ poles $\mu_{j_1},\dots,\mu_{j_\ell}$ each of order 1, and since $f(s;\mathbf{u}) = 0$ has $\ell+2$ roots, two of which are non–real, it has exactly one root in each of the intervals $(-\mu_{j_i}, -\mu_{j_{i-1}})$ for $i = 1,\dots,\ell$, where $j_0 = 0$.

By the definiteness of $\mathbf{K}$ it holds that $\mathbf{u}^T\mathbf{T}(0)\mathbf{u} = \mathbf{u}^T\mathbf{K}\mathbf{u} > 0$, and therefore, due to the underdamping assumption, the domain of definition $\mathcal{D}_1$ of the Rayleigh functional p_1 consists of all $\mathbf{u} \in \mathbb{R}^N$ such that $\lim_{s\to-\mu_1+0} \mathbf{u}^T\mathbf{T}(s)\mathbf{u} < 0$.

For $\mathbf{u}^T\mathbf{C}_1\mathbf{u} > 0$ it holds

$$\lim_{s\to-\mu_1+0} s\frac{\mu_1}{s+\mu_1}\mathbf{u}^T\mathbf{C}_1\mathbf{u} = -\infty, \quad \text{hence } \lim_{s\to\mu_1+0} f(s;\mathbf{u}) = -\infty,$$

and therefore $\mathbf{u} \in \mathcal{D}_1$.

For $\mathbf{u}^T\mathbf{C}_1\mathbf{u} = 0$ choose the minimal $j \in \{2,\dots,n\}$ such that $\mathbf{u}^T\mathbf{C}_j\mathbf{u} > 0$. Then $f(s;\mathbf{u}) = \mathbf{u}^T\mathbf{T}(s)\mathbf{u}$ is continuous in $(-\mu_j, 0)$, and as for the case $j = 1$

$$\lim_{s\to-\mu_j+0} s\frac{\mu_j}{s+\mu_j}\mathbf{u}^T\mathbf{C}_j\mathbf{u} = -\infty, \quad \text{hence } \lim_{s\to\mu_j+0} f(s;\mathbf{u}) = -\infty,$$

and $\mathbf{u} \in \mathcal{D}_1$ if and only $f(-\mu_1;\mathbf{u}) < 0$.

From

$$\frac{\partial}{\partial s} f(s;\mathbf{u})\Big|_{s=0} = \sum_{j=1}^{n} \mathbf{u}^T\mathbf{C}_j\mathbf{u} > 0,$$

it follows that

$$\sup_{\mathbf{u}\in\mathcal{D}_1} p_1(\mathbf{u}) < 0.$$

Moreover, $\mathcal{D}_1$ contains an r_1 dimensional subspace of $\mathbb{R}^N$, and therefore I_1 contains at least r_1 real eigenvalues $\lambda_j^{(1)}$ allowing for a variational characterization (2.6).

The exact number of eigenvalues in I_1 is obtained from the generalization of Sylvester's law in Theorem 2.2, (i): If $(n_+^{(1)}, n_-^{(1)}, n_0^{(1)})$ is the inertia of $T(-\mu_1+\varepsilon)$ for sufficiently small $\varepsilon > 0$, then $n_-^{(1)}$ is the number of eigenvalues in I_1. The inertia can be obtained easily from an LDL^H factorization of $T(-\mu_1+\varepsilon)$.

From Theorem 2.1, (iv) we obtain

THEOREM 3.2. *Assume that the general conditions on* $\mathbf{M}$, $\mathbf{K}$, *and* μ_j, $j = 1..n$ *are satisfied, and assume that all matrices* $\mathbf{C}_j$ *are positive semi–definite, and let* $(n_+^{(1)}, n_-^{(1)}, n_0^{(1)})$ *be the inertia of* $T(-\mu_1+\varepsilon)$ *for sufficiently small* $\varepsilon > 0$.

Then I_1 *contains at least* $n_-^{(1)}$ *eigenvalues, and all eigenvalues* $\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \cdots$ *in the interval* $I_1 = (-\mu_1, 0)$ *can be characterized as*

$$\lambda_\ell^{(j)} = \max_{V\subset\mathcal{D}_1, \dim V=\ell} \inf_{\mathbf{u}\in V} p_1(\mathbf{u}). \tag{3.5}$$

The following example demonstrates that I_1 may contain more than r_1 eigenvalues.

EXAMPLE 3.3. *Let*

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \mathbf{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mu := [1, 2].$$

Then for $\mathbf{u} := [1, 0]^T$

$$f_1(s) := \mathbf{u}^T T(s)\mathbf{u} = s^2 + \frac{1}{1+s} + 1$$

has the roots $s_1^{(1)} = -0.5698$ *and* $s_{2/3}^{(1)} = -0.2151 \pm 1.3071i$, *and for* $\mathbf{v} := [0,1]^T$

$$f_2(s) := \mathbf{v}^T T(s)\mathbf{v} = s^2 + \frac{2s}{2+s} + k$$

has the roots $s_1^{(2)} = -0.9459$, $s_{2/3}^{(2)} = -0.5270 \pm 1.2748i$ *for* $k = 0.9$, *and* $s_1 = -0.4785$, $s_{2/3} = -0.7608 \pm 1.0456i$ *for* $k = 0.4$. *Since* $T(s)\mathbf{u} = f_1(s)\mathbf{u}$ *and* $T(s)\mathbf{v} = f_2(s)\mathbf{v}$ *holds,* $\lambda_j^{(1)}$ *and* $\lambda_j^{(2)}$ *are the real eigenvalues of* T *demonstrating, that the interval* I_1 *contains more than* r_1 *eigenvalues and there are no eigenvalues in* I_2.

By Theorem 3.2 the eigenvalues corresponding to $\mathbf{u}^T\mathbf{C}_1\mathbf{u} > 0$ *are guaranteed to be in* I_1. *For* $k = 0.4$ *this example demonstrates that additional eigenvalues (corresponding to the denominator* $\mu_j + s$ *for* $j \geq 2$*) can commingle with these eigenvalues.*

We now consider an interval I_j for some $j > 2$. If $\mathbf{u}^T\mathbf{D}_i\mathbf{u} = 0$ for $i = 1, \ldots, n$ then $f(s;\mathbf{u}) = s^2\mathbf{u}^T\mathbf{M}\mathbf{u} + \mathbf{u}^T\mathbf{K}\mathbf{u} > 0$ for $s < 0$, and $\mathbf{u} \notin \mathcal{D}_j$.

Assume that $\mathbf{u} \in \mathcal{D}_j$, and let $i_1 := \max\{i \leq j-1 : \mathbf{u}^T\mathbf{C}_i\mathbf{u} > 0\}$ where $\mathbf{C}_0 := \mathbf{K}$. Then it follows from the continuity of $f(\cdot;\mathbf{u})$ in $(-\mu_j, -\mu_{i_1})$ that $\lim_{s\to-\mu_{i_1}} f(s;\mathbf{u}) > 0$, and similarly to the first interval the underdamping condition implies that condition (A_2) is satisfied. Hence, all eigenvalues in I_j can be characterized as maxmin values of the Rayleigh functional p_j.

The number of eigenvalues and the enumeration according to the maxmin characterization is obtained from Theorem 2.2:
If $(n_+(\sigma), n_-(\sigma), n_0(\sigma))$ and $(n_+(\tau), n_-(\tau), n_0(\tau))$ are the inertia of $T(\sigma)$ and $T(\tau)$, respectively, where $\mu_j < \sigma < \tau < \mu_{j-1}$ then $T(\cdot)$ has $n_-(\sigma) - n_-(\tau)$ eigenvalues $\lambda_{n_-(\sigma)+1} \geq \cdots \geq \lambda_{n_-(\tau)}$ in the interval (σ, τ).

4. Numerical examples. The proof of the maxmin characterization Theorem 2.1 reveals that the subspace for which the maximum in (2.5) is attained is the invariant subspace of $\mathbf{T}(\lambda_j)$ which is spanned by the eigenvectors of the matrix $\mathbf{T}(\lambda_j)$ corresponding to its jth smallest eigenvalue, and that the minimum is attained for every eigenvector of $\mathbf{T}(\lambda_j)\mathbf{u} = \kappa\mathbf{u}$ corresponding to its eigenvalue $\kappa = 0$. This suggests the following method called safeguarded iteration for computing the jth eigenvalue of $\mathbf{T}(\cdot)$.

Algorithm 1 Safeguarded iteration

Require: initial vector $\mathbf{x}_0 \in \mathcal{D}$

1: compute $\sigma_0 = p(\mathbf{x}_0)$
2: **for** $k = 1, 2, \ldots$ until convergence **do**
3: determine an eigenvector $\mathbf{x}_k$ corresponding to the j smallest eigenvalue of $\mathbf{T}(\sigma_{k-1})$
4: determine Rayleigh functional $\sigma_k := p(\mathbf{x}_k)$, i.e. solve $\mathbf{x}_k^H\mathbf{T}(\sigma_k)\mathbf{x}_k = 0$ for σ_k
5: **end for**

The following theorem contains the convergence properties of the safeguarded iteration (cf. [21]).

THEOREM 4.1. *Let* $J \subset \mathbb{R}$ *be an open interval, let* $\mathbf{T}(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, *be a family of Hermitian matrices depending continuously on the parameter* $\lambda \in J$ *such that the conditions* (A_1) *and* (A_2) *are satisfied.*

(i) If $\lambda_1 := \sup_{\mathbf{x}\in\mathcal{D}} p(\mathbf{x}) \in J$ *and* $\mathbf{x}_0 \in \mathcal{D}$ *then the safeguarded iteration for* $j = 1$ *converges globally and monotonically increasing to* λ_1.

(ii) If $T(\lambda)$ is holomorphic on a neighborhood $U \subset \mathbb{C}$ of a jth eigenvalue of $\mathbf{T}(\cdot)$ and λ_j is a simple eigenvalue, then the safeguarded iteration converges locally and quadratically to λ_j.

A globally convergent variant of the safeguarded iteration was presented in [25].

For computing real eigenvalues and corresponding eigenvectors of the viscoelastic eigenvalue problem the safeguarded iteration requires the following tasks: Assume that we are given an initial or the current vector $\mathbf{u}$ of the eigenvector when computing the i largest eigenvalue $\lambda_i^{(j)}$ in the interval I_j. First we evaluate the Rayleigh functional $p(\mathbf{u})$ solving the real equation $f(\lambda;\mathbf{u}) := \mathbf{u}^T\mathbf{T}(\lambda)\mathbf{u} = 0$. Multiplying $f(\lambda;\mathbf{u})$ by all appearing denominators one obtains a polynomial of degree at most $2+n$, and since n is usually quite small all its roots can be determined easily, and its unique root $p(\mathbf{u})$ in I_j can be determined (alternatively, Newton's method can be applied easily to $f(\lambda;\mathbf{u}) = 0$. However, if the wanted root $p(\mathbf{u})$ is close to the boundary of I_j it may happen, that the initial value is not in the zone of attraction of Newton's method. In this case we can make use of Theorem 2.2 and bisection to improve the initial guess and enforce convergence). Once the approximation $p(\mathbf{u})$ to the wanted eigenvalue has been found the linear eigenvalue problem $T(p(\mathbf{u}))\mathbf{v} = \kappa\mathbf{v}$ is solved , and the eigenvector corresponding to the jth smallest eigenvalue κ is chosen as new approximation $\mathbf{u}$ to the wanted eigenvector.

In order to show the reliability and efficiency of our method, we implement it for some realistic physical examples.

EXAMPLE 4.2. The following example is taken from Adhikari and Wagner [5]. Let $\mu = [1,5]$, and

$$\mathbf{M} = 3\mathbf{I},\ \mathbf{K} = 2\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},\ \mathbf{C}_1 = 0.6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{C}_2 = 0.2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then $r_1 := \operatorname{rank}(\mathbf{C}_1) = 2$ implies that there are at least two real eigenvalues in $I_1 = (-1,0)$, and Sylvester's law implies that I_1 contains exactly two eigenvalues in I_1 and another one in $I_2 = (-5,-1)$.

Determining initial values for the eigenvalues by the first order approximation in Lazaro [14] one obtains the following eigenvalue approximations by the safeguarded iteration in I_1 (the relative errors indicate the quadratic convergence)

j	λ_j	rel. error
0	-0.8850009423012485	2.33 E-2
1	-0.8650016393893549	1.71 E-4
2	-0.8648535236454781	8.53 E-9
3	-0.8648535162652167	

and

j	λ_j	rel. error
0	-0.9353368996719444	3.11 E-3
1	-0.9324383219856075	5.37 E-6
2	-0.9324333103801842	1.63 E-11
3	-0.9324333103649718	

Notice, that we did not initialize the iterations by approximations to eigenvectors but to the eigenvalues. Therefore, the monotone convergence to $\lambda_1^{(1)}$ in the table on the left holds only after the first iterate λ_1.

For the eigenvalue in I_2 we obtain

j	λ_j	rel. error
0	-4.890528934075012	3.31 E-3
1	-4.874418912730874	1.67 E-7
2	-4.874418097646541	5.46 E-16
3	-4.874418097646538	

EXAMPLE 4.3. Usually, non–state–space approaches for viscoelastic problems are based on some restrictive physical assumptions. For instance, the approach that has been developed in [14] is based on linearization of $T(\lambda)$ at some μ_j, and it is only applicable for lightly damped systems where the real eigenvalues can be shown to be in proximities of the poles μ_j.

Adhikari and Pascual [2, 3] transform $T(\cdot)$ to modal coordinates of the undamped system $KX = MX\Omega^2$, i.e. $\tilde{T} = I + \lambda \sum_{j=1}^{m} \frac{\mu_j}{\lambda+\mu_j} X^T C_j X + \Omega^2$, and then they neglect the off–diagonal entries of $X^T C_j X$. It is assumed that the non–proportionality is small, i.e. that the off-diagonal entries of $\tilde{T}$ are small compared to its diagonal entries.

This example demonstrates that the safeguarded iteration succeeds also in cases when such methods fail regarding their restrictive physical assumptions.

Again we consider the matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}_1$ and μ of the last example but we replace $\mathbf{C}_2$ with

$$\mathbf{C}_2 = 5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

The safeguarded iteration converges globally to the largest eigenvalue in $I_1 = (-1, 0)$. Choosing the initial vector at random we obtain the following iteration history

j	λ_j	rel. error
0	-0.5745285588902163	1.28E-01
1	-0.5120406384543340	5.66E-03
2	-0.5091675848745895	1.78E-05
3	-0.5091585213016412	1.80E-10
4	-0.5091585212098354	1.09E-15

Continuing with the eigenvector u corresponding to the second eigenvalue of $T(\lambda_1)$, $\lambda_1 = -0.5091585212098354$, and aiming at a second eigenvalue in $I_1 = (-1, 0)$ one gets

j	λ_j	rel. error
0	-0.9713917624952569	5.81E-02
1	-0.9214037635827200	3.60E-03
2	-0.9181304185737342	3.77E-05
3	-0.9180957791399321	4.40E-09
4	-0.9180957751045256	7.26E-16

Proceeding with the eigenvector corresponding to the smallest eigenvalue of $T(\lambda_2)$, $\lambda_2 = -0.9180957751045256$, and aiming at a first eigenvalue (in the sense of Section 2) in $I_2 = (-5, -1)$ one obtains

j	λ_j	rel. error
0	-4.697038664163345	3.59E+00
1	-1.163682728784905	1.38E-01
2	-1.054801858216535	3.14E-02
3	-1.032346569236416	9.45E-03
4	-1.024628584720732	1.90E-03
5	-1.022809231521676	1.21E-04
6	-1.022685713203887	5.69E-07
7	-1.022685131138450	1.26E-11
8	-1.022685131125515	

So the safeguarded iteration is able to determine all eigenvalues safely.

Lazaro's [14] first order approximations to eigenvalues close to the pole $\mu_1 = -1$ are -0.92239 and -1.0262, and -4.8800 to an eigenvalue close to the pole $\mu_2 = -5$.

The method of Adhikari and Pascual [2] yields the following approximations to eigenvalues

$$-0.8006,\ -0.8787,\ -0.9256,\ -1.4140,\ -4.0570, \text{and}\ -4.9292.$$

EXAMPLE 4.4. We consider a four-degree-of-freedom non-viscously damped system considered in [18] and [24] which has the following form

$$\mathbf{M} = 100\,\mathrm{diag}[1,2,3,4], \quad \mathbf{K} = 50000\,\mathbf{C}, \quad \mathbf{C}_1 = \mathbf{C}_2 = 50\,\mathbf{C}, \quad \mu = [50, 70]$$

where

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Since $\mathbf{C}_1$ and $\mathbf{C}_2$ are both nonsingular the problem has four real eigenvalues in each of the intervals $I_1 := (0, 50)$ and $I_2 := (50, 70)$.

Real eigenvalues can be calculated with high order precision solving the characteristic equation $det(\mathbf{T}(\lambda)) = 0$ by root finding algorithms. This way one gets

	$\lambda^{(1)}$	$\lambda^{(2)}$	
$\lambda_1^{(1)}$	-49.1401549067203	$\lambda_1^{(2)}$	-69.0659868415395
$\lambda_2^{(1)}$	-49.5396268709059	$\lambda_2^{(2)}$	-69.5199921387925
$\lambda_3^{(1)}$	-49.7452598611765	$\lambda_3^{(2)}$	-69.7394903026942
$\lambda_4^{(1)}$	-49.9286161974936	$\lambda_4^{(2)}$	-69.9281794704229

Initializing the safeguarded iteration with a random vector (namely in MATLAB rand('twister',5489); $\mathbf{u}$=randn(4,1)) we obtained the following approximations $\lambda_j^{(1)}$ to the eigenvalues in I_1 and $\lambda_j^{(2)}$ to the ones in I_2 and the corresponding relative errors:

j	$\lambda_j^{(1)}$	rel.err.	$\lambda_j^{(2)}$	rel.err.
1	-49.14015490672037	1.45E-15	-69.06598684153966	2.26E-15
2	-49.53962687090588	4.30E-16	-69.51999213879260	1.43E-15
3	-49.74525986117655	1.00E-15	-69.73949030269418	2.03E-16
4	-49.92861619749364	8.53E-16	-69.92817947042275	2.24E-15

The convergence behavior of safeguarded iteration for the largest eigenvalues in I_1 and I_2 is given in the next table

j	$\lambda_j^{(1)}$	rel.err.	$\lambda_j^{(2)}$	rel.err.
0	-49.63970412945238	6.19E-03	-69.73955417360676	2.70E-03
1	-49.21342383480226	6.11E-04	-69.91403489875896	2.02E-04
2	-49.14111828211603	1.04E-05	-69.92806651050697	1.62E-06
3	-49.14015490672026	3.16E-09	-69.92817946305229	1.05E-10
4	-49.14015490672037	1.45E-15	-69.92817947042275	2.26E-15

The convergence behavior demonstrates the quadratic convergence of the method, and it is typical for all eight real eigenvalues: after three to five iteration steps the full machine accuracy is reached.

Notice that the initial eigenvalue approximation $\lambda_0^{(i)}$, $i = 1, 2$ is already quite accurate although we chose the initial vector at random. The reason is that the value of the Rayleigh functional lies in the interval $[\lambda_1^{(i)}, \lambda_4^{(i)}]$ which contains all eigenvalues, and the relative length $\lambda_1^{(i)} - \lambda_4^{(i)}$ of which is only $1.57E-02$ for $i = 1$ and $1.23E-02$ for $i = 2$.

Singh [24] solved the problem iteratively where in each iteration step $\mathbf{T}(\lambda)$ was approximated by a third order Taylor expansion and the occurring polynomial eigenvalue problem of degree three was solved by linearization. He reports the following approximations to the real eigenvalues:

$$-49.1402,\ -49.7453,\ -49.9286,\ -53.7052,\ -65.4264,\ -69.0660,\ -69.7395,\ -69.9282,$$

requiring 121, 10, 8, 31, 27, 13, 15 and 462 iteration steps. Notice, that in the interval I_1 only 3 approximate eigenvalues were found while Theorem 3.1 guarantees 4 eigenvalues in I_1.

5. Conclusions. The computation of eigenvalues of viscoelastic systems with exponential damping model requires the solution of a nonlinear eigenvalue problem. If the corresponding matrices are symmetric, then its eigenvalues allow for a variational characterization, and the safeguarded iteration applies to compute the eigenvalues very efficiently, even if the problem is strongly vicoelastic or strongly non–proportional. The safeguarded iteration converges quadratically and its convergence is monotone for extreme eigenvalues. Numerical examples demonstrate the efficiency of the approach.

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