A nonlinear eigenvalue optimization problem: Optimal potential functions

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Abstract

In this paper we study the following optimal shape design problem: Given an open connected set $\Omega \subset \mathbb{R}^N$ and a positive number $A \in (0, |\Omega|)$. Find a measurable subset $D \subset \Omega$ with $|D| = A$ such that the minimal eigenvalue of $-\text{div}(\zeta(\lambda, x)\nabla u) + \alpha\chi_D u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$ is as small as possible. This sort of nonlinear eigenvalue problems arises in the study of some quantum dots taking into account an electron effective mass. We establish the existence of a solution, and we determine some qualitative aspects of the optimal configurations. A numerical algorithm is proposed to obtain an approximate description of the optimizer.

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1. Introduction

Let Ω be a bounded, connected, open set in $\mathbb{R}^N$ with smooth boundary. Assume that A is a given positive number, $0 < A < |\Omega|$, where $|.|$ denotes the Lebesgue measure. Given a measurable set $D \subset \Omega$ with $|D| = A$, consider the following nonlinear eigenvalue problem

$$-\text{div}(\zeta(\lambda, x)\nabla u) + \alpha\chi_D u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

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In this paper, λ is the principal eigenvalue or the smallest positive eigenvalue of (1.1) and $u = u(x)$ is a corresponding eigenfunction.

We are interested in the cases that $\zeta(\lambda, x)$ is a nonlinear function of the parameter λ. Indeed, equation (1.1) can be regarded as a nonlinear elliptic eigenvalue problem such that the nonlinearity originates from the nonlinear dependence on the eigenparameter.

Such nonlinear eigenvalue problems appear as the Hamiltonian equation governing some quantum dot nanostructures, where $\zeta(\lambda, x)$ corresponds to the effective mass of the carrier (electron or hole) and the surrounding matrix, $\alpha\chi_D$ is the potential function, λ is the ground state energy and u is the wave function [5, 9, 20, 25]. A real physical phenomenon modeled by equation (1.1) is the heterostructures of different semiconductors where the electron effective mass depends on both the energy and position [5, 9].

It is known that the ground state energy of (1.1) depends on the set D, the region with potential α, and we use the notation $\lambda(D)$ as we want to emphasize this dependence. To determine the system's potential which gives the minimum ground state energy, we consider the following optimization problem

$$\inf_{\substack{D\subset\Omega \\ |D|=A}} \lambda(D). \tag{1.2}$$

Let us recall here that nonlinear eigenvalue problems and optimization problems have many applications in engineering and applied sciences and these problems have been intensively attractive to mathematicians in the past decades [17]. However, it should be mentioned that the majority of the investigated nonlinear models are nonlinear in their differential operator part [10, 11, 12, 13]. We note that equation (1.1) has nonlinear dependence on the parameter λ and such systems have been under less attention in this field of study [4, 21, 22].

Taking advantage of a variational characterization of the eigenvalues of a nonlinear eigenvalue problem [26, 28], we derive in Section 2 the existence of an optimal ground state energy under certain conditions on the function ζ. Next we prove in Section 3 qualitative properties of the optimal shape $\hat{D}$. Namely, the optimal set contains a tubular neighborhood of the boundary $\partial\Omega$, and if Ω is simply connected and α is sufficiently small, then $\hat{D}$ is connected. For the special case of a ball Ω centered at the origin we verify under symmetry conditions on ζ that $\Omega \setminus \hat{D}$ is also a ball centered at the origin. Section 4 is devoted to the numerical solution of the nonlinear eigenvalue and the shape optimization problem. We propose a numerical method for the solution of the nonlinear eigenvalue problem.

The optimal configuration $\hat{D}$ is determined by a gradient type method. For this purpose we derive a formula for the shape derivative of the eigenvalue. The paper closes with some numerical examples and concluding remarks.

2. Existence result for optimization problem (1.2)

This section is devoted to prove the existence of a solution of problem (1.2). We take advantage of a variational characterization of the ground state energy which follows immediately from a generalization of the minmax characterization of the eigenvalues of Poincaré to eigenvalue problems depending nonlinearly on the eigenparameter given in [26, 27, 28].

A variational formula is derived for a more general equation where instead of $\alpha\chi_D$ the potential function is $v(x) \geq 0$ in $L^\infty(\Omega)$. Multiplying (1.1) by $\varphi \in H_0^1(\Omega)$ and integrating by parts, one gets the following variational formulation of (1.1): Find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$, $u \neq 0$ such that

$$a(\lambda, u, \varphi) := \int_\Omega \zeta(\lambda, x)\nabla u \cdot \nabla\varphi dx + \int_\Omega vu\varphi dx = \lambda \int_\Omega u\varphi dx =: \lambda b(u, \varphi), \quad (2.1)$$

for all φ in $H_0^1(\Omega)$.

We assume that for every $\lambda \geq 0$

$$\zeta(\lambda, \cdot) \in C(\bar{\Omega}) \quad \text{and} \quad \zeta(\lambda, .) \geq \theta_\lambda > 0. \quad (2.2)$$

Then the bilinear form $a(\lambda, \cdot, \cdot)$ is $H_0^1(\Omega)$-elliptic, continuous and symmetric. Further, $b(\cdot, \cdot)$ is a symmetric, completely continuous and positive definite bilinear form on $H_0^1(\Omega)$.

We further require that

$$\zeta(\cdot, x) \text{ is a continuous function} \quad (2.3)$$

such that a is uniformly continuous,

$$\zeta(0, \cdot) > 0, \quad \text{a.e. in } \Omega. \quad (2.4)$$

and that $\zeta(\lambda, x)$ is a decreasing function with respect to λ for every $x \in \Omega$, i.e.

$$\zeta(\lambda_1, \cdot) \geq \zeta(\lambda_2, \cdot) \text{ a.e. in } \Omega \text{ for } \lambda_1, \lambda_2 \geq 0 \text{ with } \lambda_1 < \lambda_2. \quad (2.5)$$

Then for fixed $u \in H_0^1(\Omega)$, $u \neq 0$ the real equation

$$f(\lambda, u) := \lambda b(u, u) - a(\lambda, u, u) = 0$$

has a unique solution $\mathcal{P}(u) > 0$, which we call the Rayleigh functional of problem (2.1) (notice that for the linear eigenvalue problem $a(u, \varphi) = \lambda b(u, \varphi)$ this is just the Rayleigh quotient), and the minmax characterization in [28] applies, i.e. the variational eigenvalue problem 2.1 has a countable number of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and

$$\lambda_j = \min_{dim V = j} \max_{u \in V, u \neq 0} \mathcal{P}(u), \quad j = 1, 2, \ldots.$$

In particular it holds:

Theorem 2.1. *Suppose that conditions* (2.2)*,* (2.3)*,* (2.4) *and* (2.5) *hold. Then the principal eigenvalue of* (1.1) *allows for a variational formulation*

$$\lambda = \min_{\substack{w \in H_0^1(\Omega) \\ \|w\|_{L^2(\Omega)} = 1}} \mathcal{P}_v(w) = \int_\Omega \zeta(\lambda, x) \|\nabla u\|^2 dx + \int_\Omega v u^2 dx, \tag{2.6}$$

with u as the associated eigenfunction.

We gain some insight into the eigenfunction of problem (1.1) associated with the principal eigenvalue from the following lemma.

Lemma 2.2. *Let u be an eigenfunction corresponding to the first eigenvalue of* (1.1) *then*

(i) $u \in H_0^1(\Omega) \cap H^2(\Omega) \cap C^{1,\delta}(\overline{\Omega})$ *for some* $\delta \in (0, 1)$*,*

(ii) $u > 0$ *in* Ω*,*

(iii) *u is unique up to a constant factor,*

Proof. (i) follows from standard regularity results for elliptic partial differential equations, see [14].

(ii) In view of (2.6), we can regard $|u|$ as an eigenfunction. Applying Harnack's inequality [14], leads us to the fact that eigenfunctions associated to λ have a constant sign.

(iii) Let $\widetilde{u}$ be an eigenfunction of (1.1) corresponding to λ. According to part (ii), we have $\int_\Omega \widetilde{u} dx > 0$ and so there exists a real constant τ such that $\int_\Omega u - \tau \widetilde{u} dx = 0$. But since $u - \tau \widetilde{u}$ is also a solution of (1.1) associated to the principal eigenvalue λ and $\int_\Omega u - \tau \widetilde{u} dx = 0$, one arrives at $u \equiv \tau \widetilde{u}$.

□

Having the variational formulation (2.6) for the first eigenvalue, we devote the rest of this section to demonstrate that the optimization problem (1.2) has a solution.

Theorem 2.3. *Assume that the conditions* (2.2)*,* (2.3)*,* (2.4) *and* (2.5) *are satisfied. Then, the minimization problem* (1.2) *is solvable, i.e. there exists* $\hat{D} \subset \Omega$ *with* $|\hat{D}| = A$*, such that*

$$\hat{\lambda} = \lambda(\hat{D}) = \inf_{\substack{D\subset\Omega\\ |D|=A}} \lambda(D).$$

Proof. In view of Theorem 2.1, there exists a real number $\hat{\lambda} \geq 0$ and a decreasing sequence $\lambda_k = \lambda(D_k)$ such that

$$\hat{\lambda} = \inf_{\substack{D\subset\Omega\\ |D|=A}} \lambda(D) = \lim_{k\to\infty} \lambda(D_k) = \lim_{k\to\infty} \int_\Omega \zeta(\lambda_k, x)\|\nabla u_k\|^2 dx + \alpha \int_\Omega \chi_{D_k} u_k^2 dx,$$

where u_k is the positive eigenfunction corresponding to λ_k normalized such that $\|u_k\|_{L^2(\Omega)} = 1$. We see that the sequence $\{\chi_{D_k}\}_1^\infty$ is bounded in $L^\infty(\Omega)$. Hence there is a subsequence (still denoted by $\{\chi_{D_k}\}_1^\infty$) converging to $0 \leq \eta \leq 1$ in $L^\infty(\Omega)$ with respect to the weak star topology. Recall that $\int_\Omega \eta dx = A$. Regarding (2.2), (2.5) and (2.6), we have

$$\lambda_k \geq \int_\Omega \zeta(\lambda_k, x)\|\nabla u_k\|^2 dx \geq \int_\Omega \zeta(\lambda_1, x)\|\nabla u_k\|^2 dx \geq \theta_{\lambda_1} \int_\Omega \|\nabla u_k\|^2 dx,$$

which leads us to the fact that $\{u_k\}_1^\infty$ is a bounded sequence in $H_0^1(\Omega)$. Consequently, there is a subsequence (still denoted by $\{u_k\}_1^\infty$) converging weakly to $\hat{u}$ in $H_0^1(\Omega)$. The compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ (see [14]) yields that $\{u_k\}_1^\infty$ converges strongly to $\hat{u}$ in $L^2(\Omega)$.

In summary, we have

$$\chi_{D_k} \rightharpoonup \eta \quad \text{in} \quad L^\infty(\Omega), \tag{2.7}$$

$$u_k \rightharpoonup \hat{u}, \quad \text{in} \quad H_0^1(\Omega), \qquad u_k \to \hat{u} \quad \text{in} \quad L^2(\Omega). \tag{2.8}$$

Employing (2.7) and (2.8) it is straightforward to check that for all $\phi \in H_0^1(\Omega)$ we have

$$\int_\Omega \chi_{D_k} u_k \phi dx \to \int_\Omega \eta \hat{u} \phi dx, \qquad \int_\Omega u_k \phi dx \to \int_\Omega \hat{u} \phi dx. \tag{2.9}$$

Moreover,

$$\begin{aligned}
&\left|\int_\Omega \zeta(\lambda_k, x)\nabla u_k \nabla\phi - \zeta(\hat{\lambda}, x)\nabla\hat{u}\ \nabla\phi dx\right| \leq \\
&\left|\int_\Omega (\zeta(\lambda_k, x) - \zeta(\hat{\lambda}, x))\nabla u_k \nabla\phi dx\right| + \left|\int_\Omega (\nabla u_k - \nabla\hat{u})\nabla\phi\zeta(\hat{\lambda}, x)dx\right| \leq \\
&\|\zeta(\lambda_k, x) - \zeta(\hat{\lambda}, x)\|_{L^\infty(\Omega)} \int_\Omega \|\nabla u_k \nabla\phi\| dx + \left|\int_\Omega (\nabla u_k - \nabla\hat{u})\nabla\phi\zeta(\hat{\lambda}, x)dx\right|,
\end{aligned} \tag{2.10}$$

where the expression on the right-hand side of the last inequality converge to zero when $k \to \infty$ by (2.3) and (2.8). Using (2.1), (2.9) and (2.10), we conclude that for all $\phi \in H_0^1(\Omega)$ we have

$$\int_\Omega \zeta(\hat{\lambda}, x)\nabla\hat{u}\ \nabla\phi dx + \alpha \int_\Omega \eta\hat{u}\phi dx = \hat{\lambda} \int_\Omega \hat{u}\phi dx,$$

as $k \to \infty$. Denoting the functional corresponding to a set D in Theorem 2.1 by $\mathcal{P}_D$, we observe that

$$\begin{aligned}
\hat{\lambda} &= \int_\Omega \zeta(\hat{\lambda}, x)\|\nabla\ \hat{u}\|^2 dx + \alpha \int_\Omega \eta\hat{u}^2 dx = \inf_{\substack{D\subset\Omega \\ |D|=A}} \lambda(D) = \inf_{\substack{D\subset\Omega \\ |D|=A}} \min_{\substack{w\in H_0^1(\Omega) \\ \|w\|_{L^2(\Omega)}=1}} \mathcal{P}_D(w) \\
&= \inf_{\substack{D\subset\Omega \\ |D|=A}} \min_{\substack{w\in H_0^1(\Omega) \\ \|w\|_{L^2(\Omega)}=1}} \int_\Omega \zeta(\mathcal{P}_D(w), x)\|\nabla w\|^2 dx + \alpha \int_\Omega \chi_D w^2 dx \\
&\leq \int_\Omega \zeta(\mathcal{P}_D(\hat{u}), x)\|\nabla\hat{u}\|^2 dx + \alpha \int_\Omega \chi_D \hat{u}^2 dx \\
&\leq \int_\Omega \zeta(\hat{\lambda}, x)\|\nabla\hat{u}\|^2 dx + \alpha \int_\Omega \chi_D \hat{u}^2 dx,
\end{aligned}$$

invoking (2.6) and (2.5). This leads us to the fact that

$$\int_\Omega \eta\hat{u}^2 dx \leq \int_\Omega \chi_D \hat{u}^2 dx, \tag{2.11}$$

for all sets $D \subset \Omega,\ |D| = A$.

Now the bathtube principle, [19], yields that there exists a set $\hat{D} \subset \Omega,\ |\hat{D}| = A$ such that

$$\int_\Omega \chi_{\hat{D}} \hat{u}^2 dx \le \int_\Omega \eta \hat{u}^2 dx. \tag{2.12}$$

Regarding the fact that

$$\hat{\lambda} = \inf_{\substack{D\subset\Omega \\ |D|=A}} \lambda(D) = \inf_{\substack{D\subset\Omega \\ |D|=A}} \min_{\substack{w\in H_0^1(\Omega) \\ \|w\|_{L^2(\Omega)}=1}} \mathcal{P}_D(w),$$

we see that $\hat{\lambda} \le \mathcal{P}_{\hat{D}}(\hat{u})$. Then, in view of (2.12) and (2.5) it is observed that

$$\begin{aligned} \hat{\lambda} = & \int_\Omega \zeta(\hat{\lambda}, x)\|\nabla \hat{u}\|^2 dx + \alpha \int_\Omega \eta \hat{u}^2 dx \ge \int_\Omega \zeta(\mathcal{P}_{\hat{D}}(\hat{u}), x)\|\nabla \hat{u}\|^2 dx + \alpha \int_\Omega \chi_{\hat{D}} \hat{u}^2 dx \\ = & \mathcal{P}_{\hat{D}}(\hat{u}) \ge \lambda(\hat{D}) \ge \hat{\lambda}, \end{aligned}$$

which yields that $\hat{\lambda} = \lambda(\hat{D})$. This completes the proof. □

3. Qualitative properties of the optimal shape

In this section we study some qualitative properties of solutions of the optimization problem (1.2). We show that the optimal set $\hat{D}$ contains a tubular neighborhood of the boundary $\partial\Omega$ and particularly, if Ω is simply connected and α is less than a certain value, then $\hat{D}$ is connected. These results extend the result of [8] to the nonlinear eigenvalue problem (1.1). We first consider a general region Ω before we show that for a ball Ω centered at the origin under certain symmetry conditions on ζ the optimal set $\hat{D}$ is a spherical shell region.

3.1. General region Ω

In this section we use the notation $\hat{\lambda}(\alpha)$ when we want to emphasize the dependence of the optimal energy on α. Moreover, $\mathcal{P}_D^\alpha$ is used in order to accent the dependence of the Rayleigh functional on D and α.

The next theorem reveals that the optimal domain $\hat{D}$ contains a tubular neighborhood of the boundary $\partial\Omega$.

Theorem 3.1. *Let $\lambda(\hat{D})$ be an optimal energy and $\hat{u}$ an associated eigenfunction. There is a number $t \ge 0$ such that*

$$\hat{D} = \{x \in \Omega : \ \hat{u}(x) \le t\}, \tag{3.1}$$

where

$$t = \sup\{s \in \mathbb{R} : |\{x \in \Omega : \ \hat{u}(x) \le s\}| \le A\}. \tag{3.2}$$

Proof. From (2.6) we obtain

$$\begin{aligned}\hat{\lambda} = \quad & \lambda(\hat{D}) = \int_\Omega \zeta(\hat{\lambda}, x)\|\nabla \hat{u}\|^2 dx + \alpha \int_\Omega \chi_{\hat{D}} \hat{u}^2 dx = \inf_{\substack{D\subset\Omega \\ |D|=A}} \min_{\substack{w\in H_0^1(\Omega) \\ \|w\|_{L^2(\Omega)}=1}} \mathcal{P}_D(w) \leq \mathcal{P}_D(\hat{u}) = \\ = \quad & \int_\Omega \zeta(\mathcal{P}_D(\hat{u}), x)\|\nabla \hat{u}\|^2 dx + \alpha \int_\Omega \chi_D \hat{u}^2 dx.\end{aligned} \tag{3.3}$$

Since $\hat{\lambda} \leq \mathcal{P}_D(\hat{u})$, then $\zeta(\hat{\lambda}, x) \geq \zeta(\mathcal{P}_D(\hat{u}), x)$ by (2.5) and so we have

$$\int_\Omega \zeta(\hat{\lambda}, x)\|\nabla \hat{u}\|^2 dx \geq \int_\Omega \zeta(\mathcal{P}_D(\hat{u}), x)\|\nabla \hat{u}\|^2 dx. \tag{3.4}$$

Employing (3.3) and (3.4), we obtain

$$\int_\Omega \chi_{\hat{D}} \hat{u}^2 dx \leq \int_\Omega \chi_D \hat{u}^2 dx,$$

for every $D \subset \Omega$ with $|D| = A$. Now, from bathtub principle, [19], it follows that

$$\{x \in \Omega : \ \hat{u}(x) < t\} \subset \hat{D} \subseteq \{x \in \Omega : \ \hat{u}(x) \leq t\}, \tag{3.5}$$

where

$$t = \sup\{s \in \mathbb{R} : |\{x \in \Omega : \ \hat{u}(x) \leq s\}| \leq A\}.$$

Setting $B = \{x \in \Omega : \ \hat{u}(x) = t\} \cap \hat{D}^c$, we observe that

$$-\mathrm{div}(\zeta(\hat{\lambda}, x)\nabla \hat{u}) = 0,$$

almost everywhere on B regarding Lemma 7.7 from [14]. Hence,

$$(\hat{\lambda} - \alpha\chi_{\hat{D}}\hat{u}) = \hat{\lambda}\hat{u} = 0,$$

almost everywhere on B. Recalling that $\hat{u} > 0$ and $\hat{\lambda} > 0$, we see that B has measure zero and consequently

$$\hat{D} = \{x \in \Omega : \ \hat{u}(x) \leq t\}.$$

□

Next we prove that for every simply connected Ω and sufficiently small α the optimal set $\hat{\Omega}$ is simply connected as well. To this end we denote by μ the principal eigenvalue of (1.1) with $\alpha = 0$:

$$-\operatorname{div}(\zeta(\mu, x)\nabla u) = \mu u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.6}$$

and by ψ the corresponding positive L^2-normalized eigenfunction.

Lemma 3.2. *The function*

$$g(\alpha) = \hat{\lambda}(\alpha) - \alpha$$

is continuous and decreasing, and it holds that

$$\lim_{\alpha\to+\infty} g(\alpha) = -\infty.$$

Proof. For $j = 1, 2$ let $\hat{\lambda}(\alpha_j)$ be the minimizer of problem (1.2), let $\hat{D}_j$ be the corresponding set according to Theorem 2.3 and u_j a corresponding normalized eigenfunction. In view of the definition of Rayleigh functional, it is straightforward to check that

$$\mathcal{P}_D^{\alpha_1}(u) < \mathcal{P}_D^{\alpha_2}(u) \quad \text{whenever} \quad \alpha_1 < \alpha_2, \tag{3.7}$$

and consequently $\hat{\lambda}(\alpha_1) \le \hat{\lambda}(\alpha_2)$.

Let $\alpha_2 \le \alpha_1$, then it follows from (2.5), (2.6), (3.7) and Theorem 2.3

$$\begin{aligned}
\hat{\lambda}(\alpha_1) &\le \mathcal{P}_{\hat{D}_2}^{\alpha_1}(u_2) \\
&= \int_\Omega \zeta(\mathcal{P}_{\hat{D}_2}^{\alpha_1}(u_2), x)\|\nabla u_2\|^2 dx + \alpha_2 \int_\Omega \chi_{\hat{D}_2} u_2^2 dx - \alpha_2 \int_\Omega \chi_{\hat{D}_2} u_2^2 dx + \alpha_1 \int_\Omega \chi_{\hat{D}_2} u_2^2 dx \\
&\le \int_\Omega \zeta(\mathcal{P}_{\hat{D}_2}^{\alpha_2}(u_2), x)\|\nabla u_2\|^2 dx + \alpha_2 \int_\Omega \chi_{\hat{D}_2} u_2^2 dx + (\alpha_1 - \alpha_2) \int_\Omega \chi_{\hat{D}_2} u_2^2 dx.
\end{aligned}$$

Hence,

$$0 \le \hat{\lambda}(\alpha_1) - \hat{\lambda}(\alpha_2) \le (\alpha_1 - \alpha_2) \int_\Omega \chi_{\hat{D}_2} u_2^2 dx, \tag{3.8}$$

and likewise

$$0 \le \hat{\lambda}(\alpha_2) - \hat{\lambda}(\alpha_1) \le (\alpha_2 - \alpha_1) \int_\Omega \chi_{\hat{D}_1} u_1^2 dx, \tag{3.9}$$

when $\alpha_1 < \alpha_2$. Thus

$$|\hat{\lambda}(\alpha_1) - \hat{\lambda}(\alpha_2)| \le |\alpha_1 - \alpha_2| \max\left(\int_\Omega \chi_{\hat{D}_1} u_1^2 dx, \int_\Omega \chi_{\hat{D}_2} u_2^2 dx \right),$$

which proves the continuity of g.

Next we show that g is decreasing. Since $\int_\Omega \chi_{\hat{D}_2} u_2^2 dx \leq 1$, it follows from (3.8) for $\alpha_1 \geq \alpha_2$

$$\hat{\lambda}(\alpha_1) - \hat{\lambda}(\alpha_2) \leq (\alpha_1 - \alpha_2) \int_\Omega \chi_{\hat{D}_2} u_2^2 dx \leq \alpha_1 - \alpha_2,$$

and therefore

$$g(\alpha_1) = \hat{\lambda}(\alpha_1) - \alpha_1 \leq \hat{\lambda}(\alpha_2) - \alpha_2 = g(\alpha_2).$$

Finally we prove $g(\alpha)$ tends to $-\infty$ as $\alpha \to +\infty$. From the definition of the Rayleigh functional and (2.5), we obtain

$$\begin{aligned} \hat{\lambda}(\alpha) - \mu \quad &\leq \quad \mathcal{P}^\alpha_{\hat{D}}(\psi) - \mu = \int_\Omega \zeta(\mathcal{P}^\alpha_{\hat{D}}(\psi), x) \|\nabla \psi\|^2 dx + \alpha \int_\Omega \chi_{\hat{D}} \psi^2 dx - \mu \\ &\leq \quad \int_\Omega \zeta(\mu, x) \|\nabla \psi\|^2 dx + \alpha \int_\Omega \chi_{\hat{D}} \psi^2 dx - \mu = \alpha \int_\Omega \chi_{\hat{D}} \psi^2 dx, \end{aligned}$$

invoking the fact that $\mu < \mathcal{P}^\alpha_{\hat{D}}(\psi)$. This yields that $\hat{\lambda}(\alpha) - \mu \leq C\alpha$ where $C < 1$ or

$$\hat{\lambda}(\alpha) - \alpha \leq \mu + (C - 1)\alpha,$$

which gives the assertion. $\square$

Recall that $g(0) = \mu > 0$ and then Lemma 3.2 yields the following corollary.

Corollary 3.3. *There is a unique constant $\bar{\alpha}$ where $\hat{\lambda}(\bar{\alpha}) = \bar{\alpha}$ and $\hat{\lambda}(\alpha) - \alpha > 0$ when $\alpha < \bar{\alpha}$ and $\hat{\lambda}(\alpha) - \alpha < 0$ when $\alpha > \bar{\alpha}$.*

Now, we state the main result of this section.

Theorem 3.4. *Let $\alpha < \bar{\alpha}$, then every connected component B of the interior of $\hat{D}$ touches the boundary of Ω, i.e. $\bar{B} \cap \partial\Omega \neq \emptyset$. Particularly, if Ω is simply connected and $\alpha < \bar{\alpha}$, then $\hat{D}$ is connected.*

Proof. Let us assume that this is false. Then there is an open connected set $B \subseteq \hat{D} = \{x \in \Omega : \hat{u}(x) \leq t\}$ such that $\partial B \subset \{x \in \Omega : \hat{u}(x) \geq t\}$. The function $\hat{u}$ attains a minimum on $\bar{B}$ and so the minimum should be attained in an interior point $x_0 \in B$. In view of Corollary 3.3, $\alpha < \bar{\alpha}$ yields that $\alpha - \hat{\lambda}(\alpha) < 0$ and hence

$$\operatorname{div}(\zeta(\hat{\lambda}, x)\nabla \hat{u}) = (\alpha - \hat{\lambda}(\alpha))\hat{u} < 0, \tag{3.10}$$

on B.

Employing the strong maximum principle (Theorem 8.19 of [14]), we conclude that $\hat{u}$ is constant in B. This contradicts (3.10) in view of Lemma 7.7 of [14]. Consequently, every connected component B of the interior of $\hat{D}$ touches the boundary and this leads us to the assertions of the theorem.

□

3.2. Optimal shape design for a ball

In this subsection we determine analytically the optimal solution of (1.2) when Ω is a ball centered at the origin in $\mathbb{R}^n$.

Our aim is to show that under certain symmetry conditions on $\zeta(\lambda,\cdot)$ the minimizer of (1.2) is an annulus with $\partial\Omega$ as a subset of its boundary.

For $a_\lambda(x) := \sqrt{\zeta(\lambda,x)}$ we assume that

$$a_\lambda \text{ is radially symmetric, i.e. } a_\lambda(x) = a_\lambda(r) \text{ where } r = \|x\|, \tag{3.11}$$

$$a_\lambda(r) \text{ is nondecreasing,} \tag{3.12}$$

and

$$(a_\lambda(r^{\frac{1}{n}}) - a_\lambda(0))r^{1-\frac{1}{n}},\ r \geq 0, \text{is convex.} \tag{3.13}$$

Note that (3.13) is satisfied, if $a_\lambda(r)$, $r \geq 0$, is nondecreasing and convex [7]. As an example, $a_\lambda(\|x\|) = \sqrt{\zeta(\lambda,x)} = (1 + e^{-\lambda}\|x\|)$ satisfies (3.11), (3.12), and (3.13).

Let us here recall definitions of the Schwarz decreasing and increasing rearrangements of a given function. If $f : \Omega \to \mathbb{R}$ is a Lebesgue measurable function then we denote by $f^* : \Omega \to \mathbb{R}$ and $f_* : \Omega \to \mathbb{R}$ the Schwarz decreasing and increasing rearrangements of f, respectively. It means that, f^* and f_* are rearrangements of f such that f^* is a radial decreasing function, whereas f_* is a radial increasing function. Next we state some well known rearrangement inequalities.

Lemma 3.5. *Suppose that Ω is a ball centered at the origin in $\mathbb{R}^n$. Then*

$$\int_\Omega f^* g_* dx \leq \int_\Omega fg dx \leq \int_\Omega f^* g^* dx,$$

where f and g are non-negative measurable functions.

Proof. See [15]. □

The main result of this subsection reads as follows.

Theorem 3.6. *Let* $\Omega = \{x \in \mathbb{R}^n : \|x\| < r_1\}$ *be the ball, and and assume that* $a_\lambda(x) = \sqrt{\zeta(\lambda, x)}$ *satisfies* (3.11), (3.12) *and* (3.13). *Then there is an optimal solution of* (1.2) *which is a spherical shell region*

$$D^* = \{x \in \Omega : r_2 < \|x\| < r_1\}.$$

Proof. Assume that D is a solution of (1.2) associated with u and $\lambda(D)$. Using Theorem 3.1 from [7] and Lemma 3.5, we have

$$\int_\Omega \zeta(\lambda, x)\|\nabla u\|^2 dx + \alpha \int_\Omega \chi_D u^2 dx \geq \int_\Omega \zeta(\lambda, x)\|\nabla u^*\|^2 dx + \alpha \int_\Omega \chi_{D_*} {u^*}^2 dx. \quad (3.14)$$

Hence, from the definition of the Rayleigh functional, we see

$$\lambda(D) \geq \mathcal{P}_{D^*}(u^*) \geq \lambda(D^*),$$

and so D^* is a minimizer. □

4. Numerical Methods

From the physical point of view, it is important to know the shape of the optimal set $\hat{D}$ in case that Ω is not a ball. To this end, there must be a numerical approach to determine the optimal shape design. In this section we will develop a numerical method to solve the shape optimization problem (1.2).

4.1. Numerical solution of the nonlinear eigenvalue problem

First, we describe a numerical method to solve the nonlinear eigenvalue problem (1.1). We assume that D is fixed and denote by Λ_i, the i-th Dirichlet-Laplacian eigenvalue of Ω and a corresponding eigenfunction by ϕ_i, normalized such that $\|\phi_i\|_{L^2(\Omega)} = 1$.

The eigenvalue problem (1.1) is equivalent to the following problem

$$\begin{cases} -\mathrm{div}(\zeta(\lambda, x)\nabla u_{\Omega\backslash D}) = \lambda u_{\Omega\backslash D} & \text{in } \Omega\backslash D \\ -\mathrm{div}(\zeta(\lambda, x)\nabla u_D) + \alpha u_D = \lambda u_D & \text{in } D \\ u_D = u_{\Omega\backslash D} & \text{on } \partial D \\ \frac{\partial u_D}{\partial n} = \frac{\partial u_{\Omega\backslash D}}{\partial n} & \text{on } \partial D \\ u_{\Omega\backslash D} = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

where we denoted by u_D and $u_{\Omega\backslash D}$ the restrictions of an eigenfunction respectively to D and $\Omega\backslash D$ and assumed that n is the unitary normal vector pointing towards the exterior of D.

We will consider the approximation involving Laplacian eigenfunctions of Ω,

$$u_{\Omega\backslash D}(x) \approx \tilde{u}_{\Omega\backslash D}(x) = \sum_{i=1}^{N^{\Omega\backslash D}} \alpha_i \phi_i(x), \tag{4.2}$$

for some $N^{\Omega\backslash D} \in \mathbb{N}$, which by construction satisfy the boundary conditions of the problem (4.1). The approximation of u_D will be made by a Kansa type method of fundamental solutions (cf. [3]). We take a fundamental solution of the Helmholtz equation in $\mathbb{R}^2$,

$$\Phi_\lambda(x) = \frac{i}{4} H_0^{(1)}\left(\sqrt{\lambda}\,\|x\|\right), \tag{4.3}$$

where $H_0^{(1)}$ is a Hankel function of the first kind and $\|.\|$ denotes the Euclidean norm and consider the approximation

$$u_D(x) \approx \tilde{u}_D(x) = \sum_{i=1}^{N^F} \sum_{j=1}^{N^{MFS}} \beta_{i,j} \Phi_{\kappa_i}(x - y_j), \tag{4.4}$$

where N^F is the number of test frequencies κ_i and y_j are N^{MFS} source points, following the distribution described in [1, 2]. Given a sample of points x_i, $i = 1, ..., M$ (almost) uniformly distributed on the boundary ∂D, the source points are given by

$$y_j = x_j + \delta n_j, \tag{4.5}$$

where n_j is the unitary outward normal vector to ∂D at the point x_j and δ is a positive parameter.

The eigenvalue problem (4.1) is solved by imposing the PDEs in D and $\Omega\backslash D$ and the conditions at the interface ∂D. Consider the sets of points

$$X^D = \left\{x_m^D \in D, m = 1, ..., M^D\right\},$$

$$X^{\Omega\backslash D} \left\{x_m^{\Omega\backslash D} \in \Omega\backslash D, m = 1, ..., M^{\Omega\backslash D}\right\}$$

and

$$X^{\partial D} = \left\{x_m^{\partial D} \in \partial D, m = 1, ..., M^{\partial D}\right\}.$$

Note that we have

$$\operatorname{div}(\zeta(\lambda, x)\nabla u) = \nabla_x \zeta(\lambda, x).\nabla u + \zeta(\lambda, x)\Delta u,$$

where

$$\nabla_x \zeta(\lambda, x) := \left(\frac{\partial \zeta(\lambda, x)}{\partial x_1}, ..., \frac{\partial \zeta(\lambda, x)}{\partial x_N} \right).$$

We will impose that $\tilde{u}_{\Omega\backslash D}$ satisfies the PDE at the points $x_m^{\Omega\backslash D}$, thus

$$\sum_{i=1}^{N} \alpha_i \left[-\nabla_x \zeta(\lambda, x_m^{\Omega\backslash D}).\nabla \phi_i(x_m^{\Omega\backslash D}) + \left(\Lambda_i \zeta(\lambda, x_m^{\Omega\backslash D}) - \lambda \right) \phi_i(x_m^{\Omega\backslash D}) \right] = 0, \ m = 1, ..., M^{\Omega\backslash D}. \tag{4.6}$$

For the points x_m^D, we obtain

$$\sum_{i=1}^{N^F} \sum_{j=1}^{N^{MFS}} \beta_{i,j} \left[-\nabla_x \zeta(\lambda, x_m^D).\nabla \Phi_{\kappa_i}(x_m^D - y_j) + \left(\kappa_i \zeta(\lambda, x_m^D) - \lambda + \alpha \right) \Phi_{\kappa_i}(x_m^D - y_j) \right] = 0, \ m = 1, ..., M^D. \tag{4.7}$$

The conditions at the interface are

$$\sum_{i=1}^{N^{\Omega\backslash D}} \alpha_i \phi_i(x_m^{\partial D}) = \sum_{i=1}^{N^F} \sum_{j=1}^{N^{MFS}} \beta_{i,j} \Phi_{\kappa_i}(x_m^{\partial D} - y_j), \ m = 1, ..., M^{\partial D} \tag{4.8}$$

and

$$\sum_{i=1}^{N^{\Omega\backslash D}} \alpha_i \frac{\partial \phi_i(.)}{\partial n}(x_m^{\partial D}) = \sum_{i=1}^{N^F} \sum_{j=1}^{N^{MFS}} \beta_{i,j} \frac{\partial \Phi_{\kappa_i}(. - y_j)}{\partial n}(x_m^{\partial D}), \ m = 1, ..., M^{\partial D}. \tag{4.9}$$

The equations (4.13), (4.7), (4.8) and (4.9) can be written together as a nonlinear matrix eigenvalue problem:

$$\mathbf{A}(\lambda)\mathbf{v} = \mathbf{0}, \tag{4.10}$$

where $\mathbf{v}$ is a vector containing all the coefficients α_i and $\beta_{i,j}$.

The numerical approximations for the eigenvalues are the values of λ for which we have a nontrivial solution of (4.10) and can be calculated by an algorithm involving the generalized singular value decomposition that was studied in [6].

4.2. Shape optimization

Now, we will study the dependence of the smallest eigenvalue λ, which will be denoted by $\lambda(\Omega, D)$, in terms of perturbations of D and Ω. This is a more general setting than we have in the shape optimization problem (1.2), where we assumed that the domain Ω is fixed. By $W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ we will denote the set of bounded

Lipschitz maps from $\mathbb{R}^N$ into itself and for each deformation field V, we consider an application $\Psi(t)$ such that $\Psi : t \in [0, T[\to W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ is differentiable at 0 with $\Psi(0) = I$ and $\Psi'(0) = V$, where I is the identity.

We will use the notation $O_t = \Psi(t)(O)$, for an arbitrary domain O, $\lambda(t) = \lambda(\Omega_t, D_t)$ and $u(t)$ is a corresponding eigenfunction satisfying $\|u\|_{L^2(\Omega_t)} = 1$. By u' we will denote the derivative of $u(t)$ at $t = 0$ and by n the unitary vector that is normal to the boundaries ∂D or $\partial\Omega$ and is oriented towards the exterior of the corresponding domain. Thus, for a sufficiently small $\epsilon > 0$, $x + \epsilon n \in \Omega \backslash \bar{D}$, $\forall x \in \partial D$.

We know that if O is an open set with Lipschitzian boundary and we define

$$J(t) = \int_{O_t} y(t,x)dx,$$

for some C^1 function y, then ([23, 24])

$$J'(0) = \int_O \frac{\partial y}{\partial t}(0,x)dx + \int_{\partial O} y(0,x)V.nds_x. \tag{4.11}$$

From now on we will assume that $\int_\Omega \frac{\partial \zeta}{\partial \lambda}(\lambda(0), x) \|\nabla u\|^2 dx \neq 1$. Thus, we have

Theorem 4.1. *The functional $\lambda(t)$ is differentiable at $t = 0$ and*

$$\lambda'(0) = \frac{-\int_{\partial\Omega} \zeta(\lambda(0), x) \left(\frac{\partial u}{\partial n}\right)^2 V.nds_x + \alpha \int_{\partial D} u^2 V.nds_x}{1 - \int_\Omega \frac{\partial \zeta}{\partial \lambda}(\lambda(0), x) \|\nabla u\|^2 dx}. \tag{4.12}$$

Proof. The eigenfunction is normalized,

$$\int_{\Omega_t} u^2(t)dx = 1$$

and differentiating we obtain

$$\int_\Omega 2uu'dx + \int_{\partial\Omega} u^2 V.nds_x = 0 \Rightarrow \int_\Omega 2uu'dx = -\int_{\partial\Omega} u^2 V.nds_x. \tag{4.13}$$

On the other hand, the eigenvalue $\lambda(t)$ satisfies

$$\int_{\Omega_t} \zeta(\lambda(t), x) \|\nabla u(t)\|^2 dx + \alpha \int_{D_t} u(t)^2 dx = \lambda(t) \int_{\Omega_t} u(t)^2 dx,$$

and differentiating,

$$\int_\Omega \lambda'(0)\frac{\partial\zeta}{\partial\lambda}(\lambda(0),x)\left\|\nabla u\right\|^2 dx+\int_\Omega \zeta(\lambda(0),x)2\nabla u.\nabla u' dx+\int_{\partial\Omega}\zeta(\lambda(0),x)\left\|\nabla u\right\|^2 V.nds_x$$

$$+\alpha\int_D 2uu'dx+\alpha\int_{\partial D}u^2V.nds_x=\lambda'(0)\int_\Omega u^2dx+\lambda(0)\left(\int_\Omega 2uu'dx+\int_{\partial\Omega}u^2V.nds_x\right)=\lambda'(0),$$

using (4.13). Thus,

$$\lambda'(0)=\frac{\int_\Omega \zeta(\lambda(0),x)2\nabla u.\nabla u'dx+\int_{\partial\Omega}\zeta(\lambda(0),x)\left\|\nabla u\right\|^2 V.nds_x+\alpha\int_D 2uu'dx+\alpha\int_{\partial D}u^2V.nds_x}{1-\int_\Omega \frac{\partial\zeta}{\partial\lambda}(\lambda(0),x)\left\|\nabla u\right\|^2 dx}. \tag{4.14}$$

We recall the identity

$$\vec{F}.\nabla\varphi=\nabla.\left(\varphi\vec{F}\right)-\left(\nabla.\vec{F}\right)\varphi \tag{4.15}$$

and we have

$$\int_\Omega \zeta(\lambda(0),x)2\nabla u.\nabla u'dx=2\int_\Omega(\zeta(\lambda(0),x)\nabla u).\nabla u'dx \underset{(4.15)}{=}$$

$$2\int_\Omega \nabla.\left[u'\zeta(\lambda(0),x)\nabla u\right]-2\int_\Omega u'\nabla.\left[\zeta(\lambda(0),x)\nabla u\right] \underset{\substack{\textit{divergence theorem}\\ \textit{and } (1.1)}}{=}$$

$$2\int_{\partial\Omega}u'\zeta(\lambda(0),x)\frac{\partial u}{\partial n}ds_x+2\int_\Omega u'\left(\lambda(0)u-\alpha\chi_D u\right)dx=$$

$$2\int_{\partial\Omega}u'\zeta(\lambda(0),x)\frac{\partial u}{\partial n}ds_x-\lambda(0)\int_{\partial\Omega}u^2V.nds_x-2\alpha\int_D uu'dx,$$

by equation (4.13).

We note that $u'=-\frac{\partial u}{\partial n}V.n$, on $\partial\Omega$ (eg. [16]), and using the boundary conditions of the nonlinear eigenvalue problem we have,

$$\int_\Omega \zeta(\lambda(0),x)2\nabla u.\nabla u'dx=-2\int_{\partial\Omega}\zeta(\lambda(0),x)\left(\frac{\partial u}{\partial n}\right)^2 V.nds_x-2\alpha\int_D uu'dx$$

and using (4.14) we obtain

$$\lambda'(0)=\frac{-2\int_{\partial\Omega}\zeta(\lambda(0),x)\left(\frac{\partial u}{\partial n}\right)^2 V.nds_x+\int_{\partial\Omega}\zeta(\lambda,x)\left\|\nabla u\right\|^2 V.nds_x+\alpha\int_{\partial D}u^2V.nds_x}{1-\int_\Omega \frac{\partial\zeta}{\partial\lambda}(\lambda(0),x)\left\|\nabla u\right\|^2 dx}=$$

$$\frac{-\int_{\partial\Omega}\zeta(\lambda(0),x)\left(\frac{\partial u}{\partial n}\right)^2 V.nds_x + \alpha\int_{\partial D} u^2 V.nds_x}{1-\int_\Omega \frac{\partial\zeta}{\partial\lambda}(\lambda(0),x)\,\|\nabla u\|^2\,dx},$$

using the boundary conditions of the nonlinear eigenvalue problem. □

Remark 4.1. Not9ce that for $\alpha = 0$ and $\zeta \equiv 1$ the formula (4.12) recovers the well known Hadamard formula for the shape derivative of a Dirichlet Laplacian eigenvalue.

In the shape optimization problem (1.2) we assume that Ω is fixed. Thus, on $\partial\Omega$ we have $V.n \equiv 0$ which implies that

$$\lambda'(0) = \frac{\alpha\int_{\partial D} u^2 V.nds_x}{1-\int_\Omega \frac{\partial\zeta}{\partial\lambda}(\lambda(0),x)\,\|\nabla u\|^2\,dx}. \tag{4.16}$$

Moreover, we assume that the volume of D is fixed. To deal with this volume constraint we used the Hadamard derivative for the volume. If we define $\mathcal{V}(t) = |D_t|$, we have

$$\mathcal{V}'(0) = \int_{\partial D} V.nds_x. \tag{4.17}$$

As a consequence it is immediate to obtain an optimality condition which states that the boundary of the optimizer belongs to a level set of the corresponding eigenfunction.

Proposition 4.2. *Assume that D^* is the solution of the shape optimization problem (1.2), with the corresponding eigenvalue λ^*. Then, there exists a unique positive normalized eigenfunction u^* and a constant C satisfying the overdetermined problem*

$$\begin{cases} -\operatorname{div}(\zeta(\lambda,x)\nabla u) + \alpha\chi_D u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u = C & \text{on } \partial D^*. \end{cases} \tag{4.18}$$

Proof. The domain D^* minimizes λ among all domains with the same volume. Thus, there exists a Lagrange multiplier Θ for which

$$\lambda'(0) = \Theta\mathcal{V}'(0),$$

which (according to (4.16) and (4.17)) can be written as

$$\frac{\alpha\int_{\partial D^*} u^2 V.nds_x}{1-\int_\Omega \frac{\partial\zeta}{\partial\lambda}(\lambda(0),x)\,\|\nabla u\|^2\,dx} = \Theta\int_{\partial D^*} V.nds_x$$

which implies that

$$\int_{\partial D^*}\left(\frac{\alpha u^2}{1-\int_\Omega \frac{\partial\zeta}{\partial\lambda}(\lambda(0),x)\,\|\nabla u\|^2\,dx}-\Theta\right)V.nds_x=0,$$

for all perturbation fields $V\in W^{1,\infty}(\mathbb{R}^N)$ and the conclusion follows with

$$C=\sqrt{\frac{\Theta\left(1-\int_\Omega \frac{\partial\zeta}{\partial\lambda}(\lambda(0),x)\,\|\nabla u\|^2\,dx\right)}{\alpha}}.$$

□

Remark 4.2. The Proposition 4.2 could also be derived immediately from Theorem 3.1.

5. Numerical Considerations for Planar Problems

In this section we present some numerical results for different subsets $\Omega\subset\mathbb{R}^2$.

To parameterize the domains for the shape optimization algorithm, we consider the functions

$$\gamma_1(t)=a_0^{(1)}+\sum_{j=1}^{M}a_j^{(1)}\cos(jt)+\sum_{j=1}^{M}b_j^{(1)}\sin(jt)$$

and

$$\gamma_2(t)=a_0^{(2)}+\sum_{j=1}^{M}a_j^{(2)}\cos(jt)+\sum_{j=1}^{M}b_j^{(2)}\sin(jt),$$

for some $M\in\mathbb{N}$ and the vector $C\in\mathbb{R}^{4M+2}$, whose components are the coefficients of these expansions,

$$C=\left(a_0^{(1)},a_1^{(1)},...,a_M^{(1)},b_1^{(1)},...,b_M^{(1)},a_0^{(2)},a_1^{(2)},...,a_M^{(2)},b_1^{(2)},...,b_M^{(2)}\right).$$

The class of planar admissible domains is the set

$$\mathbf{V}=\left\{V\subset\mathbb{R}^2:\partial V=(\gamma_1(t),\gamma_2(t)):t\in[0,2\pi[\text{ is a Jordan curve}\right\}$$

and the shape optimization (1.2) is solved by searching for optimal vectors C corresponding to domains in $\mathbf{V}$. For this purpose, we use a gradient type method to solve a sequence of minimization problems of the functionals

$$J_m(t)=\lambda(t)+\eta_m\left(\mathcal{V}(t)-A\right)^2$$

for an increasing sequence of parameters $0 < \eta_1 < \eta_2 < ....$ Recall that $0 < A < |\Omega|$ is the area of D. The calculation of the shape gradient is straightforward using (4.16) and (4.17).

This approach does not allow to change the topology of the region D during the optimization process. However, in all the numerical experiments, the algorithm allowed to determine the optimal domain $\hat{D}$, provided the initial guess for the optimization and $\hat{D}$ are homeomorphic to each other. In each experiment, we run the optimization algorithm several times, with some different types of topologies and compare the eigenvalues obtained in all the simulations to decide which one is the optimal configuration. Note also that, at least for a small parameter α, a good initial guess is $D = \{x \in \Omega : \psi(x) \leq t\}$ where t is determined using (3.2) and ψ is the solution of (3.6).

5.1. Numerical Examples

In this section we present some numerical results for the solution of the shape optimization (1.2). Typically we use the following parameters $N^{\Omega\backslash D} = 250$, $N^F = 9$, $N^{MFS} = 100$ and $M^{\partial D} = 200$. The gradient type method usually converge to the solution in less than 50 iterations. In the first example Ω is the square $[-1, 1] \times [-1, 1]$, we take $|D| = 2$ and

$$\zeta(\lambda, x) = 2 + e^{-\lambda}\left(x_1^2 + x_2^2\right).$$

Figure 1 shows the contour plots of the eigenfunctions associated with the optimal configurations obtained for $\alpha = 1, 50, 500$. In each picture we marked some contour lines of the eigenfunction and applied a low contrast color scale in the region D that allow to distinguish the regions D and $\Omega\backslash D$. The optimal eigenvalue is also provided in the legend of each figure. We can observe that for large α, the eigenfunction is localized in the region $\Omega\backslash D$.

The second example is a non convex smooth domain, whose boundary can be parameterized by $\{(1 + 1/2\cos(2t) - 1/10\cos(4t))(\cos(t), \sin(t)) : t \in [0, 2\pi[\}$ and we considered

$$\zeta(\lambda, x) = 1 + \frac{1}{2}\cos\left(\frac{x_1 x_2}{\lambda}\right).$$

In this case, a closed form for the eigenfunctions on Ω is not known, and we used the Method of Fundamental Solutions to approximate the first 200 Laplacian eigenfunctions of Ω (as in [1, 2]).

Figure 2 shows some optimal configurations obtained for $\alpha = 1, 10, 50$ and $|D| = 1, 2$. Again we note that for large parameters α, the eigenfunctions are localized in the region $\Omega\backslash D$. Moreover, the boundary of D belongs to some level set

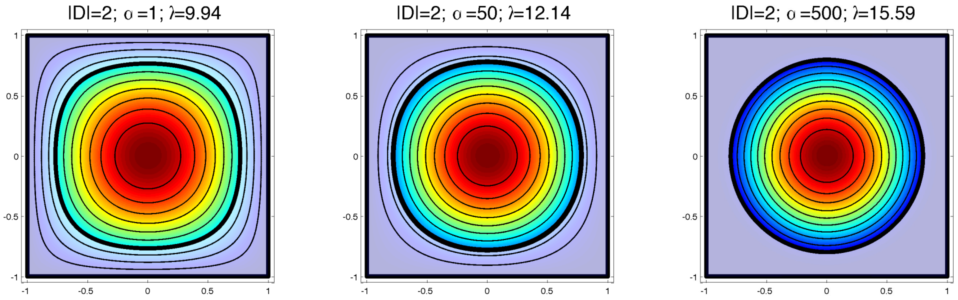

Figure 1: Contour plots of the eigenfunctions associated to the smallest eigenvalue of the optimal configuration, for $\alpha = 1, 50, 50$.

of the eigenfunction, as was proved in Proposition 4.2. We can also observe that in some cases, the optimal configuration does not preserve the two axis of symmetry of the region Ω. This *symmetry breaking phenomenon* was already reported in the composite membrane eigenvalue problem (cf. [8]).

The last example is an annular domain $\Omega = B_2 \backslash B_{1/2}$, where B_R denotes the disk centered at the origin with radius R, and let

$$\zeta(\lambda, x) = 1 + \frac{1}{x_1^2 + x_2^2 + \lambda}.$$

Figure 3 shows the contour plots of the eigenfunctions associated to the optimal configurations obtained for $\alpha = 1, 10, 50$ and $|D| = 4, 10$. Note that in the first case ($|D| = 4$ and $\alpha = 1$), the region D is disconnected, while in the remaining cases it is connected.

6. Conclusions

In this paper we have considered an optimization problem associated with the nonlinear eigenvalue problem (1.1). The nonlinearity originates from the nonlinear dependence on the eigenparameter. Indeed, we are searching for a domain $\hat{D} \subset \Omega$, $|\hat{D}| = A$, which minimizes the ground state $\lambda(D)$.

Taking advantage of the variational characterization of the ground state energy, we have proved that there is an optimal domain $\hat{D}$. Then the qualitative properties of the optimal domain have been addressed and it has been established that this set contains a tubular neighborhood of the boundary $\partial\Omega$, and if Ω is simply connected and α is sufficiently small, then $\hat{D}$ is connected. We have determined analytically the optimal domain when Ω is a ball centered at the origin.

From the physical point of view, it is important to know the shape of the optimal set $\hat{D}$ in case that Ω is not a ball. To this end, a numerical method for the solution of the optimization problem has been proposed. A formula for the shape derivative of the eigenvalue has been derived and a gradient type method has been developed in order to determine the optimal configuration.

Numerical results demonstrate the efficiency of the method. Running the numerical method with different initializers, our numerical method can capture the optimal configuration in less that 50 iterations for all examples. Our numerical results reveal that for non convex domains symmetry breaking phenomenon happens and in these cases the optimal solutions are not unique.

Our numerical results illustrate some of the theoretical results of Section 3. For example, the first two numerical examples correspond to simply connected regions Ω. Accordingly to Theorem 3.4, for small parameter α, the optimal domain $\hat{D}$ is connected. We note that in all the simulations that we considered with simply connected domains Ω, we obtained connected domains $\hat{D}$. This is no longer true if we consider non simply connected domains, as shown in the first plot of Figure 3. Moreover, in all the numerical simulations we obtained that $\hat{D}$ is a tubular neighborhood of the boundary, which illustrate the result of Theorem 3.1.

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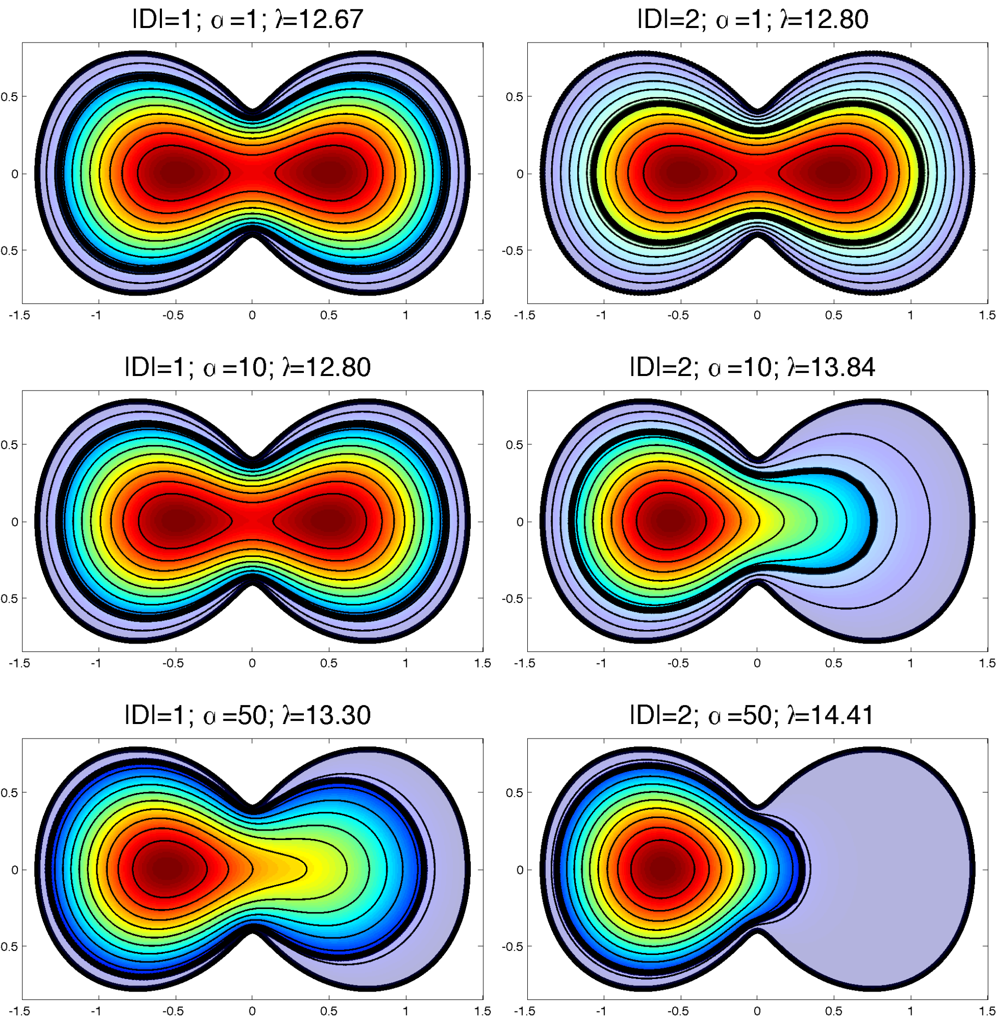

Figure 2: Contour plots of the eigenfunctions associated to the smallest eigenvalue of a non convex domain the, for $\alpha = 1, 10, 50$ and $|D| = 1, 2$.

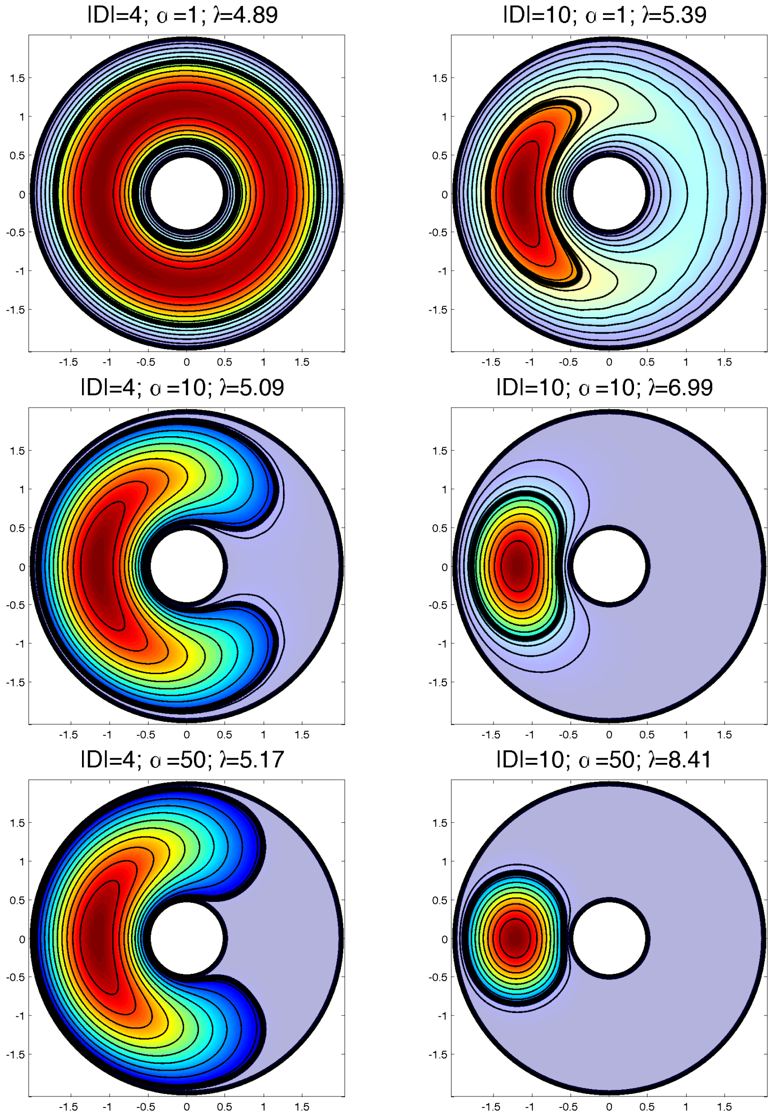

Figure 3: Plots of the eigenfunctions associated to the smallest eigenvalue, for $\alpha = 1, 10, 50$ and $|D| = 4, 10$.