On the Spectrum and Numerical Range of Tridiagonal Random Operators

Raffael Hagger*

July 21, 2014

Abstract

In this paper we derive an explicit formula for the numerical range of tridiagonal random operators using limit operator techniques. Although the numerical range is also of independent interest, for example in semigroup theory, we mainly use it for the natural upper bound to the spectrum that it provides here. For normal operators, this bound is sharp in the sense that the numerical range is the convex hull of the spectrum. But in general, this bound can be arbitrarily bad. For tridiagonal pseudo-ergodic operators, however, we show that this bound is sharp in the sense stated above. This result is a bit surprising because a random operator is pseudo-ergodic almost surely and thus a random tridiagonal operator has this property almost surely. Furthermore we introduce a somewhat combinatorical method to compute the numerical range of an arbitrary tridiagonal operator. Although this method can be applied to finite matrices as well, we concentrate on the infinite matrix case here. To demonstrate the procedure, we compute the numerical range of the square of certain pseudo-ergodic operators. As a result we obtain a better upper bound to the spectrum of these operators than the one obtained by the numerical range alone. In particular, we get a new upper bound to the spectrum of the Feinberg-Zee hopping sign model. We also deduce an easy formula for the numerical abscissa of 2-periodic tridiagonal operators without using the Fourier transform.

2010 Mathematics Subject Classification: Primary 47B80; Secondary 47A10, 47A12, 47B36.
Keywords: random operator, spectrum, numerical range, tridiagonal, pseudo-ergodic

1 Introduction

Throughout this paper we consider the Hilbert space $\mathbf{X} = \ell^2(\mathbb{I})$ with $\mathbb{I} \in \{\mathbb{N}, \mathbb{Z}\}$ and the set of all bounded linear operators $\mathbf{X} \to \mathbf{X}$, denoted by $\mathcal{L}(\mathbf{X})$. The set of all compact operators $\mathbf{X} \to \mathbf{X}$ will be denoted by $\mathcal{K}(\mathbf{X})$.

We want to think of $\mathcal{L}(\mathbf{X})$ as the space of (one-sided or two-sided) infinite matrices. Operators in $\mathcal{L}(\mathbf{X})$ are identified with infinite matrices in the following way. Let $\langle \cdot, \cdot \rangle$ be a scalar product defined on $\mathbf{X}$ and let $\{e_i\}_{i \in \mathbb{I}}$ be a corresponding orthonormal basis, i.e. $\langle e_i, e_j \rangle = \delta_{i,j}$ for all $i, j \in \mathbb{I}$. Let $A \in \mathcal{L}(\mathbf{X})$. Then the entry $A_{i,j}$ is given by $\langle Ae_j, e_i \rangle$. The matrix $(A_{i,j})_{i,j \in \mathbb{I}}$, in the following again denoted by A, acts on a vector $v \in \mathbf{X}$ in the natural way. If v_j is the j-th component of v, then the i-th component of Av is given by $\sum\limits_{j \in \mathbb{I}} A_{i,j} v_j$. This identification of operators and matrices on $\mathbf{X}$

*raffael.hagger@tuhh.de

is an isomorphism (see e.g. [11, Section 1.3.5]). Therefore we do not distinguish between operators and matrices. As usual, the vector $(A_{i,j})_{j\in\mathbb{I}} \in \mathbf{X}$ is called the *i-th row* and $(A_{i,j})_{i\in\mathbb{I}} \in \mathbf{X}$ is called the *j-th column* of A. For $k \in \mathbb{Z}$, the vector $(A_{i+k,i})_{i\in\mathbb{I}} \in \mathbf{X}$ is called the *k-th diagonal* of A or the *diagonal with index k*. In this paper we are interested in operators with specifically constrained diagonals. Note that the orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ is arbitrary but fixed. We do not consider a change of basis here. A change of basis obviously changes the matrix structure and in particular our diagonals. In applications it may be an interesting task to find a suitable basis such that the operator at hand matches specific constraints. In this paper, however, we are not interested in this kind of task and therefore we fix the orthogonal basis at this point and for the rest of the paper. This also fixes the matrix representation of a given operator.

Let $A \in \mathcal{L}(\mathbf{X})$. Then A is called a *band operator* if only a finite number of diagonals are non-zero. The set of all band operators will be denoted by $\mathrm{BO}(\mathbf{X})$. A is said to have *band-width* w if all diagonals with index $|k| > w$ vanish. The set of these operators is denoted by $\mathrm{BO}_w(\mathbf{X})$. Furthermore we call A *tridiagonal* if $A \in \mathrm{BO}_1(\mathbf{X})$.

We consider the following subclasses. Let $n \leq m$ be integers and let $U_n, \dots, U_m \subset \mathbb{C}$ be non-empty compact sets. Then we define

$$M(U_n, \dots, U_m) = \{A \in \mathcal{L}(\mathbf{X}) : A_{i+k,i} \in U_k \text{ if } n \leq k \leq m \text{ and } A_{i+k,i} = 0 \text{ otherwise}\},$$

i.e. the k-th diagonal only contains elements of U_k. To distinguish between one-sided and two-sided infinite matrices, we denote the one-sided variants by $M_+(U_n, \dots, U_m)$. Similarly we denote the set of all finite square matrices with this property by $M_{fin}(U_n, \dots, U_m)$. If $A \in M(U_n, \dots, U_m)$ satisfies $A_{i,j} = A_{i+p,j+p}$ for all $i, j \in \mathbb{I}$ and some $p \geq 1$, then A is called p-periodic and the set of all of these operators will be denoted by $M_{per,p}(U_n, \dots, U_m)$ ($M_{per,p,+}(U_n, \dots, U_m)$ respectively). In the special case $p = 1$ these operators are usually called Laurent and Toeplitz operators respectively and therefore we define $L(U_n, \dots, U_m) := M_{per,1}(U_n, \dots, U_m)$ and $T(U_n, \dots, U_m) := M_{per,1,+}(U_n, \dots, U_m)$.

$A \in M(U_n, \dots, U_m)$ is called a *random operator* if the entries of every diagonal $k \in \{n, \dots m\}$ are chosen randomly w.r.t. some probability measure on U_k. Finally, pseudo-ergodic operators are defined as follows. Let $P_{k,l}$ be the orthogonal projection onto $\operatorname{span}\{e_k, \dots, e_l\}$. Then $A \in M(U_n, \dots, U_m)$ (or $A \in M_+(U_n, \dots, U_m)$) is called *pseudo-ergodic* if for all $\varepsilon > 0$ and all $B \in M_{fin}(U_n, \dots, U_m)$ there exist k and l such that $\|P_{k,l}AP_{k,l} - B\| \leq \varepsilon$. In other words, every finite square matrix of this particular kind can be found up to epsilon when moving along the diagonal of a pseudo-ergodic operator. Note that if all of the U_k are discrete, one can simply put $\varepsilon = 0$ in the definition. At first sight, it is not easy to see why one may want to consider operators of this type, but in fact, pseudo-ergodic operators are closely related to random operators. Under certain conditions on the probability measure (e.g. scaled Lebesgue measure on each of the U_k), one can show that a random operator is pseudo-ergodic almost surely (for details see [12, Section 5.5.3]). Therefore the definition of pseudo-ergodic operators is a nice circumvention of probabilistic arguments when dealing with random operators. We will make use of this fact for the rest of the paper and just mention here that every statement that holds for a pseudo-ergodic operator, holds for a random operator almost surely. We denote the set of pseudo-ergodic operators by $\Psi E(U_n, \dots, U_m)$ and $\Psi E_+(U_n, \dots, U_m)$ respectively. Furthermore we call a sequence $(h_i)_{i\in\mathbb{I}}$, $\mathbb{I} \in \{\mathbb{N}, \mathbb{Z}\}$ *pseudo-ergodic* if the corresponding diagonal operator is pseudo-ergodic with respect to the set $\{h_i : i \in \mathbb{I}\}$. Note that an operator with pseudo-ergodic diagonals is not necessarily pseudo-ergodic, but the diagonals of a pseudo-ergodic operator are always pseudo-ergodic themselves. The notion of pseudo-ergodic operators goes back to Davies [4].

Limit operators are an important tool when treating band operators. Let $\mathbb{I} = \mathbb{Z}$, $k \in \mathbb{Z}$ and define the k-th shift operator V_k by $(V_kx)_j = x_{j-k}$ for all $x \in \mathbf{X}$. In the notation above, V_k is a Laurent operator with $n = m = k$ and $U_k = \{1\}$. Now let $A \in \mathcal{L}(\mathbf{X})$ and let $h := (h_m)_{m\in\mathbb{N}} \subset \mathbb{Z}$ be a sequence tending to infinity such that the $*$-strong[1] limit $A_h := \lim\limits_{m\to\infty} V_{-h_m}AV_{h_m}$ exists. Then A_h is called a *limit operator* of A. For $A \in \mathcal{L}(\ell^2(\mathbb{N}))$, we look at the limit operators of $\tilde{A} = I \oplus A \in \mathcal{L}(\ell^2(\mathbb{Z}))$ and only consider sequences tending to $+\infty$. In both cases the set of all limit operators of A is called the *operator spectrum* of A and denoted by $\sigma^{\mathrm{op}}(A)$. Note that limit operators are usually defined using $\mathcal{P}$-convergence, but in order to keep things simple, we just note that in the cases considered here $\mathcal{P}$-convergence and $*$-strong convergence coincide by [11, Section 1.6.3]. Here are some basic properties of limit operators that we will need in the following (see e.g. [11, Proposition 3.4, Corollary 3.24]):

Proposition 1. *Let $A, B \in \mathrm{BO}(\mathbf{X})$, $h := (h_m)_{m\in\mathbb{N}} \subset \mathbb{Z}$ a sequence tending to infinity, p a polynomial in both z and $\bar{z}$. Then the following statements hold:*

- *There exists a subsequence $g := (g_m)_{m\in\mathbb{N}}$ of h such that A_g and B_g exist.*
- *If A_h and B_h exist, so does $(A + B)_h$ and $(A + B)_h = A_h + B_h$.*
- *If A_h and B_h exist, so does $(AB)_h$ and $(AB)_h = A_hB_h$.*
- *If A_h exists, so does $(A^*)_h$ and $(A^*)_h = (A_h)^*$.*
- $\sigma^{\mathrm{op}}(p(A)) = p(\sigma^{\mathrm{op}}(A)) := \{p(A_h) : A_h \in \sigma^{\mathrm{op}}(A)\}$.
- *If A_h exists, then $\|A_h\| \leq \|A\|$.*
- *If $A \in \mathcal{K}(\mathbf{X})$, then $A_h = 0$.*

We call an operator A *Fredholm* if $\ker(A)$ and $\mathrm{im}(A)^\perp$ are both finite-dimensional. As usual we define the *spectrum*

$$\mathrm{sp}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$$

and the *essential spectrum*

$$\mathrm{sp}_{\mathrm{ess}}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}.$$

After introducing all the notation, we can cite the main theorem of limit operator theory (which holds in much more generality than stated and needed here).

Theorem 2. *(e.g. [12, Corollary 5.26])*
Let $A \in \mathrm{BO}(\mathbf{X})$. Then

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{B\in\sigma^{\mathrm{op}}(A)} \mathrm{sp}(B).$$

In order to apply this theorem to pseudo-ergodic operators, we use the following result that describes the operator spectrum of pseudo-ergodic operators.

Proposition 3. *Let $U_n, \ldots, U_m$ be non-empty and compact and let $A \in \Psi E(U_n, \ldots, U_m)$ or $A \in \Psi E_+(U_n, \ldots, U_m)$. Then $\sigma^{\mathrm{op}}(A) = M(U_n, \ldots, U_m)$.*

[1] A sequence of operators is called $*$-strongly convergent if the sequence as well as the sequence of adjoints converges in the strong operator topology, i.e. elementwise on $\mathbf{X}$ (and $\mathbf{X}^*$).

Proof. For diagonal operators on $\ell^2(\mathbb{Z})$, this is Corollary 3.70 in [11]. The proof easily carries over to the case of band operators and also to the one-sided infinite case. $\square$

Using this, we get the following corollary.

Corollary 4. *Let $U_n, \ldots, U_m$ be non-empty and compact and let $A \in \Psi E(U_n, \ldots, U_m)$. Then*

$$\mathrm{sp}_{\mathrm{ess}}(A) = \bigcup_{B \in M(U_n,\ldots,U_m)} \mathrm{sp}(B).$$

Corollary 4 in particular implies that all pseudo-ergodic operators w.r.t to the sets $U_n, \ldots, U_m$ have the same essential spectrum. Furthermore we have that the spectrum and the essential spectrum coincide for all $A \in \Psi E(U_n, \ldots, U_m)$ since $\mathrm{sp}_{\mathrm{ess}}(A) \subset \mathrm{sp}(A)$ and $\Psi E(U_n, \ldots, U_m) \subset M(U_n, \ldots, U_m)$. Thus Corollary 4 can be rewritten in the following way:

Corollary 5. *Let $U_n, \ldots, U_m$ be non-empty and compact. It holds*

$$\mathrm{sp}(A) = \bigcup_{B \in M(U_n,\ldots,U_m)} \mathrm{sp}(B) \tag{1}$$

for all $A \in \Psi E(U_n, \ldots, U_m)$.

In particular we see that the spectrum of a pseudo-ergodic operator only depends on the sets $U_n, \ldots, U_m$. Furthermore, equation (1) provides a somewhat easy method to obtain lower bounds for the spectrum of $A \in \Psi E(U_n, \ldots, U_m)$. Indeed, we can take any operator $B \in M(U_n, \ldots, U_m)$ with a known spectrum and get a lower bound for the spectrum of A. For example the spectrum of a periodic operator B can be computed via the Fourier transform.

Theorem 6. *([5, Theorem 4.4.9])*
Let $U_n, \ldots, U_m$ be non-empty and compact, $p \in \mathbb{N}$, $B \in M_{per,p}(U_n, \ldots, U_m)$ and let $B_k \in \mathcal{L}(\mathbb{C}^p)$ be defined by $(B_k)_{i,j} = B_{i+kp,j}$ for all $i, j \in \{1, \ldots, p\}$ and $k \in \mathbb{Z}$. Then

$$\mathrm{sp}(B) = \bigcup_{\theta \in [0,2\pi)} \mathrm{sp}\left(\sum_{k\in\mathbb{Z}} B_k e^{-ik\theta}\right). \tag{2}$$

Note that the sum in equation (2) is actually finite because B is assumed to be a band operator. Thus there is no need to worry about convergence. The right-hand side of (2) can be evaluated analytically ($p \le 4$) or numerically to arbitrary precision. Computing the spectra of all periodic operators in $M(U_n, \ldots, U_m)$ yields a nice lower bound to the spectrum of $A \in \Psi E(U_n, \ldots, U_m)$. A natural question is whether considering periodic operators is enough. The answer is obviously YES for diagonal operators and (not so obviously) YES for bidiagonal operators under the assumption $0 \notin U_1$ (see [13] and observe that section 3 carries over to the general bidiagonal case if one assumes $0 \notin U_1$) and NO if the assumption $0 \notin U_1$ is dropped (choose e.g. $U_1 = \{0, 1\}$, $U_k = \{0\}$ for $k \neq 1$ and compare with [13, Theorem 1.1]). For three or more diagonals, the story gets much more complicated. Of course the answer is still NO in general, but it is conjectured that the answer should be YES in some particular cases (e.g. [2],[3]).

In this paper we want to give some upper bounds to the spectrum of tridiagonal pseudo-ergodic operators. It is well-known that the numerical range $N(A) := \mathrm{clos}\,\{\langle Ax, x\rangle : x \in \mathbf{X}, \|x\| = 1\}$ is an upper bound for the spectrum of every operator. However, this bound can be arbitrarily bad even for finite matrices as the simple example $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ shows. The spectrum of A is given by $\mathrm{sp}(A) = \{0\}$ while its numerical range is the closed ball of radius $a/2$. However, for normal operators we have $N(A) = \mathrm{conv}(\mathrm{sp}(A))$. Thus the numerical range is a quite good upper bound if A is normal. Suprisingly the same is true for tridiagonal pseudo-ergodic operators (which are generally non-normal, of course) as we will show in section 2. Furthermore we introduce a somewhat combinatorical method to obtain an even better upper bound in some cases using second order numerical ranges and a version of the Schur Test in section 3. We use this method and the concept of higher order numerical ranges (see e.g. [5, Section 9.4]) to obtain a better upper bound to the spectrum of the Feinberg-Zee hopping sign model (see [2], [6] or [10] for an overview). We prove a special case in section 3 and refer to the appendix for the complete proof. In section 4 we give some concluding remarks and have a look at open problems regarding pseudo-ergodic operators.

2 The Numerical Range

Definition 7. Let $A \in \mathcal{L}(\mathbf{X})$. Then the *numerical range* is defined as

$$N(A) := \mathrm{clos}\left\{\langle Ax, x\rangle : x \in \mathbf{X}, \|x\| = 1\right\},$$

the *numerical radius* is defined as

$$r(A) := \max\left\{|z| : z \in N(A)\right\}$$

and for $\varphi \in [0, 2\pi)$ the *(rotated) numerical abscissa* is defined as

$$r_\varphi(A) := \max\left\{\mathrm{Re}\ z : z \in N(e^{i\varphi}A)\right\}.$$

Note that the numerical range is usually defined without the closure (and denoted by $W(A)$) and the closure appears when it comes to giving an upper bound to the spectrum. However, we will find it useful to take the closure right here in the definition. Also note that for finite matrices the set $\{\langle Ax, x\rangle : x \in \mathbf{X}, \|x\| = 1\}$ is always closed, but for infinite matrices this is usually not the case.

The following results are well-known and also hold in arbitrary Hilbert spaces.

Theorem 8. *Let $A \in \mathcal{L}(\mathbf{X})$. Then $\mathrm{sp}(A) \subset N(A)$.*

Theorem 9. *(Hausdorff-Toeplitz)*
Let $A \in \mathcal{L}(\mathbf{X})$. Then its numerical range $N(A)$ is convex.

Theorem 10. *Let $A \in \mathcal{L}(\mathbf{X})$. Then*

$$\rho(A) \leq r(A) \leq \|A\|,$$

where $\rho(A)$ stands for the spectral radius of A. If A is normal, then equality holds. Furthermore, we have

$$\mathrm{conv}(\mathrm{sp}(A)) \subset N(A)$$

with equality if A is normal.

In general, it is difficult to compute the spectrum of an operator explicitly. Even in the tridiagonal case not much is known. For the numerical range, the story seems to be a lot easier, at least for tridiagonal pseudo-ergodic operators as the following theorem shows.

Theorem 11. *Let U_{-1}, U_0 and U_1 be non-empty and compact. Then for $A \in \Psi E(U_{-1}, U_0, U_1)$ the following formula holds:*

$$N(A) = \operatorname{conv}\left(\bigcup_{B\in L(U_{-1},U_0,U_1)} \operatorname{sp}(B)\right) = \operatorname{conv}\left(\bigcup_{\substack{u_k\in U_k,\\ k=-1,0,1}} \left\{u_{-1}e^{i\theta} + u_0 + u_1 e^{-i\theta} : \theta \in [0, 2\pi)\right\}\right).$$

Proof. The second equality follows immediately from Theorem 6. Thus we focus on the proof of the first equality.

Let $A \in \Psi E(U_{-1}, U_0, U_1)$. The idea is to compute the numerical abscissa $r_\varphi(A)$ for every angle $\varphi \in [0, 2\pi)$ and compare it with the abscissae of the Laurent operators. Since by Theorem 9 both sides of the first assertion are convex, proving the equality of all abscissae is enough to prove the theorem. This idea can also be used to compute numerical ranges of finite matrices numerically. But let us first observe that one direction follows immediately from Corollary 5, Theorem 9 and Theorem 10. Indeed, the spectrum of every operator in $L(U_{-1}, U_0, U_1)$ is contained in the spectrum of A by Corollary 5[2], which is of course contained in the numerical range of A by Theorem 10. Taking the convex hull on both sides yields

$$N(A) \supset \operatorname{conv}\left(\bigcup_{B\in L(U_{-1},U_0,U_1)} \operatorname{sp}(B)\right).$$

by Theorem 9.

Conversely, let $z_0 := \sup_{B\in M(U_{-1},U_0,U_1)} r(B)$. It is clear that z_0 is finite because the norm of an operator B is bounded by the sum of the maximal elements of its diagonals. In our case this means

$$r(B) \le \|B\| \le \max_{\substack{u_k\in U_k\\ k=-1,0,1}} (|u_{-1}| + |u_0| + |u_1|). \tag{3}$$

As a consequence we have $N(e^{i\varphi}B + z_0 I) \subset \mathbb{C}_{\mathrm{Re}\ge 0}$ for all $B \in M(U_{-1}, U_0, U_1)$ and $\varphi \in [0, 2\pi)$. It follows

$$\begin{aligned} r_\varphi(A) &= \sup_{\|x\|=1} \operatorname{Re}\left\langle e^{i\varphi}Ax, x\right\rangle \\ &= \sup_{\|x\|=1} \operatorname{Re}\left\langle (e^{i\varphi}A + z_0 I)x, x\right\rangle - z_0 \\ &= \sup_{\|x\|=1} \left|\operatorname{Re}\left\langle (e^{i\varphi}A + z_0 I)x, x\right\rangle\right| - z_0 \\ &= \frac{1}{2}\sup_{\|x\|=1} \left|\left\langle (e^{i\varphi}A + e^{-i\varphi}A^* + 2z_0 I)x, x\right\rangle\right| - z_0 \\ &= \frac{1}{2}\left\|e^{i\varphi}A + e^{-i\varphi}A^* + 2z_0 I\right\| - z_0, \end{aligned}$$

[2]Pseudo-ergodicity is actually a bit too strong here, we only need that all Laurent operators are contained in the operator spectrum of A, i.e. $A \in M(U_{-1}, U_0, U_1)$ such that $L(U_{-1}, U_0, U_1) \subset \sigma^{\mathrm{op}}(A)$.

where we used Theorem 10 in the last line. Using (3) we arrive at

$$\begin{aligned} r_\varphi(A) &= \frac{1}{2}\left\|e^{i\varphi}A + e^{-i\varphi}A^* + 2z_0 I\right\| - z_0 \\ &\le \max_{\substack{u_{-1}\in U_{-1}\\ u_1\in U_1}} \left|e^{i\varphi}u_1 + e^{-i\varphi}\overline{u_{-1}}\right| + \frac{1}{2}\max_{u_0\in U_0}\left|e^{i\varphi}u_0 + e^{-i\varphi}\overline{u_0} + 2z_0\right| - z_0. \end{aligned} \tag{4}$$

Fix $w_{-1} \in U_{-1}$, $w_0 \in U_0$ and $w_1 \in U_1$ such that the maximum in (4) is attained, i.e.

$$\max_{\substack{u_{-1}\in U_{-1}\\ u_1\in U_1}} \left|e^{i\varphi}u_1 + e^{-i\varphi}\overline{u_{-1}}\right| = \left|e^{i\varphi}w_1 + e^{-i\varphi}\overline{w_{-1}}\right|$$

and

$$\max_{u_0\in U_0}\left|e^{i\varphi}u_0 + e^{-i\varphi}\overline{u_0} + 2z_0\right| = \left|e^{i\varphi}w_0 + e^{-i\varphi}\overline{w_0} + 2z_0\right|.$$

It is not hard to see that the spectrum of a tridiagonal Laurent operator $L(v_{-1}, v_0, v_1)$ (to simplify the notation we identify the set $L(v_{-1}, v_0, v_1) := L(\{v_{-1}\}, \{v_0\}, \{v_1\})$ with its only element) is given by an ellipse with center v_0 and half-axes $\left||v_{-1}| \pm |v_1|\right|$ (see e.g. [14]). If in addition $C := L(v_{-1}, v_0, v_1)$ is self-adjoint, then clearly $\rho(C) = \|C\| = |v_0| + |v_{-1}| + |v_1|$. In our case, if we put $B := L(w_{-1}, w_0, w_1)$ and $C = e^{i\varphi}B + e^{-i\varphi}B^* + 2z_0 I$, we get

$$\left\|e^{i\varphi}B + e^{-i\varphi}B^* + 2z_0 I\right\| = 2\left|e^{i\varphi}w_1 + e^{-i\varphi}\overline{w_{-1}}\right| + \left|e^{i\varphi}w_0 + e^{-i\varphi}\overline{w_0} + 2z_0\right|$$

and therefore

$$r_\varphi(A) \le \frac{1}{2}\left\|e^{i\varphi}B + e^{-i\varphi}B^* + 2z_0 I\right\| - z_0.$$

From here, we can go all the steps backwards to finish the proof:

$$\begin{aligned} r_\varphi(A) &\le \frac{1}{2}\left\|e^{i\varphi}B + e^{-i\varphi}B^* + 2z_0 I\right\| - z_0 \\ &= \frac{1}{2}\sup_{\|x\|=1}\left|\left\langle (e^{i\varphi}B + e^{-i\varphi}B^* + 2z_0 I)x, x\right\rangle\right| - z_0 \\ &= \sup_{\|x\|=1}\left|\operatorname{Re}\left\langle (e^{i\varphi}B + z_0 I)x, x\right\rangle\right| - z_0 \\ &= \sup_{\|x\|=1}\operatorname{Re}\left\langle (e^{i\varphi}B + z_0 I)x, x\right\rangle - z_0 \\ &= r_\varphi(B). \end{aligned}$$

Doing this for every $\varphi \in [0, 2\pi)$ and using the observation at the beginning of the proof, we get

$$N(A) \subset \operatorname{conv}\left(\bigcup_{B\in L(U_{-1},U_0,U_1)} N(B)\right),$$

which is not yet what we wanted. But Laurent operators are normal and thus

$$N(A) \subset \operatorname{conv}\left(\bigcup_{B\in L(U_{-1},U_0,U_1)} \operatorname{conv}(\operatorname{sp}(B))\right) = \operatorname{conv}\left(\bigcup_{B\in L(U_{-1},U_0,U_1)} \operatorname{sp}(B)\right).$$

□

Combining Corollary 5, Theorem 10 and Theorem 11 we get the following corollary.

Corollary 12. *Let U_{-1}, U_0 and U_1 be non-empty and compact and let $A \in \Psi E(U_{-1}, U_0, U_1)$. Then*

$$N(A) = \operatorname{conv}(\operatorname{sp}(A)).$$

This property of tridiagonal pseudo-ergodic operators is quite remarkable because it is rarely seen on non-normal operators. We do not know if pseudo-ergodic operators with more than three diagonals share this property, but we do know that Theorem 11 is wrong if the tridiagonality assumption is dropped (see Example 17). To show this, we use the following theorem that is very related to Theorem 2. Again this theorem can be proved (without further effort) in much more generality than we state it here.

Theorem 13. *Let $A \in \mathrm{BO}(\mathbf{X})$. Then*

$$\bigcap_{K \in \mathcal{K}(\mathbf{X})} N(A+K) = \operatorname{conv}\left(\bigcup_{B \in \sigma^{\mathrm{op}}(A)} N(B) \right).$$

This is actually a corollary of the following theorem.

Theorem 14. *([9])*
Let $A \in \mathrm{BO}(\mathbf{X})$. Then

$$\inf_{K \in \mathcal{K}(\mathbf{X})} \|A+K\| = \max_{B \in \sigma^{\mathrm{op}}(A)} \|B\|.$$

Proof. (Theorem 13)
Like in the proof of Theorem 11 it is enough to consider the numerical abscissae. Using the same arguments as used there, we get

$$\begin{aligned} \inf_{K \in \mathcal{K}(\mathbf{X})} r_\varphi(A+K) &\leq \frac{1}{2} \inf_{K \in \mathcal{K}(\mathbf{X})} \left\| e^{i\varphi}(A+K) + e^{-i\varphi}(A^*+K^*) + 2z_0 I \right\| - z_0 \\ &= \frac{1}{2} \inf_{\substack{K \in \mathcal{K}(\mathbf{X}) \\ K=K^*}} \left\| e^{i\varphi}A + e^{-i\varphi}A^* + K + 2z_0 I \right\| - z_0 \end{aligned}$$

for all $z_0 \in \mathbb{R}$. For a self-adjoint operator C, the norm $\|C+K\|$ is minimized by a self-adjoint operator K. This can be seen as follows:

$$\begin{aligned} \|C+K\| &\geq \sup_{\|x\|=1} |\langle (C+K)x, x\rangle| \\ &= \sup_{\|x\|=1} \left| \left\langle \left(C + \frac{K+K^*}{2} \right) x, x \right\rangle + \left\langle \left(\frac{K-K^*}{2} \right) x, x \right\rangle \right| \\ &\geq \sup_{\|x\|=1} \left| \left\langle \left(C + \frac{K+K^*}{2} \right) x, x \right\rangle \right| \\ &= \left\| C + \frac{K+K^*}{2} \right\|, \end{aligned}$$

where we used Theorem 10 and the fact that $\left\langle\left(C+\frac{K+K^*}{2}\right)x,x\right\rangle \in \mathbb{R}$ and $\left\langle\left(\frac{K-K^*}{2}\right)x,x\right\rangle \in i\mathbb{R}$ for all $x \in \mathbf{X}$. Therefore we have

$$\begin{aligned}
\inf_{K\in\mathcal{K}(\mathbf{X})} r_\varphi(A+K) &\le \frac{1}{2}\inf_{K\in\mathcal{K}(\mathbf{X})}\left\|e^{i\varphi}A+e^{-i\varphi}A^*+K+2z_0I\right\|-z_0\\
&=\frac{1}{2}\max\left\{\|B\| : B\in\sigma^{\mathrm{op}}(e^{i\varphi}A+e^{-i\varphi}A^*+2z_0I)\right\}-z_0\\
&=\frac{1}{2}\max\left\{\left\|e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I\right\| : B\in\sigma^{\mathrm{op}}(A)\right\}-z_0
\end{aligned}$$

for all $z_0 \in \mathbb{R}$, where we used Theorem 14 and some basic properties of the operator spectrum (see Proposition 1). Choosing $z_0 \ge \|A\|$ we get

$$\frac{1}{2}\max\left\{\left\|e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I\right\| : B\in\sigma^{\mathrm{op}}(A)\right\}-z_0=\max\left\{r_\varphi(B) : B\in\sigma^{\mathrm{op}}(A)\right\}$$

and thus

$$\inf_{K\in\mathcal{K}(\mathbf{X})} r_\varphi(A+K)\le\max\left\{r_\varphi(B) : B\in\sigma^{\mathrm{op}}(A)\right\}.$$

Conversely, fix a compact $K \in \mathcal{L}(\mathbf{X})$ and set $z_0 := \max\{\|A\|, \|A+K\|\}$. Similarly as before we get

$$\begin{aligned}
r_\varphi(A+K) &= \frac{1}{2}\left\|e^{i\varphi}(A+K)+e^{-i\varphi}(A^*+K^*)+2z_0I\right\|-z_0\\
&\ge\frac{1}{2}\max\left\{\|B\| : B\in\sigma^{\mathrm{op}}(e^{i\varphi}A+e^{-i\varphi}A^*+2z_0I)\right\}-z_0\\
&=\frac{1}{2}\max\left\{\left\|e^{i\varphi}B+e^{-i\varphi}B^*+2z_0I\right\| : B\in\sigma^{\mathrm{op}}(A)\right\}-z_0\\
&=\max\left\{r_\varphi(B) : B\in\sigma^{\mathrm{op}}(A)\right\}.
\end{aligned}$$

This implies

$$\inf_{K\in\mathcal{K}(\mathbf{X})} r_\varphi(A+K)\ge\max_{B\in\sigma^{\mathrm{op}}(A)} r_\varphi(A)$$

and thus

$$\inf_{K\in\mathcal{K}(\mathbf{X})} r_\varphi(A+K)=\max_{B\in\sigma^{\mathrm{op}}(A)} r_\varphi(A).$$

□

If we apply these results to pseudo-ergodic operators, we get the following corollaries[3]

Corollary 15. *Let $U_n, \ldots, U_m$ be non-empty and compact. It holds*

$$N(A)=\bigcup_{B\in M(U_n,\ldots,U_m)} N(B)$$

for all $A \in \Psi E(U_n, \ldots, U_m)$.

[3]These already follow from the weaker statements $\|A_h\| \le \|A\|$ and $N(A_h) \subset N(A)$ for $A_h \in \sigma^{\mathrm{op}}(A)$ (see Proposition 1 and Lemma 20)

Note that taking the convex hull is obviosuly not necessary here. We also have the following corollary of Theorem 14:

Corollary 16. *Let $U_n, \ldots, U_m$ be non-empty and compact. It holds*

$$\|A\| = \max_{B \in M(U_n,\ldots,U_m)} \|B\|$$

for all $A \in \Psi E(U_n, \ldots, U_m)$.

Now we are able to give a counterexample to Theorem 11 if the tridiagonality assumption is dropped.

Example 17. Let $U_{-2} = \{i\}$, $U_{-1} = \{\pm i\}$, $U_0 = \{0\}$, $U_1 = \{i\}$ and $U_2 = \{i\}$ and let $A \in \Psi E(U_{-2}, \ldots, U_2)$. In Figure 1 we can see the boundaries of the numerical ranges of all operators in $M_{per,3}(U_{-2}, \ldots, U_2)$ (red) and the spectra of both Laurent operators (blue).

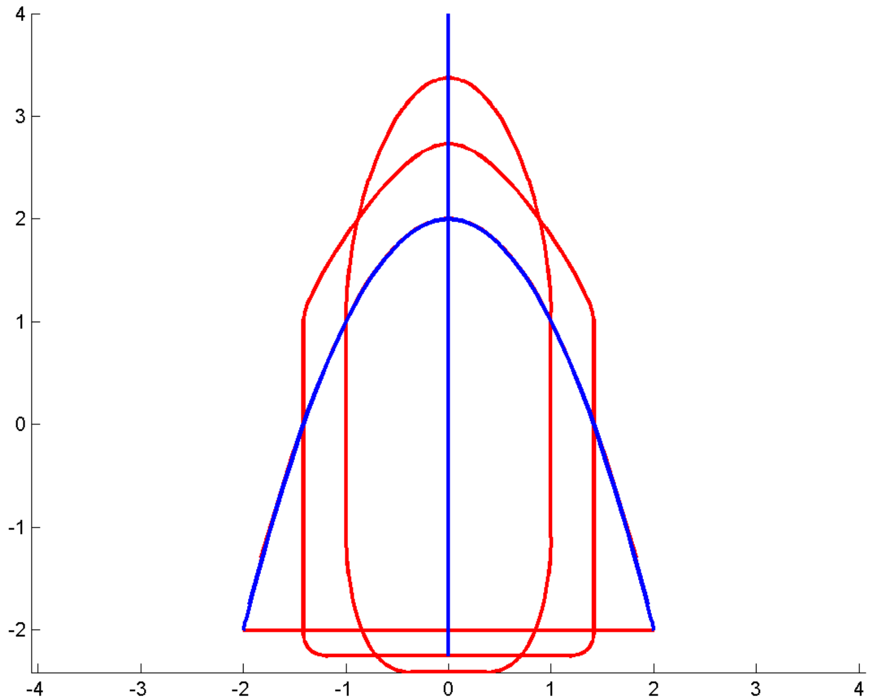

Figure 1: The boundaries of the numerical ranges of all 3-periodic operators (red) and the spectra of both Laurent operators (blue).

Note that due to symmetry, we only see 4 different numerical ranges rather than the expected 8 (and 2 of them are actually the ones from the Laurent operators). Clearly, some of the numerical ranges exceed the convex hull of the spectra of the Laurent operators. But these numerical ranges are contained in the numerical range of the respective pseudo-ergodic operator by Corollary 15. Thus Theorem 11 does not hold in this example of a 5-diagonal pseudo-ergodic operator.

We have seen that the numerical range, besides being an upper bound to the spectrum, has some benefits over the spectrum, especially in the case of tridiagonal pseudo-ergodic operators. There are some more desired properties which are satisfied by the numerical range but not by the spectrum. For example if we tried to compute the spectrum of an infinite matrix explicitly, one of our first attempts would be to cut out finite square matrices of larger and larger size (called finite section method or short FSM), compute their eigenvalues and hope that they converge in some particular way. Unfortunately this naïve attempt fails more often than not. A well-known counterexample is the two-sided shift. Let $\{e_i\}_{i\in\mathbb{Z}}$ be the usual orthonormal basis of $\ell^2(\mathbb{Z})$. Then the shift V is defined by $e_i \mapsto e_{i-1}$ for all $i \in \mathbb{Z}$. As an infinite matrix V has ones on the first superdiagonal and zeros everywhere else. So if we cut out finite square matrices, we end up getting matrices with ones on the first superdiagonal and zeros everywhere else. Thus the only eigenvalue of all of these matrices is 0. But the spectrum of the two-sided shift V is the unit circle. No matter what convergence we consider, there is no way the eigenvalues of the finite matrices converge to the spectrum of V. Many different approaches to (numerically) compute the spectrum of an infinite matrix have been considered by various mathematicians, but none of them has proven to be satisfactory yet. For the numerical range, however, the story is again much different as Theorem 19 shows. But first we have to clarify what we mean by convergence of set sequences.

Definition 18. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C}$. Then we define

$$\limsup_{n\to\infty} M_n := \{m \in \mathbb{C} : m \text{ is an accumulation point of a sequence } (m_n)_{n\in\mathbb{N}}, m_n \in M_n\},$$
$$\liminf_{n\to\infty} M_n := \{m \in \mathbb{C} : m \text{ is the limit of a sequence } (m_n)_{n\in\mathbb{N}}, m_n \in M_n\}.$$

The *Hausdorff metric* for compact sets $A, B \subset \mathbb{C}$ is defined as

$$h(A,B) := \max\left\{\max_{a\in A}\min_{b\in B}|a-b|, \max_{b\in B}\min_{a\in A}|a-b|\right\}.$$

Furthermore we define $\lim\limits_{n\to\infty} M_n$ as the limit of the sequence $(M_n)_{n\in\mathbb{N}}$ w.r.t. the Hausdorff metric.

$\lim\limits_{n\to\infty} M_n$ exists if and only if $\limsup\limits_{n\to\infty} M_n = \liminf\limits_{n\to\infty} M_n$ and in this case we have

$$\lim_{n\to\infty} M_n = \limsup_{n\to\infty} M_n = \liminf_{n\to\infty} M_n$$

(see [8, Proposition 3.6]).

Theorem 19. *Let $A \in \mathcal{L}(\mathbf{X})$ and let $(P_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathbf{X})$ be a sequence of orthogonal projections converging to the identity in the strong operator topology. Further assume $P_n \neq 0$ and $P_{n+1}P_n = P_n = P_nP_{n+1}$ for all $n \in \mathbb{N}$. Consider the finite sections $A_n := P_nAP_n$. Then $N(A_n) \to N(A)$ w.r.t the Hausdorff metric and $N(A) = \operatorname{clos}\left(\bigcup\limits_{n\in\mathbb{N}} N(A_n)\right)$.*

Note that there is some ambiguity in the notation $N(A_n)$ because we interpret A_n as an operator on $\mathbf{X}$. This leads to the undesired fact that 0 is always in $N(A_n)$ (and thus possibly some more undesired points since numerical ranges are convex) if we are really strict with the definition. However, we want to understand $N(A_n)$ as the numerical range of A_n restricted to $\operatorname{im}(P_n)$ without changing the notation. This will hopefully not cause confusion. To prove this theorem, we need the following lemma that looks very similar to the Banach-Steinhaus theorem.

Lemma 20. *Let $A \in \mathcal{L}(\mathbf{X})$ and let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathbf{X})$ be a sequence that converges to A in weak operator topology. Then $N(A) \subset \liminf_{n\to\infty} N(A_n)$.*

Proof. (Lemma)
$A_n \to A$ in the weak operator topology implies $\langle (A_n - A)x, x\rangle \to 0$ for all $x \in \mathbf{X}$ as $n \to \infty$. Let $z \in N(A)$. Choose $x_1 \in \mathbf{X}$ with $\|x_1\| = 1$ such that $|z - \langle Ax_1, x_1\rangle| < 1$ and n_1 such that $|\langle (A_n - A)x_1, x_1\rangle| < 1$ for all $n \geq n_1$. For $j \in \mathbb{N}$, choose $x_{j+1} \in \mathbf{X}$ with $\|x_{j+1}\| = 1$ such that $|z - \langle Ax_{j+1}, x_{j+1}\rangle| < \frac{1}{j+1}$ and $n_{j+1} > n_j$ such that $|\langle (A_n - A)x_{j+1}, x_{j+1}\rangle| < \frac{1}{j+1}$ for all $n \geq n_{j+1}$. Of course this implies $|z - \langle A_n x_j, x_j\rangle| < \frac{2}{j}$ for all $n \geq n_j$. Now define a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ in the following way. For $n < n_1$, choose $z_n \in N(A_n)$ arbitrarily. For $j \in \mathbb{N}$ and $n_j \leq n < n_{j+1}$, choose $z_n \in N(A_n)$ such that $|z - z_n| < \frac{2}{j}$. We get $|z - z_n| \to 0$ as $n \to \infty$. Thus $N(A) \subset \liminf_{n\to\infty} N(A_n)$. □

Proof. (Theorem)
Let $x \in \operatorname{im}(P_n)$ with $\|x\| = 1$. Then we have

$$\langle A_n x, x\rangle = \langle P_n A x, x\rangle = \langle Ax, P_n x\rangle = \langle Ax, x\rangle \in N(A).$$

Thus $N(A_n) \subset N(A)$ for all $n \in \mathbb{N}$. On the other hand A_n converges weakly to A (even strongly). Thus Lemma 20 implies

$$\limsup_{n\to\infty} N(A_n) \subset N(A) \subset \liminf_{n\to\infty} N(A_n) \subset \limsup_{n\to\infty} N(A_n)$$

and thus

$$N(A) = \liminf_{n\to\infty} N(A_n) = \limsup_{n\to\infty} N(A_n) = \lim_{n\to\infty} N(A_n).$$

The second statement easily follows as well. □

For pesudo-ergodic operators, Theorem 19 yields the following two corollaries.

Corollary 21. *Let $U_n, \ldots, U_m$ be non-empty and compact. It holds*

$$N(A) = \operatorname{clos}\left(\bigcup_{B\in M_{fin}(U_n,\ldots,U_m)} N(B)\right)$$

for all $A \in \Psi E(U_n, \ldots, U_m)$.

Proof. Let $A \in \Psi E(U_n, \ldots, U_m)$. By Theorem 19, we have $N(A) = \operatorname{clos}\left(\bigcup_{n\in\mathbb{N}} N(A_n)\right)$ for finite sections $A_n = P_n A P_n$ with P_n as in Theorem 19. By choosing the sequence $(P_n)_{n\in\mathbb{N}}$ appropriately and by the pseudo-ergodic property, we can actually get every finite square matrix of this kind as a finite section of A. Thus

$$N(A) = \operatorname{clos}\left(\bigcup_{B\in M_{fin}(U_n,\ldots,U_m)} N(B)\right)$$

holds. Alternatively one could also argue that due to Corollary 15 all pseudo-ergodic operators of this kind have the same numerical range. Using this, one can get every finite square matrix with the same sequence $(P_n)_{n\in\mathbb{N}}$ (but different A). □

Corollary 22. *Let $U_n, \ldots, U_m$ be non-empty and compact. It holds*

$$N(A) = \operatorname{clos}\left(\bigcup_{B \in M_{per}(U_n,\ldots,U_m)} N(B) \right)$$

for all $A \in \Psi E(U_n, \ldots, U_m)$.

Proof. Let $A \in \Psi E(U_n, \ldots, U_m)$. It is not difficult to find a sequence $A_n \in M_{per}(U_n, \ldots, U_m)$ that converges weakly to A (even strongly). Thus by Lemma 20 and Corollary 15 we have

$$N(A) \subset \liminf_{n\to\infty} N(A_n) \subset \operatorname{clos}\left(\bigcup_{B \in M_{per}(U_n,\ldots,U_m)} N(B) \right) \subset N(A).$$

□

So the question whether it is enough to consider periodic operators can be answered affirmatively in the case of the numerical range.

So far we only considered the numerical range of two-sided pseudo-ergodic operators (i.e. $A \in \Psi E(U_n, \ldots, U_m)$ rather than $A \in \Psi E_+(U_n, \ldots, U_m)$). Theorem 19 now allows an easy inclusion of one-sided pseudo-ergodic operators. Namely, they have exactly the same numerical range.

Corollary 23. *Let $U_n, \ldots, U_m$ be non-empty and compact. It holds $N(A) = N(A_+)$ for all $A \in \Psi E(U_n, \ldots, U_m)$ and $A_+ \in \Psi E_+(U_n, \ldots, U_m)$.*

Proof. This can actually be seen in various different ways. We will make use of Theorem 19 here. Since we know that all $A \in \Psi E(U_n, \ldots, U_m)$ have the same numerical range, we can choose one of them according to our requirements. So let $A_+ \in \Psi E_+(U_n, \ldots, U_m)$ and choose $A \in \Psi E(U_n, \ldots, U_m)$ such that $A_+ = P_{\mathbb{N}} A P_{\mathbb{N}}$, where $P_{\mathbb{N}}$ denotes the projection onto span $\{e_1, e_2, \ldots\}$. By Proposition 3 and Theorem 13, we clearly have $N(A) \subset N(A_+)$. On the other hand $P_{\mathbb{N}}$ is of course a viable projection for Theorem 19 if we choose the other projections accordingly (in fact, we do not even have to choose them). Thus $N(A_+) \subset N(A)$. □

Again, this is a feature of the numerical range that the spectrum does not have. To see this consider for example Laurent and Toeplitz matrices which are a special case of pseudo-ergodic operators if we choose all sets to be singletons.

3 A Method to compute Numerical Ranges explicitly

The method introduced here is based on a reformulation of the Schur Test that is known [15], but apparently not used very often. However, it seems to fit perfectly here. Our motivation is to give a better upper bound to the spectrum of the Feinberg-Zee hopping sign model [6] than the one given by the numerical range. In other words, we want to compute a better upper bound to the spectrum of a (the) pseudo-ergodic operator given by $U_{-1} = \{1\}$, $U_0 = \{0\}$ and $U_1 = \{\pm 1\}$. It is conjectured by the authors of [2] that the spectrum is given by the black set in Figure 2. In [1] and [3] it is proved that the unit disk is contained in the spectrum. The red square indicates the boundary of the numerical range.

The following lemma is a small extension to [15, Proposition 1] which is basically a reformulation of the Schur Test lemma.

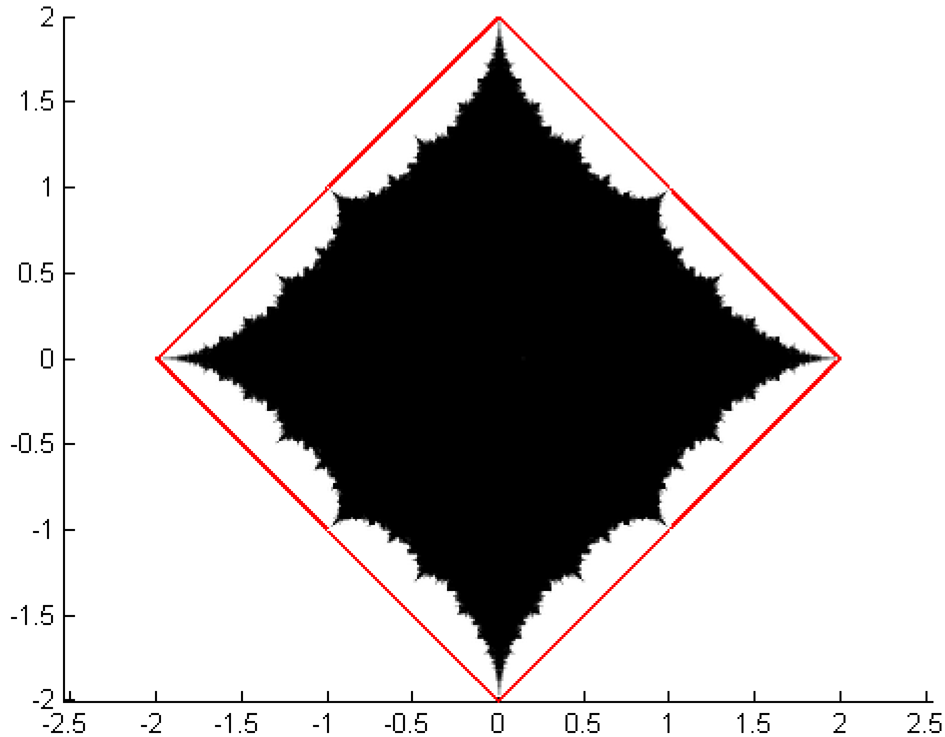

Figure 2: The conjectured spectum of [2] (black) and the boundary of the numerical range (red).

Lemma 24. *Let* $\mathbb{I} = \{1, \ldots, n\}$ *with* $n = \infty$ *if* $\mathbb{I} = \mathbb{N}$ *and let* $A \in \mathcal{L}(\ell^2(\mathbb{I}))$ *be tridiagonal. If there is a constant* $N > \sup_{i \in I} \text{Re } A_{i,i}$ *and a sequence* $(g_i)_{i \in \mathbb{I}}$ *with the following properties*

- $0 \leq g_i \leq 1$ *for all* $i \in \mathbb{I}$,
- *the following inequality holds for all* $1 \leq i \leq n-1$:

$$\frac{\left|A_{i,i+1} + \overline{A_{i+1,i}}\right|^2}{4(N - \text{Re } A_{i,i})(N - \text{Re } A_{i+1,i+1})} \leq g_{i+1}(1 - g_i), \tag{5}$$

then $r_0(A) \leq N$.

Proof. Let $B := \frac{1}{2}(A + A^*)$. Then

$$r_0(A) = \sup_{\|x\|=1} \text{Re } \langle Ax, x \rangle = \sup_{\|x\|=1} \langle \ \frac{1}{2}(A + A^*)x, x \ \rangle = r_0(B).$$

Furthermore we have $\text{Re } A_{i,i} = B_{i,i}$ and $\frac{1}{2}(A_{i,i+1} + \overline{A_{i+1,i}}) = B_{i,i+1}$ for all $i \in \mathbb{I}$ so that condition (5) simplifies to

$$\frac{\left|B_{i,i+1}\right|^2}{(N - B_{i,i})(N - B_{i+1,i+1})} \leq g_{i+1}(1 - g_i).$$

Let T be unitary and diagonal with the following entries:

$$T_{1,1} = 1 \qquad \text{and} \qquad T_{i+1,i+1} = \operatorname{sign}(B_{i,i+1})\, T_{i,i} \quad (i \in \{1, \dots, n-1\}).$$

Then $C := TBT^*$ is real and symmetric with $r_0(C) = r_0(B) = r_0(A)$, $C_{i,i+1} = |B_{i,i+1}| \geq 0$ and $C_{i,i} = B_{i,i}$ for all $i \in \mathbb{I}$. If there is an $i \in \mathbb{I}$ such that $C_{i,i+1} = C_{i+1,i} = 0$ then we can write $C = C_1 \oplus C_2$. But then $r_0(C) = \max\{r_0(C_1), r_0(C_2)\}$ and we can investigate C_1 and C_2 separately. Thus we can assume that $C_{i,i+1} = C_{i+1,i} > 0$ for all $i \in \mathbb{I}$. Furthermore $r_0(C + \lambda I) = r_0(C) + \lambda$ for all $\lambda \in \mathbb{R}$, so shifting the diagonal just shifts N and $r_0(C) = r_0(A)$ by the same amount. Thus we can further assume that C only has positive entries on its three diagonals. With these assumptions C meets the requirements for the Schur Test. Let $(x_i)_{i\in\mathbb{I}}$ be the sequence given by

$$x_1 = 1 \qquad \text{and} \qquad x_{i+1} = \frac{C_{i+1,i}}{(N - C_{i+1,i+1})g_{i+1}} x_i.$$

Then x is a positive sequence and we have

$$\begin{aligned}
(Cx)_i &= C_{i,i-1}x_{i-1}(1-\delta_{i,1}) + C_{i,i}x_i + C_{i,i+1}x_{i+1}(1-\delta_{i,n}) \\
&= C_{i-1,i}\frac{(N - C_{i,i})g_i}{C_{i-1,i}} x_i(1-\delta_{i,1}) + C_{i,i}x_i + C_{i,i+1}\frac{C_{i+1,i}}{(N - C_{i+1,i+1})g_{i+1}} x_i(1-\delta_{i,n}) \\
&\leq ((N - C_{i,i})g_i + C_{i,i} + (N - C_{i,i})(1-g_i))x_i \\
&= Nx_i,
\end{aligned}$$

where we used condition (5) in line 2. The Schur Test then yields $N \geq \|C\| = r_0(C) = r_0(A)$. □

The next lemma is a small extension to [15, Proposition 2] and in some sense the converse of the previous lemma.

Lemma 25. *Let $\mathbb{I} = \{1, .., n\}$ with $n = \infty$ if $\mathbb{I} = \mathbb{N}$ and let $A \in \mathcal{L}(\ell^2(\mathbb{I}))$ be tridiagonal. If $r_0(A) > \sup_{i \in I} \operatorname{Re} A_{i,i}$, then there exists a sequence $(g_i)_{i\in\mathbb{I}}$ with the following properties:*

- $0 \leq g_i \leq 1$ *for all* $i \in \mathbb{I}$
- $g_i = 0$ *if and only if* $i = 1$ *or* $A_{i-1,i} + \overline{A_{i,i-1}} = 0$
- $g_i = 1$ *only if* $i = n$ *or* $A_{i,i+1} + \overline{A_{i+1,i}} = 0$
- *the following equality holds for all* $1 \leq i \leq n-1$*:*

$$\frac{\left|A_{i,i+1} + \overline{A_{i+1,i}}\right|^2}{4(r_0(A) - \operatorname{Re} A_{i,i})(r_0(A) - \operatorname{Re} A_{i+1,i+1})} = g_{i+1}(1 - g_i). \tag{6}$$

Proof. Again we can set $B := \frac{1}{2}(A + A^*)$ and $C := TBT^*$ as above, simplifying property (6) to

$$\frac{C_{i,i+1}^2}{(r_0(C) - C_{i,i})(r_0(C) - C_{i+1,i+1})} = g_{i+1}(1 - g_i) \tag{7}$$

for all $1 \leq i \leq n-1$. Furthermore we can assume again that C only has positive entries on its three diagonals. Let n be finite first. By the theorem of Perron-Frobenius [7], C has a strictly positive eigenvector x (i.e. $x_i > 0$ for all $i \in \{1, \ldots, \}$) corresponding to the eigenvalue $r_0(C)$. Now choose the sequence $(g_i)_{i\in\mathbb{I}}$ as follows:

$$g_1 = 0 \qquad \text{and} \qquad g_{i+1} = \frac{C_{i+1,i}x_i}{(r_0(C) - C_{i+1,i+1})x_{i+1}} \tag{8}$$

for $1 \le i \le n-1$. Thus we have $0 < g_i$ for all $1 < i \le n$. Using

$$C_{i+1,i}x_i + C_{i+1,i+1}x_{i+1} < C_{i+1,i}x_i + C_{i+1,i+1}x_{i+1} + C_{i+1,i+2}x_{i+2} = r_0(C)x_{i+1}$$

we also get $g_i < 1$ for all $1 \le i < n$. Furthermore we have the equality

$$C_{n,n-1}x_{n-1} + C_{n,n}x_n = r_0(C)x_n$$

and thus $g_n = 1$. So there is only property (7) left to prove:

$$\begin{aligned} g_{i+1}(1-g_i) &= \frac{C_{i+1,i}x_i}{(r_0(C) - C_{i+1,i+1})x_{i+1}} \frac{(r_0(C) - C_{i,i})x_i - C_{i,i-1}x_{i-1}(1-\delta_{i,1})}{(r_0(C) - C_{i,i})x_i} \\ &= \frac{C_{i+1,i}}{(r_0(C) - C_{i+1,i+1})x_{i+1}} \frac{C_{i,i+1}x_{i+1}}{r_0(C) - C_{i,i}} \\ &= \frac{C_{i,i+1}^2}{(r_0(C) - C_{i,i})(r_0(C) - C_{i+1,i+1})}. \end{aligned}$$

This proves the lemma in the case $n < \infty$. To prove the lemma also in the case $\mathbb{I} = \mathbb{N}$, we use a finite section approach. Let P_n be the projection onto span $\{e_1, \ldots, e_n\}$ and define $C^{(n)} := P_nCP_n$, interpreted as a finite matrix. Furthermore let $x^{(n)}$ be a positive eigenvector of $C^{(n)}$ corresponding to the eigenvalue $r_0(C^{(n)})$ filled up with zeros such that $(x_i^{(n)})_{i\in\mathbb{N}}$ is an infinite sequence. W.l.o.g. we can assume $x_1^{(n)} = 1$ for all $n \in \mathbb{N}$. Then by Theorem 19 we get

$$r_0(C)x_1^{(n)} \ge r_0(C^{(n)})x_1^{(n)} = C_{1,1}^{(n)}x_1^{(n)} + C_{1,2}^{(n)}x_2^{(n)} = C_{1,1}x_1^{(n)} + C_{1,2}x_2^{(n)}$$

and

$$r_0(C)x_i^{(n)} \ge r_0(C^{(n)})x_i^{(n)} = C_{i,i-1}^{(n)}x_{i-1}^{(n)} + C_{i,i}^{(n)}x_i^{(n)} + C_{i,i+1}^{(n)}x_{i+1}^{(n)} = C_{i,i-1}x_{i-1}^{(n)} + C_{i,i}x_i^{(n)} + C_{i,i+1}x_{i+1}^{(n)}$$

for all $n \in \mathbb{N}$ and $1 < i < n$. Thus all the sequences $(x_i^{(n)})_{n\in\mathbb{N}}$ are bounded by induction (not necessarily uniformly of course). Choose a convergent subsequence $(x_1^{(n_{k_1})})_{k_1\in\mathbb{N}}$ of $(x_1^{(n)})_{n\in\mathbb{N}}$ and denote the limit by x_1 (of course $x_1 = 1$, but never mind). Now if x_i is defined, we choose a convergent subsequence $(x_{i+1}^{(n_{k_{i+1}})})_{k_{i+1}\in\mathbb{N}}$ of $(x_i^{(n_{k_i})})_{k_i\in\mathbb{N}}$ and denote the limit by x_{i+1}. With these choices of sequences we have

$$r_0(C^{n_{k_2}})x_1^{(n_{k_2})} = C_{1,1}^{(n_{k_2})}x_1^{(n_{k_2})} + C_{1,2}^{(n_{k_2})}x_2^{(n_{k_2})} = C_{1,1}x_1^{(n_{k_2})} + C_{1,2}x_2^{(n_{k_2})}$$

and

$$\begin{aligned} r_0(C^{n_{k_{i+1}}})x_i^{(n_{k_{i+1}})} &= C_{i,i-1}^{(n_{k_{i+1}})}x_{i-1}^{(n_{k_{i+1}})} + C_{i,i}^{(n_{k_{i+1}})}x_i^{(n_{k_{i+1}})} + C_{i,i+1}^{(n_{k_{i+1}})}x_{i+1}^{(n_{k_{i+1}})} \\ &= C_{i,i-1}x_{i-1}^{(n_{k_{i+1}})} + C_{i,i}x_i^{(n_{k_{i+1}})} + C_{i,i+1}x_{i+1}^{(n_{k_{i+1}})} \end{aligned}$$

for all $k_i \in \mathbb{N}$ and $1 < i < n_{k_{i+1}}$ and thus

$$C_{1,1}x_1 + C_{1,2}x_2 = r_0(C)x_1$$

and

$$C_{i,i-1}x_{i-1} + C_{i,i}x_i + C_{i,i+1}x_{i+1} = r_0(C)x_i$$

for all $i \in \mathbb{N}$ since $\lim_{k_{i+1}\to\infty} r_0(C^{(n_{k_{i+1}})}) = r_0(C)$. Using $C_{i,i-1} > 0$, $C_{i,i+1} > 0$ and $x_i \geq 0$ for all $i \in \mathbb{N}$, we can conclude that $x_i > 0$ for all $i \in \mathbb{N}$. Now we can define a sequence $(g_i)_{i\in\mathbb{N}}$ as in (8) and the desired properties follow similarly as in the finite-dimensional case. □

Regarding the condition $r_0(A) > \sup_{i\in\mathbb{I}} \operatorname{Re} A_{i,i}$ we have the following remark.

Remark 26. Obviously we always have $r_0(A) \geq \sup_{i\in\mathbb{I}} \operatorname{Re} A_{i,i}$ since $\operatorname{Re} \langle Ae_i, e_i\rangle = \operatorname{Re} A_{i,i}$. So assume $r_0(A) = \operatorname{Re} A_{j,j}$ for some $j \in \mathbb{I}$. Let $B := \frac{1}{2}(A + A^*) + \lambda I$ and $\lambda \in \mathbb{R}$ large enough such that B is positive definite. Then

$$\begin{aligned}
r_0(A) &= r_0(B) - \lambda \\
&= \|B\| - \lambda \\
&\geq \|Be_j\| - \lambda \\
&= \sqrt{|B_{j,j-1}|^2 (1 - \delta_{j,1}) + |B_{j,j}|^2 + |B_{j,j+1}|^2} - \lambda \\
&= \sqrt{|B_{j,j-1}|^2 (1 - \delta_{j,1}) + (r_0(A) + \lambda)^2 + |B_{j,j+1}|^2} - \lambda
\end{aligned}$$

and thus $|B_{j,j-1}|^2 (1 - \delta_{j,1}) = |B_{j,j+1}|^2 = 0$. In other words, B can be written as $B_1 \oplus B_2$, where B_1 is a 1×1-block and

$$r_0(A) = r_0(B) = \max\{r_0(B_1), r_0(B_2)\} = \max\{\operatorname{Re} A_{j,j}, r_0(B_2)\}.$$

Thus the condition $r_0(A) > \sup_{i\in\mathbb{I}} \operatorname{Re} A_{i,i}$ is not really a restriction.

So if we are given a concrete tridiagonal $n \times n$ matrix A and assuming $\frac{1}{2}(A + A^*)$ does not have block structure, we can make an educated guess $N > \sup_{i\in\mathbb{I}} \operatorname{Re} A_{i,i}$ for $r_0(A)$ and start computing the sequence recursively by the prescription

$$g_1 = 0 \qquad \text{and} \qquad g_{i+1} = \frac{\left|A_{i,i+1} + \overline{A_{i+1,i}}\right|^2}{4(1 - g_i)(N - \operatorname{Re} A_{i,i})(N - \operatorname{Re} A_{i+1,i+1})}$$

for $1 \leq i \leq n - 1$. Depending on the outcome of the sequence, we can decide whether $N < r_0(A)$, $N = r_0(A)$ or $N > r_0(A)$:

- if $g_i > 1$ (including ∞) for some $1 \leq i \leq n$, we have $N < r_0(A)$,
- if $g_i \leq 1$ for all $1 \leq i \leq n$, we have $N \geq r_0(A)$,
- if $g_i \leq 1$ for all $1 \leq i \leq n$ and $g_n = 1$, we have $N = r_0(A)$.

While the first two assertions are quite obvious from Lemma 24 and Lemma 25, the third assertion needs some clarification. So assume we got a sequence $(g_i)_{i\in\mathbb{I}}$ with $g_i \leq 1$ for all $1 \leq i \leq n$ and $g_n = 1$. By Lemma 24, we get that $r_0(A) \leq N$. Assume $r_0(A) < N$. By Lemma 25, we then get another sequence (h_i) with the properties $h_1 = 0$, $0 < h_i < 1$ for $1 < i < n$, $0 < h_n \leq 1$ and

$$\frac{\left|A_{i,i+1} + \overline{A_{i+1,i}}\right|^2}{4(r_0(A) - \text{Re } A_{i,i})(r_0(A) - \text{Re } A_{i+1,i+1})} = h_{i+1}(1 - h_i).$$

But since

$$\frac{\left|A_{i,i+1} + \overline{A_{i+1,i}}\right|^2}{4(r_0(A) - \text{Re } A_{i,i})(r_0(A) - \text{Re } A_{i+1,i+1})} > \frac{\left|A_{i,i+1} + \overline{A_{i+1,i}}\right|^2}{4(N - \text{Re } A_{i,i})(N - \text{Re } A_{i+1,i+1})}$$

we have $h_i > g_i$ for all $1 < i \leq n$. Thus $h_n > 1$, a contradiction.

Using bisection, we could then narrow down the numerical abscissa $r_0(A)$ as precise as we want although this may become a bit unstable, because a small change in N can have huge effects on g_n. Then we could go on and compute the numerical abscissae $r_\varphi(A)$ for a bunch of angles and estimate the numerical range. Of course if the matrix $\frac{1}{2}(A + A^*)$ has block structure, we can just compute the numerical abscissa of every block and take the maximum.

Although this algorithm actually works and is not the most inefficient one to compute a numerical range, it is not the best use of Lemma 24 and Lemma 25. What we are really interested in are infinite matrices and in particular pseudo-ergodic operators. Of course, this algorithm cannot quite work for infinite matrices, not to mention the question on how to actually implement an infinite matrix... Thus we have to be a bit smarter and that is where a sowewhat combinatorical argument comes in. To demonstrate this, we are considering a (two-sided) pseudo-ergodic operator A with $U_{-1} = \{1\}$, $U_0 = \{0\}$ and $U_1 = \{\pm\sigma\}$, where $0 < \sigma \leq 1$. Of course, we already know the numerical range of A by Theorem 11 and therefore we rather want to compute $N(A^2)$. A^2 is of course not tridiagonal, but we can use the special structure of A^2 to make it tridiagonal. Also note that A^2 is not pseudo-ergodic. Let $B = A^2$ and let $A_{i+1,i} =: h_i$. Then B is given by

$$\begin{aligned}
B_{i,i+2} &= A_{i,i+1}A_{i+1,i+2} = 1,\\
B_{i,i+1} &= A_{i,i+1}A_{i+1,i+1} + A_{i,i}A_{i,i+1} = 0,\\
B_{i,i} &= A_{i,i+1}A_{i+1,i} + A_{i,i}A_{i,i} + A_{i,i-1}A_{i-1,i} = h_i + h_{i-1},\\
B_{i,i-1} &= A_{i,i}A_{i,i-1} + A_{i,i-1}A_{i-1,i-1} = 0,\\
B_{i,i-2} &= A_{i,i-1}A_{i-1,i-2} = h_{i-1}h_{i-2}
\end{aligned}$$

for all $i \in \mathbb{Z}$. Thus using the decomposition

$$\mathbf{X} = \{(x_i)_{i\in\mathbb{I}} : x_{2j} = 0 \quad \forall j \in \mathbb{I}\} \oplus \{(x_i)_{i\in\mathbb{I}} : x_{2j-1} = 0 \quad \forall j \in \mathbb{I}\},$$

B can be divided into two tridiagonal operators $B = C \oplus D$, where C is given by

$$\begin{aligned}
C_{i,i+1} &= 1,\\
C_{i,i} &= h_{2i} + h_{2i-1},\\
C_{i,i-1} &= h_{2i-1}h_{2i-2}
\end{aligned} \tag{9}$$

for all $i \in \mathbb{Z}$. Although C is not pseudo-ergodic, the sequences $(C_{i,i})_{i\in\mathbb{Z}}$ and $(C_{i,i-1})_{i\in\mathbb{Z}}$ are pseudo-ergodic because $(h_i)_{i\in\mathbb{Z}}$ is. Thus we have $C \in \sigma^{\text{op}}(C)$ by Proposition 3 applied to the diagonals separately. Similarly we also have $D \in \sigma^{\mathrm{op}}(C)$ and vice versa. In particular we get $N(C) = N(D)$ by Theorem 13. Therefore it suffices to consider C from now on. Furthermore since the numerical range of $B = A^2$ does not depend on the choice of $(h_i)_{i\in\mathbb{Z}}$ as long as it is pseudo-ergodic, we can choose $(h_i)_{i\in\mathbb{Z}}$ in such a way that $(h_i)_{i\in\mathbb{N}}$ is still pseudo-ergodic. This implies $N(C) \subset N(C_+) := N(P_{\mathbb{N}} C P_{\mathbb{N}})$ by Proposition 3 and Theorem 13 again. Theorem 19 then yields $N(C_+) = N(C) = N(B) = N(A^2)$. Note that the last step is not really necessary, but it simplifies the proof a bit, since we only have to think in one direction instead of two. Next we need an educated guess (and looking back, it really was a guess) for $r_\varphi(C_+)$ in order to use Lemma 24.

Theorem 27. *Let $\sigma \in (0,1]$, $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm\sigma\}$ and $A \in \Psi E(U_{-1}, U_0, U_1)$. Then*

$$N(A^2) = \operatorname{conv}\left(\bigcup_{B \in M_{per,4}(U_{-1},U_0,U_1)} N(B^2) \right).$$

That the right-hand side is a subset of the left-hand side is obvious by Corollary 15 and the fact (see Proposition 1) that $\sigma^{\mathrm{op}}(B^2) = \sigma^{\mathrm{op}}(B)^2$. To prove the other inclusion, we want to concentrate on one particular case first in order to illustrate the spirit of the argument. The full proof can be found in the appendix. The special case we want to consider is the following:

Proposition 28. *Let $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm 1\}$ and $A \in \Psi E(U_{-1}, U_0, U_1)$. Furthermore let $B_1 \in M_{per,4}(U_{-1}, U_0, U_1)$ be an operator with period 1,1,1,1 and $B_2 \in M_{per,4}(U_{-1}, U_0, U_1)$ an operator with period -1,-1,1,1. Then*

$$r_{\frac{\pi}{3}}(A^2) = r_{\frac{\pi}{3}}(B_1^2) = r_{\frac{\pi}{3}}(B_2^2) = 2.$$

Proof. $r_{\frac{\pi}{3}}(B_1^2) = r_{\frac{\pi}{3}}(B_2^2) = 2$ is easy to see as $N(B_1^2) = [0,4]$ and $N(B_2^2) = \{z \in \mathbb{C} : |z| \le 2\}$ (see Proposition A.1 in the appendix). Thus by Corollary 15 it suffices to show $r_{\frac{\pi}{3}}(A^2) \le 2$.

Let $h_i := A_{i,i+1}$ and assume w.l.o.g. that $(h_i)_{i\in\mathbb{N}}$ is pseudo-ergodic. Let C be as in (9) and define $E := e^{\frac{i\pi}{3}} C_+$. Then $r_{\frac{\pi}{3}}(A^2) = r_0(E)$. Further define

$$\eta_i := \frac{\left|E_{i,i+1} + \overline{E_{i+1,i}}\right|^2}{4(2 - \operatorname{Re} E_{i,i})(2 - \operatorname{Re} E_{i+1,i+1})}$$

for $i \in \mathbb{N}$. Since

$$\operatorname{Re} E_{i,i} = \cos(\frac{\pi}{3})(h_{2i} + h_{2i-1}) = \frac{1}{2}(h_{2i} + h_{2i-1}) \le 1$$

for all $i \in \mathbb{N}$, we only have to find a sequence $(g_i)_{i\in\mathbb{N}}$ with

- $0 \le g_i \le 1$,
- $\eta_i \le g_{i+1}(1 - g_i)$

for all $i \in \mathbb{N}$ in order to apply Lemma 24. Note that η_i only depends on h_{2i-1}, h_{2i}, h_{2i+1} and h_{2i+2}. Thus η_i can take 16 different values:

type	$(h_{2i-1}, h_{2i}, h_{2i+1}, h_{2i+2})$	η_i
(1)	$(1,1,1,1)$	$\frac{1}{4}$
(2)	$(1,1,1,-1)$	$\frac{1}{8}$
(3)	$(1,1,-1,1)$	$\frac{3}{8}$
(4)	$(1,1,-1,-1)$	$\frac{1}{4}$
(5)	$(1,-1,1,1)$	$\frac{3}{8}$
(6)	$(1,-1,1,-1)$	$\frac{3}{16}$
(7)	$(1,-1,-1,1)$	$\frac{1}{16}$
(8)	$(1,-1,-1,-1)$	$\frac{1}{24}$
(9)	$(-1,1,1,1)$	$\frac{1}{8}$
(10)	$(-1,1,1,-1)$	$\frac{1}{16}$
(11)	$(-1,1,-1,1)$	$\frac{3}{16}$
(12)	$(-1,1,-1,-1)$	$\frac{1}{8}$
(13)	$(-1,-1,1,1)$	$\frac{1}{4}$
(14)	$(-1,-1,1,-1)$	$\frac{1}{8}$
(15)	$(-1,-1,-1,1)$	$\frac{1}{24}$
(16)	$(-1,-1,-1,-1)$	$\frac{1}{36}$

Table 1

Let us say that η_i is of type (n) if it is generated by the sequence $(h_{2i-1}, h_{2i}, h_{2i+1}, h_{2i+2})$ listed under (n), e.g. we say that η_i is of type (1) if it is generated by (1,1,1,1) etc. Now the idea is the following. If there were no η_i of type (3) or (5), the sequence $(g_i)_{i\in\mathbb{N}} = \left(\frac{1}{2}\right)_{i\in\mathbb{N}}$ would suffice the conditions. We take this sequence as a starting point and change it appropriately when necessary. Let us start with type (3). So assume η_i is of type (3). Then η_{i+1} can only be of type (9), (10),

(11) or (12). We change g_{i+1} to $\frac{3}{4}$. This implies

$$\eta_i = \frac{3}{8} = \frac{3}{4}\left(1 - \frac{1}{2}\right) = g_{i+1}(1 - g_i).$$

Now if η_{i+1} is of type (9), (10) or (12), we have

$$\eta_{i+1} \leq \frac{1}{8} = \frac{1}{2}\left(1 - \frac{3}{4}\right) = g_{i+2}(1 - g_{i+1}).$$

If η_{i+1} is of type (11), we change g_{i+2} to $\frac{3}{4}$ and get

$$\eta_{i+1} = \frac{3}{16} = \frac{3}{4}\left(1 - \frac{3}{4}\right) = g_{i+2}(1 - g_{i+1}).$$

Taking a closer look at type (11), we see that in this case $h_{2i+1} = h_{2i+3}$ and $h_{2i+2} = h_{2i+4}$ and thus we are just in the same situation as before, i.e. η_{i+2} can only be of type (9), (10), (11) or (12) again. Repeating this argument, we either end up with $g_j = \frac{3}{4}$ for all $j \geq i+1$, which is fine, or we eventually go out with $g_j = \frac{1}{2}$ for some $j \geq i+2$. Thus the case of type (3) is solved.

Now assume η_i is of type (5). Comparing type (3) and (5), it seems reasonable to do the same again but backwards. In this case η_{i-1} is of type (2), (6), (10) or (14). We change g_i to $\frac{1}{4}$. This implies

$$\eta_i = \frac{3}{8} = \frac{1}{2}\left(1 - \frac{1}{4}\right) = g_{i+1}(1 - g_i).$$

Now if η_{i-1} is of type (2), (10) or (14), we have

$$\eta_{i-1} \leq \frac{1}{8} = \frac{1}{4}\left(1 - \frac{1}{2}\right) = g_i(1 - g_{i-1}).$$

If η_{i+1} is of type (6), we change g_{i-1} to $\frac{1}{4}$ and get

$$\eta_{i-1} = \frac{3}{16} = \frac{1}{4}\left(1 - \frac{1}{4}\right) = g_i(1 - g_{i-1}).$$

Repeating this argument as before, we either end up with $g_j = \frac{1}{4}$ for all $j \leq i$, which is fine, or we eventually go out with $g_j = \frac{1}{2}$ for some $j \leq i-1$. Thus the case of type (5) is solved as well. Almost. One last thing that we have to check is that the changes in these two cases do not contradict each other. Comparing the two cases, this can only happen if η_i is of type (3) or (11), η_{i+1} is of type (10) and η_{i+2} is of type (5) or (6). But fortunately we have some space here. Doing the changes above, we get $g_{i+1} = \frac{3}{4}$ and $g_{i+2} = \frac{1}{4}$. Thus

$$\eta_{i+1} = \frac{1}{16} = \frac{1}{4}\left(1 - \frac{3}{4}\right) = g_{i+2}(1 - g_{i+1})$$

and the changes do indeed not contradict each other. Applying Lemma 24 finishes the proof. □

Theorem 27 has some important consequences. In particular it yields another upper bound to the spectrum of these pseudo-ergodic operators. Indeed, we have

$$\mathrm{sp}(A)^2 = \mathrm{sp}(A^2) \subset N(A^2)$$

and thus

$$\mathrm{sp}(A) \subset \left\{ z \in \mathbb{C} : z^2 \in N(A^2) \right\}.$$

We want to illustrate this at the example of the Feinberg-Zee hopping sign model [6]. Let $U_{-1} = \{1\}$, $U_0 = \{0\}$ and $U_1 = \{\pm 1\}$ and consider $A \in \Psi E(U_{-1}, U_0, U_1)$. Then by Theorem 27 the numerical range of A^2 is given by

$$N(A^2) = \mathrm{conv}\left(\bigcup_{B \in M_{per,4}(U_{-1},U_0,U_1)} N(B^2) \right).$$

The right hand side is illustrated in Figure 3. The square root of $N(A^2)$ as well as $N(A)$ and the best known lower bound of $\mathrm{sp}(A)$ are illustrated in Figure 4.

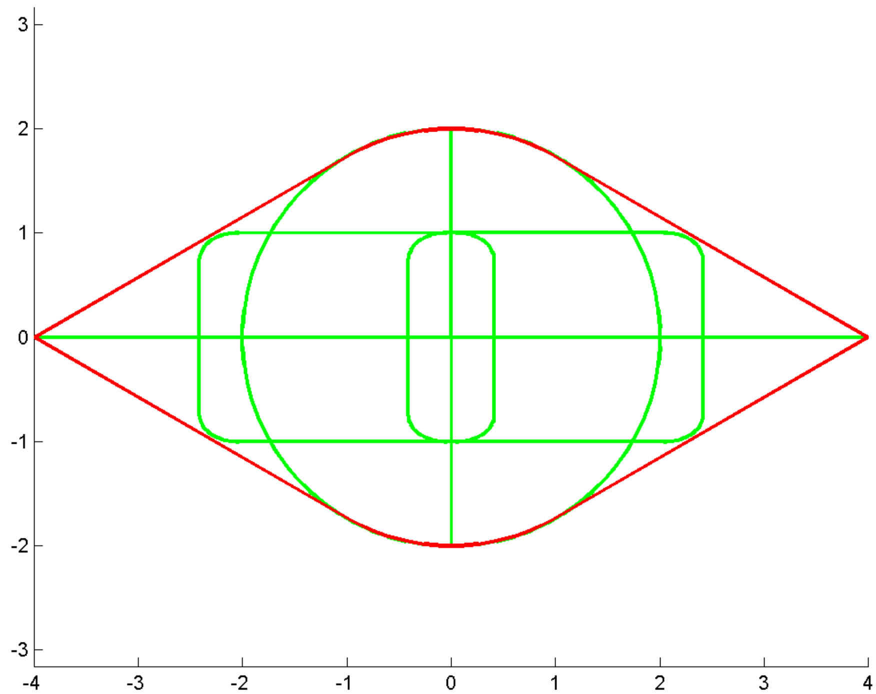

Figure 3: The boundaries of the numerical ranges of all 4-periodic operators (green) and their convex hull (red).

Let us also mention the following proposition which is used to prove Theorem 27 (see Appendix), but which is also of independent interest. It provides an easy formula for the numerical abscissa (and thus for the numerical range) of 2-periodic operators.

Proposition 29. *Let $A \in \mathcal{L}(\mathbf{X})$ be tridiagonal and 2-periodic and let $N, r_0(A) > \sup_{i \in I} \mathrm{Re}\, A_{i,i}$. Define*

$$\eta_1(A) := \frac{\left|A_{1,2} + \overline{A_{2,1}}\right|^2}{4(N - \mathrm{Re}\, A_{1,1})(N - \mathrm{Re}\, A_{2,2})}, \qquad \eta_2(A) := \frac{\left|A_{2,3} + \overline{A_{3,2}}\right|^2}{4(N - \mathrm{Re}\, A_{2,2})(N - \mathrm{Re}\, A_{3,3})}.$$

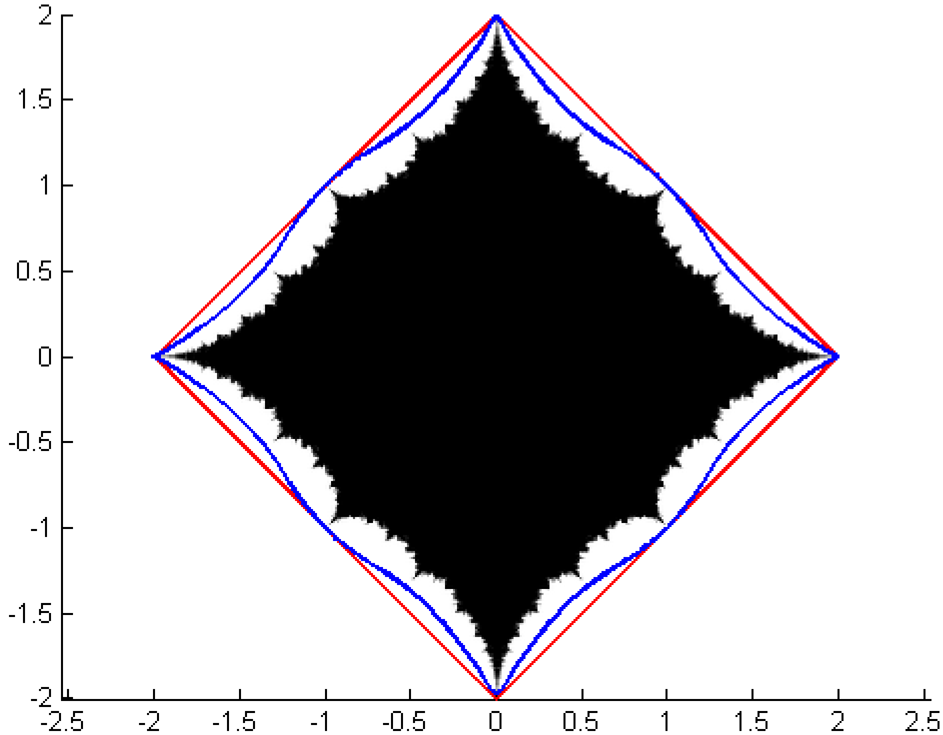

Figure 4: The boundary of the square root of $N(A^2)$ (blue), the boundary of $N(A)$ (red) and a lower bound to $\mathrm{sp}(A)$ consisting of spectra of periodic operators (black).

Then we have $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ *if and only if* $N = r_0(A)$.

In a similar way, one can also get formulas for 3- and 4-periodic operators.

4 Further Remarks and Open Problems

In Theorem 27 we gave a viable formula for the numerical range of A^2 where A is a very specific operator, i.e. we have

$$N(A^2) = \operatorname{conv}\left(\bigcup_{B \in M_{per,4}(U_{-1},U_0,U_1)} N(B^2) \right)$$

if $A \in \Psi E(U_{-1}, U_0, U_1)$, $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm\sigma\}$ and $\sigma \in (0, 1]$. This can easily be generalized to arbitrary $\sigma \in \mathbb{C}$ by a suitable similarity transformation, scaling, rotation and transposition (while $\sigma = 0$ is trivial). Also, by the same arguments as used in the proof of Theorem 27, the formula still holds if we assume $U_1 = \left\{\sigma e^{\frac{2\pi i k}{n}} : 0 \leq k \leq n-1\right\}$ and $n \in \mathbb{N}$. However, the calculations get much more tedious than in the case $n = 2$ discussed here. Moreover, a numerical analysis of several operators in $\Psi E(U_{-1}, \{0\}, U_1)$, with U_{-1} and U_1 arbitrary but compact, suggests that Theorem 27 holds in much more generality. In fact, we have not found any counterexamples yet and therefore we conjecture that Theorem 27 holds for all $A \in \Psi E(U_{-1}, \{0\}, U_1)$, where U_{-1} and U_1 are arbitrary but compact. However, the computational method introduced in section 3 is very situational and probably not suited to prove this theorem in full generality because the computations get more and more tedious the more parameters we introduce. Thus a more direct proof of Theorem 27 would be appreciated.

If we only allow real entries, the story is a bit easier. For simplicity let $U_{-1} = \{1\}$ for the moment and as always let $U_0 = \{0\}$. Then it is not hard to show that

$$\operatorname{conv}\left(\bigcup_{B \in M_{per,4}(U_{-1}, U_0, U_1)} N(B^2)\right)$$

only depends on $m := \min_{u_1 \in U_1} u_1$ and $M := \max_{u_1 \in U_1} u_1$. Furthermore if $|m + M| \geq 2$, then Theorem 27 trivially holds. The reason for this is roughly speaking that most of the numerical ranges are contained in each other in the real case. Thus in the real case Theorem 27 can probably be proved by the method introduced in section 3. However, a direct proof would again be more favourable.

Beside these generalizations, a natural question is whether we can get similar results for A^4, A^8 and so on. A closely related question is also whether one can do something similar in the non-tridiagonal case or in other words, what can be said if we allow matrix or even operator entries instead of complex numbers. Some numerical computations suggest that such an approach might be possible. However, we have also seen that Theorem 11, for example, fails in the non-tridiagonal case. Numerical considerations also suggest that the numerical ranges of A^4, A^8 and so on yield better and better upper bounds to the spectrum of $A \in \Psi E(\{1\}, \{0\}, \{\pm 1\})$ and this might also explain the somewhat fractal structure in Figure 4. However, this is only speculation and should be treated with caution.

A Appendix

Here we give a full proof of Theorem 27.

Theorem 27. *Let $\sigma \in (0, 1]$, $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm\sigma\}$ and $A \in \Psi E(U_{-1}, U_0, U_1)$. Then*

$$N(A^2) = \operatorname{conv}\left(\bigcup_{B \in M_{per,4}(U_{-1}, U_0, U_1)} N(B^2)\right).$$

That the right-hand side is a subset of the left-hand side is obvious by Corollary 15 and the fact (see Proposition 1) that $\sigma^{\mathrm{op}}(B^2) = \sigma^{\mathrm{op}}(B)^2$. To prove the other inclusion, we need to compute $N(B^2)$ for all $B \in M_{per,4}(U_{-1}, U_0, U_1)$ first. However, it turns out that it suffices to consider three of them.

Proposition A.1. *Let $\sigma \in (0, 1]$, $U_{-1} = \{1\}$, $U_0 = \{0\}$ and $U_1 = \{\pm\sigma\}$. Furthermore let $B_1 \in M_{per,4}(U_{-1}, U_0, U_1)$ be an operator with period $\sigma, \sigma, \sigma, \sigma$, $B_2 \in M_{per,4}(U_{-1}, U_0, U_1)$ an operator with period $-\sigma, -\sigma, \sigma, \sigma$ and $B_3 \in M_{per,4}(U_{-1}, U_0, U_1)$ an operator with period $-\sigma, -\sigma, -\sigma, -\sigma$.*

Then we have

$$r_\varphi(B_1^2) = 2\sigma\cos(\varphi) + \sqrt{(1+\sigma^2)^2\cos(\varphi)^2 + (1-\sigma^2)^2\sin(\varphi)^2},$$
$$r_\varphi(B_2^2) = 1+\sigma^2,$$
$$r_\varphi(B_3^2) = -2\sigma\cos(\varphi) + \sqrt{(1+\sigma^2)^2\cos(\varphi)^2 + (1-\sigma^2)^2\sin(\varphi)^2}$$

and the boundaries of $N(B_1^2)$ and $N(B_2^2)$ are given by the following parametrizations:

$$\partial N(B_1^2) : z(t) = 2\sigma + (1+\sigma^2)\cos(t) + i(1-\sigma^2)\sin(t),$$
$$\partial N(B_2^2) : z(t) = (1+\sigma^2)e^{it},$$
$$\partial N(B_3^2) : z(t) = -2\sigma + (1+\sigma^2)\cos(t) + i(1-\sigma^2)\sin(t).$$

Proof. B_1 is a Laurent operator with diagonals $(1)_{i\in\mathbb{Z}}$, $(0)_{i\in\mathbb{Z}}$ and $(\sigma)_{i\in\mathbb{Z}}$ and therefore B_1^2 is a Laurent operator with diagonals $(1)_{i\in\mathbb{Z}}$, $(0)_{i\in\mathbb{Z}}$, $(2\sigma)_{i\in\mathbb{Z}}$, $(0)_{i\in\mathbb{Z}}$ and $(\sigma^2)_{i\in\mathbb{Z}}$. Therefore the spectrum of B_1^2 is given by the ellipse $E := \{t \in [0, 2\pi) : 2\sigma + (1+\sigma^2)\cos(t) + i(1-\sigma^2)\sin(t)\}$ (see e.g. [14] or use Theorem 6 with $p = 1$). Also, Laurent operators are normal and therefore E also defines the boundary of the numerical range of B_1^2. An elementary computation yields $r_\varphi(B_1^2) = 2\sigma\cos(\varphi) + \sqrt{(1+\sigma^2)^2\cos(\varphi)^2 + (1-\sigma^2)^2\sin(t)^2}$.

B_2^2 is a 4-periodic operator that looks like this:

$$B_2^2 = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & \sigma^2 & 0 & 0 & 0 & 1 & & & & \\ & & -\sigma^2 & 0 & -2\sigma & 0 & 1 & & & \\ & & & \sigma^2 & 0 & 0 & 0 & 1 & & \\ & & & & -\sigma^2 & 0 & 2\sigma & 0 & 1 & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{10}$$

Like we did several times before, we can split B_2^2 into even (C_2) and odd (D_2) part:

$$C_2 = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & \sigma^2 & 0 & 1 & \\ & & \sigma^2 & 0 & 1 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad D_2 = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & -\sigma^2 & -2\sigma & 1 & \\ & & -\sigma^2 & 2\sigma & 1 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We see that C_2 is a Laurent operator and similarly as before we conclude that the boundary of the numerical range of C_2 is given by the ellipse $\{t \in [0, 2\pi) : (1+\sigma^2)\cos(t) + i(1-\sigma^2)\sin(t)\}$. However, D_2 is not a Laurent operator and thus it is a bit harder to compute the numerical range of D_2. We could use Theorem 6, but this calculation is quite tedious. Moreover, these 2-periodic operators come up quite a lot. Thus we rather want to cover all 2-periodic operators at once. And indeed, this is possible and will be Proposition A.2. Let $D_{2,\varphi} := e^{i\varphi}D_2$, $N := 1+\sigma^2$ and let us exclude the cases $(\sigma, \varphi) = (1, 0)$ and $(\sigma, \varphi) = (1, \pi)$ for the moment so that $\frac{1}{2}(D_{2,\varphi} + D_{2,\varphi}^*)$ does not have block structure (compare with Remark 26). In the notation of Proposition A.2, $\eta_1(D_{2,\varphi})$

and $\eta_2(D_{2,\varphi})$ are given by

$$\begin{aligned}\eta_1(D_{2,\varphi}) &= \frac{\left|e^{i\varphi} - \sigma^2 e^{-i\varphi}\right|^2}{4(1+\sigma^2+2\sigma\cos(\varphi))(1+\sigma^2-2\sigma\cos(\varphi))} \\ &= \frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{4((1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2)} \\ &= \frac{1}{4} \\ \eta_2(D_{2,\varphi}) &= \eta_1(D_{2,\varphi}) \\ &= \frac{1}{4}.\end{aligned}$$

Thus $\sqrt{\eta_1(D_{2,\varphi})} + \sqrt{\eta_2(D_{2,\varphi})} = 1$ and by Proposition A.2, $r_\varphi(D_2) = 1 + \sigma^2$ for all $\varphi \in [0, 2\pi)$ ($(\sigma, \varphi) \notin \{(1,0),(1,\pi)\}$). In the remaining two cases, $\frac{1}{2}(D_{2,\varphi} + D_{2,\varphi}^*)$ is a diagonal matrix and thus it is easily seen that $r_\varphi(D_2) = 2$ holds. Therefore we have $r_\varphi(D_2) = 1 + \sigma^2$ for all $\varphi \in [0, 2\pi)$. Now obviously $N(C_2) \subset N(D_2)$ holds and thus we get $r_\varphi(B_2^2) = 1 + \sigma^2$ for all $\varphi \in [0, 2\pi)$. A parametrization of $\partial N(B_2^2)$ is then of course given by $z(t) = (1+\sigma^2)e^{it}, t \in [0, 2\pi)$.

B_3 is the same as B_1 just with σ replaced by $-\sigma$. □

Proposition A.2. *Let $A \in \mathcal{L}(\mathbf{X})$ be tridiagonal and 2-periodic and let $N, r_0(A) > \sup_{i\in I} \operatorname{Re} A_{i,i}$. Define*

$$\eta_1(A) := \frac{\left|A_{1,2} + \overline{A_{2,1}}\right|^2}{4(N - \operatorname{Re} A_{1,1})(N - \operatorname{Re} A_{2,2})}, \qquad \eta_2(A) := \frac{\left|A_{2,3} + \overline{A_{3,2}}\right|^2}{4(N - \operatorname{Re} A_{2,2})(N - \operatorname{Re} A_{3,3})}.$$

Then we have $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ if and only if $N = r_0(A)$.

Proof. Let $N = r_0(A)$ and assume first that $A_{i,i+1} + \overline{A_{i+1,i}} \neq 0$ for all $i \in \mathbb{I}$. Note that for periodic operators we have $N(P_\mathbb{N} A P_\mathbb{N}) = N(A)$ by Theorem 13 and Theorem 19. Thus w.l.o.g. we can assume that $\mathbb{I} = \mathbb{N}$. In this case Lemma 25 yields a sequence with the properties

- $0 \leq g_i \leq 1$ for all $i \in \mathbb{N}$
- $g_i = 0$ if and only if $i = 1$
- the following equality holds for all $i \in \mathbb{N}$:

$$\frac{\left|A_{i,i+1} + \overline{A_{i+1,i}}\right|^2}{4(r_0(A) - \operatorname{Re} A_{i,i})(r_0(A) - \operatorname{Re} A_{i+1,i+1})} = g_{i+1}(1 - g_i).$$

Since A is 2-periodic, we have

$$\begin{aligned}
\eta_1(A) &= \frac{\left|A_{1,2}+\overline{A_{2,1}}\right|^2}{4(r_0(A)-\text{Re }A_{1,1})(r_0(A)-\text{Re }A_{2,2})} = g_2\\
\eta_2(A) &= \frac{\left|A_{2,3}+\overline{A_{3,2}}\right|^2}{4(r_0(A)-\text{Re }A_{2,2})(r_0(A)-\text{Re }A_{1,1})} = g_3(1-g_2)\\
\eta_1(A) &= \frac{\left|A_{1,2}+\overline{A_{2,1}}\right|^2}{4(r_0(A)-\text{Re }A_{1,1})(r_0(A)-\text{Re }A_{2,2})} = g_4(1-g_3)\\
&\vdots \qquad\qquad\qquad\qquad \ddots
\end{aligned}$$

If i is odd, we deduce

$$g_{i+2} = \frac{\eta_2(A)}{1-\frac{\eta_1(A)}{1-g_i}} = \frac{(1-g_i)\eta_2(A)}{1-g_i-\eta_1(A)}.$$

The corresponding iteration function

$$\begin{aligned}
&f : (0, 1-\eta_1(A)) \to \mathbb{R},\\
&x \mapsto \frac{(1-x)\eta_2(A)}{1-x-\eta_1(A)}
\end{aligned} \tag{11}$$

has a positive derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{(1-x)\eta_2(A)}{1-x-\eta_1(A)} = \frac{\eta_1(A)\eta_2(A)}{(1-x-\eta_1(A))^2} > 0, \tag{12}$$

since $\eta_1(A), \eta_2(A) \neq 0$ by assumption. Thus f is strictly increasing. Moreover, we have $g_3 > 0 = g_1$ and thus $(g_i)_{i\in 2\mathbb{N}-1}$ is strictly increasing, too. Furthermore $(g_i)_{i\in 2\mathbb{N}-1}$ is bounded by 1 and therefore convergent. Denote the limit of this sequence by x^*. By the fixed-point theorem, x^* has to be a fixed point of the iteration function f. After some rearranging, we get two possible candidates for a fixed point:

$$\begin{aligned}
\frac{(1-x^*)\eta_2(A)}{1-x^*-\eta_1(A)} = x^* \quad &\Leftrightarrow \quad (1-x^*)\eta_2(A) = x^*(1-x^*-\eta_1(A))\\
&\Leftrightarrow \quad (x^*)^2 - (1+\eta_2(A)-\eta_1(A))x^* + \eta_2(A) = 0\\
&\Leftrightarrow \quad x^* = \frac{1+\eta_2(A)-\eta_1(A) \pm \sqrt{(1+\eta_2(A)-\eta_1(A))^2-4\eta_2(A)}}{2}.
\end{aligned} \tag{13}$$

Of course the fixed point we are looking for has to be real and thus $(1+\eta_2(A)-\eta_1(A))^2 - 4\eta_2(A)$ has to be non-negative. It follows

$$\begin{aligned}
0 &\leq (1+\eta_2(A)-\eta_1(A))^2 - 4\eta_2(A)\\
&= 1+\eta_2(A)^2+\eta_1(A)^2+2\eta_2(A)-2\eta_1(A)-2\eta_1(A)\eta_2(A)-4\eta_2(A)\\
&= \eta_2(A)^2 - 2(1+\eta_1(A))\eta_2(A) + (1-\eta_1(A))^2.
\end{aligned}$$

Solving for $\eta_2(A)$ yields

$$\eta_2(A) \leq 1+\eta_1(A) - \sqrt{(1+\eta_1(A))^2-(1-\eta_1(A))^2} = 1+\eta_1(A)-2\sqrt{\eta_1(A)} = (1-\sqrt{\eta_1(A)})^2,$$

since $\eta_2(A) = g_3(1 - g_2) < 1$. This inequality now implies $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} \leq 1$. That this inequality is actually an equality, we will prove later.

Now assume $A_{1,2} + \overline{A_{2,1}} = 0$. Then $\eta_1(A) = g_2 = 0$ and $\eta_2(A) = g_3(1 - g_2) = g_3 \leq 1$. Thus also in this case $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} \leq 1$. The other case (i.e. $A_{2,3} + \overline{A_{3,2}} = 0$) is of course similar.

Conversely, let $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$. Of course, we can again assume that $\mathbb{I} = \mathbb{N}$. Define the sequence $(g_i)_{i \in \mathbb{N}}$ as follows:

$$
\begin{aligned}
g_1 &:= 0, \\
g_{i+1} &:= \frac{\eta_1(A)}{1 - g_i} \quad \text{if } i \text{ is odd,} \\
g_{i+1} &:= \frac{\eta_2(A)}{1 - g_i} \quad \text{if } i \text{ is even,}
\end{aligned}
$$

where $\frac{0}{0}$ is treated as 0. In order to use Lemma 24, we have to check $g_i \leq 1$ for all $i \in \mathbb{N}$. The other conditions will follow immediately. Let us consider $(g_i)_{i \in 2\mathbb{N}-1}$ and its iteration function (11) first. As seen in (13) the fixed points of f are given by

$$
x^* = \frac{1 + \eta_2(A) - \eta_1(A) \pm \sqrt{(1 + \eta_2(A) - \eta_1(A))^2 - 4\eta_2(A)}}{2}.
$$

Plugging our assumption $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ into this equation, we get

$$
\begin{aligned}
x^* &= \frac{1 + \eta_2(A) - (1 - \sqrt{\eta_2(A)})^2 \pm \sqrt{(1 + \eta_2(A) - (1 - \sqrt{\eta_2(A)})^2)^2 - 4\eta_2(A)}}{2} \\
&= \sqrt{\eta_2(A)} \pm \frac{\sqrt{4\eta_2(A) - 4\eta_2(A)}}{2} \\
&= \sqrt{\eta_2(A)}.
\end{aligned}
$$

Thus there is only one fixed point and in particular $x^* \leq 1$. If $x^* = 1$, then $\sqrt{\eta_1(A)}$ has to vanish. But this means that $g_i = 0$ for all $i \in 2\mathbb{N}$ and consequently $g_i = \eta_2(A) = 1$ for all $i \in 2\mathbb{N}+1$. So let $x^* < 1$. By (12), we have that the iteration function f is strictly increasing for $0 < x < 1 - \eta_1(A)$, while

$$
1 - \eta_1(A) = 1 - (1 - \sqrt{\eta_2(A)})^2 = 2\sqrt{\eta_2(A)} - \eta_2(A) > \sqrt{\eta_2(A)},
$$

since $\eta_2(A) < 1$. Furthermore $g_1 = 0$ and thus $g_i \leq x^* < 1$ for all $i \in 2\mathbb{N}-1$. We conclude $g_i \leq 1$ for i odd. Similarly (exchanging $\eta_1(A)$ and $\eta_2(A)$ and using the starting point $\eta_1(A) < \sqrt{\eta_1(A)} < 1$), we get $g_i \leq 1$ for i even. Furthermore we have $g_i \geq 0$ for $i \in \mathbb{N}$ and inequality (5) is fulfilled by definition. Thus $(g_i)_{i \in \mathbb{N}}$ meets all the requirements and we can use Lemma 24, which implies $r_0(A) \leq N$. So let us summarize what we have so far. We have

(i) $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} \leq 1$ if $N = r_0(A)$ and

(ii) $N \geq r_0(A)$ if $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$.

Now let $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$, $r_0(A) < N$ and assume $\max_{i \in \{1,2\}} \left| A_{i,i+1} + \overline{A_{i+1,i}} \right| \neq 0$. Then

$$\eta_1(A) \leq \frac{\left|A_{1,2} + \overline{A_{2,1}}\right|^2}{4(r_0(A) - \operatorname{Re} A_{1,1})(r_0(A) - \operatorname{Re} A_{2,2})} =: \tilde{\eta}_1(A),$$
$$\eta_2(A) \leq \frac{\left|A_{2,3} + \overline{A_{3,2}}\right|^2}{4(r_0(A) - \operatorname{Re} A_{2,2})(r_0(A) - \operatorname{Re} A_{3,3})} =: \tilde{\eta}_2(A),$$

where at least one inequality is strict and thus $\sqrt{\tilde{\eta}_1(A)} + \sqrt{\tilde{\eta}_2(A)} > 1$. But this is a contradiction to (i). If both $A_{1,2} + \overline{A_{2,1}}$ and $A_{2,3} + \overline{A_{3,2}}$ vanish, then $B := \frac{1}{2}(A + A^*)$ is diagonal and thus

$$r_0(A) = r_0(B) = \sup_{i \in \mathbb{I}} B_{i,i} = \sup_{i \in \mathbb{I}} \operatorname{Re} A_{i,i},$$

which is a contradiction to the assuption $r_0(A) > \sup_{i \in \mathbb{I}} \operatorname{Re} A_{i,i}$. Thus $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ implies $N = r_0(A)$.

Conversely let $N = r_0(A)$, $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} < 1$ and assume $\max_{i \in \{1,2\}} \left| A_{i,i+1} + \overline{A_{i+1,i}} \right| \neq 0$. Then by continuity there exists an $\varepsilon > 0$ such that

$$\sqrt{\frac{\left|A_{1,2} + \overline{A_{2,1}}\right|^2}{4(N - \varepsilon - \operatorname{Re} A_{1,1})(N - \varepsilon - \operatorname{Re} A_{2,2})}} + \sqrt{\frac{\left|A_{2,3} + \overline{A_{3,2}}\right|^2}{4(N - \varepsilon - \operatorname{Re} A_{2,2})(N - \varepsilon - \operatorname{Re} A_{3,3})}} = 1.$$

But this is a contradiction to (ii), since $N - \varepsilon < r_0(A)$. If both $A_{1,2} + \overline{A_{2,1}}$ and $A_{2,3} + \overline{A_{3,2}}$ vanish, we get the same contradiction to $r_0(A) > \sup_{i \in \mathbb{I}} \operatorname{Re} A_{i,i}$ as before. Thus $N = r_0(A)$ implies $\sqrt{\eta_1(A)} + \sqrt{\eta_2(A)} = 1$ and we proved all assertions. □

Now that we have the numerical ranges of B_1^2 (period σ, σ, σ, σ), B_2^2 (period $-\sigma$, $-\sigma$, σ, σ) and B_3^2 (period $-\sigma$, $-\sigma$, $-\sigma$, $-\sigma$), we can set

$$N(\varphi) = \max\left\{r_\varphi(B_1^2), r_\varphi(B_2^2), r_\varphi(B_3^2)\right\}. \tag{14}$$

Proposition A.3. *Let B_1, B_2 and B_3 be as before and let N be given by (14). Let $\varphi^* := \arccos(\frac{\sigma}{1+\sigma^2})$. Then N takes the following values:*

$$\begin{aligned}
N(\varphi) &= 2\sigma\cos(\varphi) + \sqrt{(1+\sigma^2)^2\cos(\varphi)^2 + (1-\sigma^2)^2\sin(\varphi)^2} && \text{if } 0 \leq \varphi \leq \varphi^*,\\
N(\varphi) &= 1 + \sigma^2 && \text{if } \varphi^* \leq \varphi \leq \pi - \varphi^*,\\
N(\varphi) &= -2\sigma\cos(\varphi) + \sqrt{(1+\sigma^2)^2\cos(\varphi)^2 + (1-\sigma^2)^2\sin(\varphi)^2} && \text{if } \pi - \varphi^* \leq \varphi \leq \pi + \varphi^*,\\
N(\varphi) &= 1 + \sigma^2 && \text{if } \pi + \varphi^* \leq \varphi \leq 2\pi - \varphi^*,\\
N(\varphi) &= 2\sigma\cos(\varphi) + \sqrt{(1+\sigma^2)^2\cos(\varphi)^2 + (1-\sigma^2)^2\sin(\varphi)^2} && \text{if } 2\pi - \varphi^* \leq \varphi \leq 2\pi.
\end{aligned} \tag{15}$$

This is equivalent to saying that $r_\cdot(B_1^2)$ is the largest in the interval $[0, \varphi^*]$, $r_\cdot(B_2^2)$ is the largest in the interval $[\varphi^*, \pi - \varphi^*]$ and so on.

Proof. By Proposition A.1, we only have to check where the graphs $r_\varphi(B_1^2)$, $r_\varphi(B_2^2)$ and $r_\varphi(B_3^2)$ intersect. Let us have a look at $r_\varphi(B_1^2)$ and $r_\varphi(B_2^2)$ first, i.e.

$$\begin{aligned} r_\varphi(B_1^2) = r_\varphi(B_2^2) \quad &\Leftrightarrow \quad 2\sigma\cos(\varphi) + \sqrt{(1+\sigma^2)^2\cos(\varphi)^2 + (1-\sigma^2)^2\sin(\varphi)^2} = 1+\sigma^2 \\ &\Leftrightarrow \quad (1+\sigma^2)^2\cos(\varphi)^2 + (1-\sigma^2)^2(1-\cos(\varphi)^2) = (1+\sigma^2-2\sigma\cos(\varphi))^2 \\ &\Leftrightarrow \quad \cos(\varphi) = \frac{\sigma}{1+\sigma^2}. \end{aligned}$$

Thus the graphs of $r_\varphi(B_1^2)$ and $r_\varphi(B_2^2)$ only intersect at $\varphi^* = \arccos(\frac{\sigma}{1+\sigma^2})$ and $2\pi - \varphi^*$. Similarly the graphs of $r_\varphi(B_2^2)$ and $r_\varphi(B_3^2)$ only intersect at $\pi - \varphi^* = \arccos(\frac{-\sigma}{1+\sigma^2})$ and $\pi + \varphi^*$. Finally, $r_\varphi(B_1^2)$ and $r_\varphi(B_3^2)$ obviously only intersect at $\pm\frac{\pi}{2}$. Plugging in some angles and using (14), one easily deduces (15). ☐

Next we want to tabularize all possible combinations for (5). Remember that A^2 looks like this:

$$\begin{aligned} (A^2)_{i,i+2} &= A_{i,i+1}A_{i+1,i+2} = 1, \\ (A^2)_{i,i+1} &= A_{i,i+1}A_{i+1,i+1} + A_{i,i}A_{i,i+1} = 0, \\ (A^2)_{i,i} &= A_{i,i+1}A_{i+1,i} + A_{i,i}A_{i,i} + A_{i,i-1}A_{i-1,i} = h_i + h_{i-1}, \\ (A^2)_{i,i-1} &= A_{i,i}A_{i,i-1} + A_{i,i-1}A_{i-1,i-1} = 0, \\ (A^2)_{i,i-2} &= A_{i,i-1}A_{i-1,i-2} = h_{i-1}h_{i-2}, \end{aligned}$$

and that A^2 can be decomposed as $A^2 = C \oplus D$, where

$$\begin{aligned} C_{i,i+1} &= 1, \\ C_{i,i} &= h_{2i} + h_{2i-1}, \\ C_{i,i-1} &= h_{2i-1}h_{2i-2}, \end{aligned}$$

and $N(A^2) = N(C)$. Moreover we showed that $N(C_+) = N(P_{\mathbb{N}}CP_{\mathbb{N}}) = N(C) = N(A^2)$. Let $E(\varphi) := e^{i\varphi}C_+$. Then $r_\varphi(A) = r_0(E(\varphi))$ for every angle $\varphi \in [0, 2\pi)$. Now for every σ and angle φ there are 16 different combinations for $(h_{2i-1}, h_{2i}, h_{2i+1}, h_{2i+2})$ in (5). Define

$$\eta_i(\varphi) := \frac{\left|E_{i,i+1}(\varphi) + \overline{E_{i+1,i}(\varphi)}\right|^2}{4(N(\varphi) - \text{Re } E_{i,i}(\varphi))(N(\varphi) - \text{Re } E_{i+1,i+1}(\varphi))},$$

for all $i \in \mathbb{N}$. Let us consider $\varphi \in [\varphi^*, \frac{\pi}{2}]$ first. For these angles, we have the following table:

type	$(h_{2i-1}, h_{2i}, h_{2i+1}, h_{2i+2})$	$\eta_i(\varphi)$
(1)	$(\sigma, \sigma, \sigma, \sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))^2}$
(2)	$(\sigma, \sigma, \sigma, -\sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))(1+\sigma^2)}$
(3)	$(\sigma, \sigma, -\sigma, \sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))(1+\sigma^2)}$
(4)	$(\sigma, \sigma, -\sigma, -\sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))(1+\sigma^2+2\sigma\cos(\varphi))}$
(5)	$(\sigma, -\sigma, \sigma, \sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))(1+\sigma^2)}$
(6)	$(\sigma, -\sigma, \sigma, -\sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2)^2}$
(7)	$(\sigma, -\sigma, -\sigma, \sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2)^2}$
(8)	$(\sigma, -\sigma, -\sigma, -\sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2+2\sigma\cos(\varphi))(1+\sigma^2)}$
(9)	$(-\sigma, \sigma, \sigma, \sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))(1+\sigma^2)}$
(10)	$(-\sigma, \sigma, \sigma, -\sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2)^2}$
(11)	$(-\sigma, \sigma, -\sigma, \sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2)^2}$
(12)	$(-\sigma, \sigma, -\sigma, -\sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2+2\sigma\cos(\varphi))(1+\sigma^2)}$
(13)	$(-\sigma, -\sigma, \sigma, \sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))(1+\sigma^2+2\sigma\cos(\varphi))}$
(14)	$(-\sigma, -\sigma, \sigma, -\sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2+2\sigma\cos(\varphi))(1+\sigma^2)}$
(15)	$(-\sigma, -\sigma, -\sigma, \sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2+2\sigma\cos(\varphi))(1+\sigma^2)}$
(16)	$(-\sigma, -\sigma, -\sigma, -\sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2+2\sigma\cos(\varphi))^2}$

Table 2

We will find the following equalities and inequalities useful:

$$0 \leq \cos(\varphi) \leq \frac{\sigma}{1+\sigma^2} < 1 \tag{16}$$

$$(1-\sigma^2)^2 + 4\sigma^2\cos(\varphi)^2 \leq (1-\sigma^2)^2 + \frac{4\sigma^4}{(1+\sigma^2)^2} = \frac{(1+\sigma^4)^2}{(1+\sigma^2)^2} \tag{17}$$

$$1+\sigma^2 - 2\sigma\cos(\varphi) \geq 1+\sigma^2 - \frac{2\sigma^2}{1+\sigma^2} = \frac{1+\sigma^4}{1+\sigma^2} \tag{18}$$

$$(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2 = (1+\sigma+2\sigma\cos(\varphi))(1+\sigma-2\sigma\cos(\varphi)) \tag{19}$$

Using these, it is not difficult to see that for $\varphi \in [\varphi^*, \frac{\pi}{2}]$ all $\eta_i(\varphi)$ in Table 2 are lower or equal to $\frac{1}{2}$ and all $\eta_i(\varphi)$ but type (3) and (5) are even lower or equal to $\frac{1}{4}$. This observation is very useful to finally construct the sequence needed for Lemma 24.

Proposition A.4. *Let $\sigma \in (0,1]$, $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm\sigma\}$ and let $A \in \Psi E(U_{-1}, U_0, U_1)$. Let $\varphi \in [\varphi^*, \frac{\pi}{2}]$ and $\eta_i := \eta_i(\varphi)$ for all $i \in \mathbb{N}$ be defined as above. Then the sequence $(g_i)_{i\in\mathbb{N}}$, defined by the following prescription, satisfies $0 \leq g_i \leq 1$ and $\eta_i \leq g_{i+1}(1-g_i)$ for all $i \in \mathbb{N}$:*

- *If η_1 is of type (5), choose $g_1 = \frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}$.*
- *If there is some $j \in \mathbb{N}$ such that η_k is of type (6) for all $k \in \{1,\dots,j\}$ and η_{j+1} is of type (5), choose $g_1 = \frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}$.*
- *If neither is true, choose $g_1 = \frac{1}{2}$.*
- *If η_i is of type (2), (6), (10) or (14) and η_{i+1} is of type (5), choose $g_{i+1} = \frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}$.*
- *If η_i is of type (2), (6), (10) or (14), there is some $j \in \mathbb{N}$ such that η_{i+k} is of type (6) for all $k \in \{1,\dots,j\}$ and η_{i+j+1} is of type (5), choose $g_{i+1} = \frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}$.*
- *If η_i is of type (3), choose $g_{i+1} = \frac{1}{2}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}$.*
- *If there is some $j \in \mathbb{N}$ such that η_{i-j} is of type (3) and η_{i-k+1} is of type (11) for all $k \in \{1,\dots,j\}$, choose $g_{i+1} = \frac{1}{2}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}$.*
- *If none of the above is true, choose $g_{i+1} = \frac{1}{2}$.*

Proof. That $0 \leq g_i \leq 1$ holds for all $i \in \mathbb{N}$ follows from (16). So it remains to prove that $\eta_i \leq g_{i+1}(1-g_i)$ holds. Above we observed that $\eta_i \leq \frac{1}{4}$ unless η_i is of type (3) or (5). Thus if the types (3) and (5) do not occur, then $\eta_i \leq g_{i+1}(1-g_i)$ is obviously satisfied It remains to investigate what happens if η_i is of type (3) or (5). Note that in these cases we can simplify η_i:

$$\begin{aligned} \eta_i &= \frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))(1+\sigma^2)} \\ &= \frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}, \end{aligned}$$

where we used (19).

Let us consider the case (3) first. Assume that $g_i = \frac{1}{2}$. Then by definition $g_{i+1} = \frac{1}{2}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}$ and

$$g_{i+1}(1-g_i) = \frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2} = \eta_i.$$

The way C_+ is defined, η_i and η_{i+1} are not independent. Indeed, η_{i+1} is defined by the sequence $(h_{2i+1}, h_{2i+2}, h_{2i+3}, h_{2i+4})$, where η_i is defined by $(h_{2i-1}, h_{2i}, h_{2i+1}, h_{2i+2})$. Thus if we fix η_i, there are only 4 possible combinations for η_{i+1}. In particular, if η_i is of type (3), then η_{i+1} has to be of type (9), (10), (11) or (12). So there are four cases:

$$\begin{aligned}
\eta_{i+1} &= \frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2-2\sigma\cos(\varphi))(1+\sigma^2)} \quad \text{(type (9))},\\
\eta_{i+1} &= \frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2)^2} \quad \text{(type (10))},\\
\eta_{i+1} &= \frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2)^2} \quad \text{(type (11))},\\
\eta_{i+1} &= \frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(1+\sigma^2+2\sigma\cos(\varphi))(1+\sigma^2)} \quad \text{(type (12))}.
\end{aligned}$$

In the first case we have $g_{i+2} = \frac{1}{2}$:

$$\begin{aligned}
g_{i+2}(1-g_{i+1}) &= \frac{1}{2} - \frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}\\
&= \frac{1}{4}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}\\
&\geq \frac{1+\sigma^4}{(1+\sigma^2)^2}\\
&\geq \eta_{i+1},
\end{aligned}$$

where we used (18) in line 2 and (17) and (18) in line 3. In the second case we have $g_{i+2} = \frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}$ if η_{i+2} is of type (5) or (6) and $g_{i+2} = \frac{1}{2}$ if not:

$$\begin{aligned}
g_{i+2}(1-g_{i+1}) &\geq \frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}\left(\frac{1}{2} - \frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}\right)\\
&= \frac{1}{4}\frac{(1+\sigma^2-2\sigma\cos(\varphi))^2}{(1+\sigma^2)^2}\\
&\geq \frac{(1+\sigma^4)^2}{(1+\sigma^2)^4}\\
&\geq \eta_{i+1},
\end{aligned}$$

where we used (18) in line 2 and (17) and (18) in line 3. In the third case we have $g_{i+2} =$

$\frac{1}{2}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}$:

$$\begin{aligned} g_{i+2}(1-g_{i+1}) &= \frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}\left(\frac{1}{2}-\frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}\right) \\ &= \frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2} \\ &= \frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(1+\sigma^2)^2} \\ &= \eta_{i+1}. \end{aligned}$$

In the fourth case we have $g_{i+2}=\frac{1}{2}$:

$$\begin{aligned} g_{i+2}(1-g_{i+1}) &= \frac{1}{2}-\frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2} \\ &= \frac{1}{4}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2} \\ &= \frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(1+\sigma^2+2\sigma\cos(\varphi))(1+\sigma^2)} \\ &= \eta_{i+1}. \end{aligned}$$

So what do we learn from this? Either $g_{i+2}\leq\frac{1}{2}$ (and we included one special case that we need afterwards) or $g_{i+2}=g_{i+1}$. Thus either we are where we started with, namely $\frac{1}{2}$, or we are in the third case, where η_{i+1} is of type (11). But in this case we have $h_{2i+1}=h_{2i+3}$ and $h_{2i+2}=h_{2i+4}$ and thus we have again the same four cases for η_{i+2} and so on. So either we end up with an infinite sequence with $g_j=g_{i+1}$ for all $j>i$ (which is impossible by pseudo-ergodicity, but would still be just fine) or we eventually go out with $g_j\leq\frac{1}{2}$ for some $j\geq i+2$. Thus by induction we are done if we can also control type (5). So let's do it.

Type (5) is essentially the same as type (3), but we have to think backwards, which is a little bit more complicated. If we have a look at the generators of (3) and (5), it is intuitively clear, why this has to be the same but backwards. Assume that η_i is of type (5). Then $g_i=\frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}$ and $g_{i+1}=\frac{1}{2}$ by definition and thus

$$g_{i+1}(1-g_i)=\frac{1}{2}-\frac{1}{4}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}=\frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}=\eta_i.$$

As already mentioned, we have to look backwards here, i.e. we want to control g_{i-1}. Now there are five cases here. The first case is $i=1$, which is trivial of course. The second case is where η_{i-1} is of type (2). In this case we have $g_{i-1}=\frac{1}{2}$:

$$\begin{aligned} g_i(1-g_{i-1}) &= \frac{1}{4}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2} \\ &\geq \frac{1+\sigma^4}{(1+\sigma^2)^2} \\ &\geq \eta_{i+1}, \end{aligned}$$

where we used (18) in line 1 and (17) and (18) in line 2. The third case is where η_{i-1} is of type (6). In this case we have $g_{i-1} = \frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}$:

$$\begin{aligned} g_i(1-g_{i-1}) &= \frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}\left(\frac{1}{2}-\frac{1}{4}\frac{1}{2}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}\right) \\ &= \frac{1}{4}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2} \\ &= \frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(1+\sigma^2)^2} \\ &= \eta_{i-1}. \end{aligned}$$

The fourth case is where η_{i-1} is of type (10). In this case we either have $g_{i-1} = \frac{1}{2}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}$ if η_{i-2} is of type (3) or (11) or $g_{i-1} = \frac{1}{2}$ if not:

$$\begin{aligned} g_i(1-g_{i-1}) &\geq \frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2}\left(\frac{1}{2}-\frac{1}{4}\frac{1+\sigma^2+2\sigma\cos(\varphi)}{1+\sigma^2}\right) \\ &= \frac{1}{4}\frac{(1+\sigma^2-2\sigma\cos(\varphi))^2}{(1+\sigma^2)^2} \\ &\geq \frac{1}{4}\frac{(1+\sigma^4)^2}{(1+\sigma)^4} \\ &\geq \eta_{i-1}, \end{aligned}$$

where we used (18) in line 2 and (17) and (18) in line 3. Note that this case matches perfectly with the second case above. The fifth case is where η_{i-1} is of type (14). In this case we have $g_{i-1} = \frac{1}{2}$:

$$\begin{aligned} g_i(1-g_{i-1}) &= \frac{1}{4}\frac{1+\sigma^2-2\sigma\cos(\varphi)}{1+\sigma^2} \\ &= \frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(1+\sigma^2+2\sigma\cos(\varphi))(1+\sigma^2)} \\ &= \eta_{i+1}. \end{aligned}$$

Again we conclude that either $g_{i-1} \geq \frac{1}{2}$ (note that the inequality is in the other direction this time, which is good!) or $g_{i-1} = g_i$. Thus either we started where we ended, namely $\frac{1}{2}$ (or even better, we started with something $\geq \frac{1}{2}$ and the sequence reduced to $\frac{1}{2}$, compare with the mentioned special case above), or we are in the third case, where η_{i-1} is of type (6). But in this case we have $h_{2i-1} = h_{2i-3}$ and $h_{2i-2} = h_{2i-4}$ and thus we again have the same four cases for η_{i-2} and so on. Thus we either end up at g_1, which is fine or we eventually have $g_j = \frac{1}{2}$ for some $j \leq i-1$. In either case we are done by induction. □

So we are done with the case $\varphi \in [\varphi^*, \frac{\pi}{2}]$. This means that there is only the case $\varphi \in [0, \varphi^*]$ left. All the other angles will follow by symmetry. Let us have a look at the table of the angles $\varphi \in [0, \varphi^*]$. Remember that we have

$$N(\varphi) = 2\sigma\cos(\varphi)+\sqrt{(1+\sigma^2)^2\cos(\varphi)^2+(1-\sigma^2)^2\sin(\varphi)^2} = 2\sigma\cos(\varphi)+\sqrt{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}$$

here and let us drop the φ in $N(\varphi)$ for the sake of readability.

type	$(h_{2i-1}, h_{2i}, h_{2i+1}, h_{2i+2})$	$\eta_i(\varphi)$
(1)	$(\sigma, \sigma, \sigma, \sigma)$	$\frac{1}{4}$
(2)	$(\sigma, \sigma, \sigma, -\sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(N-2\sigma\cos(\varphi))N}$
(3)	$(\sigma, \sigma, -\sigma, \sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(N-2\sigma\cos(\varphi))N}$
(4)	$(\sigma, \sigma, -\sigma, -\sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(N-2\sigma\cos(\varphi))(N+2\sigma\cos(\varphi))}$
(5)	$(\sigma, -\sigma, \sigma, \sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(N-2\sigma\cos(\varphi))N}$
(6)	$(\sigma, -\sigma, \sigma, -\sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4N^2}$
(7)	$(\sigma, -\sigma, -\sigma, \sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4N^2}$
(8)	$(\sigma, -\sigma, -\sigma, -\sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(N+2\sigma\cos(\varphi))N}$
(9)	$(-\sigma, \sigma, \sigma, \sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(N-2\sigma\cos(\varphi))N}$
(10)	$(-\sigma, \sigma, \sigma, -\sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4N^2}$
(11)	$(-\sigma, \sigma, -\sigma, \sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4N^2}$
(12)	$(-\sigma, \sigma, -\sigma, -\sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(N+2\sigma\cos(\varphi))N}$
(13)	$(-\sigma, -\sigma, \sigma, \sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(N-2\sigma\cos(\varphi))(N+2\sigma\cos(\varphi))}$
(14)	$(-\sigma, -\sigma, \sigma, -\sigma)$	$\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{4(N+2\sigma\cos(\varphi))N}$
(15)	$(-\sigma, -\sigma, -\sigma, \sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(N+2\sigma\cos(\varphi))N}$
(16)	$(-\sigma, -\sigma, -\sigma, -\sigma)$	$\frac{(1-\sigma^2)^2+4\sigma^2\cos(\varphi)^2}{4(N+2\sigma\cos(\varphi))^2}$

Table 3

We will find the following equalities and inequalities useful:

$$N \geq 1 + \sigma^2 \tag{20}$$

$$\cos(\varphi) \geq \frac{\sigma}{1+\sigma^2} \tag{21}$$

$$N - 2\sigma\cos(\varphi) = \sqrt{(1-\sigma^2)^2 + 4\sigma^2\cos(\varphi)^2} \geq \sqrt{(1-\sigma^2)^2 + \frac{4\sigma^4}{(1+\sigma^2)^2}} = \frac{1+\sigma^4}{1+\sigma^2} \tag{22}$$

$$N + 2\sigma\cos(\varphi) \geq \frac{1+\sigma^4}{1+\sigma^2} + \frac{4\sigma^2}{1+\sigma^2} = \frac{1+4\sigma^2+\sigma^4}{1+\sigma^2} \tag{23}$$

$$(1-\sigma^2)^2 + 4\sigma^2\cos(\varphi)^2 = (N - 2\sigma\cos(\varphi))^2 \tag{24}$$

$$(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2 \leq (1+\sigma^2)^2 - \frac{4\sigma^4}{(1+\sigma^2)^2} = \frac{(1+4\sigma^2+\sigma^4)(1+\sigma^4)}{(1+\sigma^2)^2} \tag{25}$$

Using these, it is again not difficult to see that for $\varphi \in [0, \varphi^*]$ all $\eta_i(\varphi)$ in Table 3 are lower or equal to $\frac{1}{2}$ and all $\eta_i(\varphi)$ but type (3) and (5) are lower or equal to $\frac{1}{4}$. So we have to do the same calculations as before.

Proposition A.5. *Let $\sigma \in (0,1]$, $U_{-1} = \{1\}$, $U_0 = \{0\}$, $U_1 = \{\pm\sigma\}$ and let $A \in \Psi E(U_{-1}, U_0, U_1)$. Let $\varphi \in [0, \varphi^*]$ and $\eta_i := \eta_i(\varphi)$ for all $i \in \mathbb{N}$ be defined as above. Then the sequence $(g_i)_{i\in\mathbb{N}}$, defined by the following prescription, satisfies $0 \leq g_i \leq 1$ and $\eta_i \leq g_{i+1}(1-g_i)$ for all $i \in \mathbb{N}$:*

- *If η_1 is of type (5), choose $g_1 = 1 - \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$.*
- *If there is some $j \in \mathbb{N}$ such that η_k is of type (6) for all $k \in \{1, \dots, j\}$ and η_{j+1} is of type (5), choose $g_1 = 1 - \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$.*
- *If neither is true, choose $g_1 = \frac{1}{2}$.*
- *If η_i is of type (2), (6), (10) or (14) and η_{i+1} is of type (5), choose $g_{i+1} = 1 - \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$.*
- *If η_i is of type (2), (6), (10) or (14), there is some $j \in \mathbb{N}$ such that η_{i+k} is of type (6) for all $k \in \{1, \dots, j\}$ and η_{i+j+1} is of type (5), choose $g_{i+1} = 1 - \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$.*
- *If η_i is of type (3), choose $g_{i+1} = \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$.*
- *If there is some $j \in \mathbb{N}$ such that η_{i-j} is of type (3) and η_{i-k+1} is of type (11) for all $k \in \{1, \dots, j\}$, choose $g_{i+1} = \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$.*
- *If none of the above is true, choose $g_{i+1} = \frac{1}{2}$.*

Proof. That $0 \leq g_i \leq 1$ holds for all $i \in \mathbb{N}$ follows from (20), (22) and (25). So it remains to prove that $\eta_i \leq g_{i+1}(1-g_i)$ holds. As observed above, we have $\eta_i \leq \frac{1}{4}$ unless η_i is of type (3) or (5). Thus if the types (3) and (5) do not occur, then $\eta_i \leq g_{i+1}(1-g_i)$ is obviously satisfied. So again it remains to investigate what happens if η_i is of type (3) or (5).

Let's treat type (3) first and assume that $g_i = \frac{1}{2}$. Then by definition

$$g_{i+1} = \frac{1}{2} \frac{(1+\sigma^2)^2 - 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N}$$

and

$$g_{i+1}(1 - g_i) = \frac{1}{4} \frac{(1+\sigma^2)^2 - 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N} = \eta_i.$$

Now there are four cases:

$$\begin{aligned}
\eta_{i+1} &= \frac{(1-\sigma^2)^2 + 4\sigma^2 \cos(\varphi)^2}{4(N - 2\sigma \cos(\varphi))N} \quad \text{(type (9))}, \\
\eta_{i+1} &= \frac{(1-\sigma^2)^2 + 4\sigma^2 \cos(\varphi)^2}{4N^2} \quad \text{(type (10))}, \\
\eta_{i+1} &= \frac{(1+\sigma^2)^2 - 4\sigma^2 \cos(\varphi)^2}{4N^2} \quad \text{(type (11))}, \\
\eta_{i+1} &= \frac{(1+\sigma^2)^2 - 4\sigma^2 \cos(\varphi)^2}{4(N + 2\sigma \cos(\varphi))N} \quad \text{(type (12))}.
\end{aligned}$$

In the first case we have $g_{i+2} = \frac{1}{2}$:

$$\begin{aligned}
g_{i+2}(1 - g_{i+1}) &= \frac{1}{2} - \frac{1}{4} \frac{(1+\sigma^2)^2 - 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N} \\
&= \frac{1}{4} \frac{2(N - 2\sigma \cos(\varphi))N - (1+\sigma^2)^2 + 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N} \\
&\geq \frac{1}{4} \frac{2(1+\sigma^4) - (1+\sigma^2)^2 + 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N} \\
&= \frac{1}{4} \frac{(1-\sigma^2)^2 + 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N} \\
&= \eta_{i+1},
\end{aligned}$$

where we used (20) and (22) in line 2. In the second case we have $g_{i+2} = 1 - \frac{1}{2} \frac{(1+\sigma^2)^2 - 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N}$ if η_{i+2} is of type (5) or (6) and $g_{i+2} = \frac{1}{2}$ if not:

$$\begin{aligned}
g_{i+2}(1 - g_{i+1}) &\geq \left(1 - \frac{1}{2} \frac{(1+\sigma^2)^2 - 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N}\right)\left(1 - \frac{1}{2} \frac{(1+\sigma^2)^2 - 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N}\right) \\
&= \frac{1}{4} \left(\frac{(2N(N - 2\sigma \cos(\varphi)) - (1+\sigma^2)^2 + 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N}\right)^2 \\
&\geq \frac{1}{4} \left(\frac{(N(N - 2\sigma \cos(\varphi)) + 1 + \sigma^4 - (1+\sigma^2)^2 + 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N}\right)^2 \\
&\geq \frac{1}{4} \left(\frac{(N(N - 2\sigma \cos(\varphi)) - 2\sigma N \cos(\varphi) + 4\sigma^2 \cos(\varphi)^2}{(N - 2\sigma \cos(\varphi))N}\right)^2 \\
&= \frac{1}{4} \frac{(N - 2\sigma \cos(\varphi))^2}{N^2} \\
&= \eta_{i+1},
\end{aligned}$$

where we used (20) and (22) in line 2 and (20) and (21) in line 3. In the third case we have $g_{i+2} = \frac{1}{2}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$:

$$\begin{aligned}
g_{i+2}(1-g_{i+1}) &= \frac{1}{2}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\left(1-\frac{1}{2}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\right) \\
&\geq \frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\frac{N-2\sigma\cos(\varphi)}{N} \\
&= \frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{N^2} \\
&= \eta_{i+1}
\end{aligned}$$

like in the second case. In the fourth case we have $g_{i+2} = \frac{1}{2}$:

$$\begin{aligned}
g_{i+2}(1-g_{i+1}) &= \frac{1}{2}-\frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&\geq \frac{1}{2}-\frac{1}{4}\frac{1+4\sigma^2+\sigma^4}{(1+\sigma^2)^2} \\
&= \frac{1}{4}\frac{1+\sigma^4}{(1+\sigma^2)^2} \\
&\geq \frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(N+2\sigma\cos(\varphi))N} \\
&= \eta_{i+1},
\end{aligned}$$

where we used (20), (22) and (25) in line 1 and (20), (23) and (25) in line 3. Now either $g_{i+2} \leq \frac{1}{2}$ (and we included one special case that we need afterwards) or $g_{i+2} = g_{i+1}$. Thus either we are where we started with, namely $\frac{1}{2}$, or we are in the third case, where η_{i+1} is of type (11). But in this case we have $h_{2i+1} = h_{2i+3}$ and $h_{2i+2} = h_{2i+4}$ and thus we have again the same four cases for η_{i+2} and so on. So either we end up with an infinite sequence with $g_j = g_{i+1}$ for all $j > i$ (which is impossible by pseudo-ergodicity, but would still be just fine) or we eventually go out with $g_j \leq \frac{1}{2}$ for some $j \geq i+2$. Thus by induction we are done if we can also control type (5) again.

Assume that η_i is of type (5). Then $g_i = 1 - \frac{1}{2}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$ and $g_{i+1} = \frac{1}{2}$ by definition and thus

$$g_{i+1}(1-g_i) = \frac{1}{4}\frac{(1+\sigma^2)^2-4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} = \eta_i.$$

As already mentioned, we have to look backwards here, i.e. we want to control g_{i-1}. Now there are five cases here. The first case is $i = 1$, which is trivial of course. The second case is where η_{i-1}

is of type (2). In this case we have $g_{i-1} = \frac{1}{2}$:

$$\begin{aligned}
g_i(1-g_{i-1}) &= \frac{1}{2} - \frac{1}{4}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&= \frac{1}{4}\frac{2(N-2\sigma\cos(\varphi))N - (1+\sigma^2)^2 + 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&\geq \frac{1}{4}\frac{2(1+\sigma^4) - (1+\sigma^2)^2 + 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&= \frac{1}{4}\frac{(1-\sigma^2)^2 + 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&= \eta_{i+1},
\end{aligned}$$

where we used (20) and (22) in line 2. The third case is where η_{i-1} is of type (6). In this case we have $g_{i-1} = 1 - \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$:

$$\begin{aligned}
g_i(1-g_{i-1}) &= \left(1 - \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\right)\frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&= \frac{1}{4}\frac{(2N(N-2\sigma\cos(\varphi)) - (1+\sigma^2)^2 + 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&\geq \frac{1}{4}\frac{(N(N-2\sigma\cos(\varphi)) + 1 + \sigma^4 - (1+\sigma^2)^2 + 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&\geq \frac{1}{4}\frac{(N(N-2\sigma\cos(\varphi)) - 2\sigma N\cos(\varphi) + 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&= \frac{1}{4}\frac{N-2\sigma\cos(\varphi)}{N}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N} \\
&= \frac{1}{4}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{N^2} \\
&= \eta_{i-1},
\end{aligned}$$

where we used (20) and (22) in line 2 and (20) and (21) in line 3. The fourth case is where η_{i-1} is of type (10). In this case we either have $g_{i-1} = \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}$ is of type (3) or (11) or $g_{i-1} = \frac{1}{2}$ if not:

$$\begin{aligned}
g_i(1-g_{i-1}) &\geq \left(1 - \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\right)\left(1 - \frac{1}{2}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N-2\sigma\cos(\varphi))N}\right) \\
&= \frac{1}{4}\frac{(N-2\sigma\cos(\varphi))^2}{N^2} \\
&= \eta_{i+1}
\end{aligned}$$

like in the third case. Note that this case matches perfectly with the second case above. The fifth case is where η_{i-1} is of type (14). In this case we have $g_{i-1} = \frac{1}{2}$:

$$\begin{aligned}
g_i(1 - g_{i-1}) &= \frac{1}{2} - \frac{1}{4}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N - 2\sigma\cos(\varphi))N} \\
&\geq \frac{1}{2} - \frac{1}{4}\frac{1 + 4\sigma^2 + \sigma^4}{(1+\sigma^2)^2} \\
&= \frac{1}{4}\frac{1+\sigma^4}{(1+\sigma^2)^2} \\
&\geq \frac{1}{4}\frac{(1+\sigma^2)^2 - 4\sigma^2\cos(\varphi)^2}{(N + 2\sigma\cos(\varphi))N} \\
&= \eta_{i+1},
\end{aligned}$$

where we used (20), (22) and (25) in line 1 and (20), (23) and (25) in line 3. Again we conclude that either $g_{i-1} \geq \frac{1}{2}$ or $g_{i-1} = g_i$. Thus either we started where we ended, namely $\frac{1}{2}$ (or even better, we started with something $\geq \frac{1}{2}$ and the sequence reduced to $\frac{1}{2}$, compare with the mentioned special case above), or we are in the third case, where η_{i-1} is of type (6). But in this case we have $h_{2i-1} = h_{2i-3}$ and $h_{2i-2} = h_{2i-4}$ and thus we again have the same four cases for η_{i-2} and so on. Thus we either end up at g_1, which is fine or we eventually have $g_j = \frac{1}{2}$ for some $j \leq i-1$. In either case we are done by induction. □

Thus for every angle $\varphi \in [0, \frac{\pi}{2}]$ and every $\sigma \in (0,1]$ we managed to construct a sequence $(g_i)_{i\in\mathbb{N}}$ that meets all the requirements for Lemma 24. By symmetry, we also get sequences for the remaining angles (for $\varphi \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ we have to interchange σ and $-\sigma$). So combining the Propositions A.1, A.3, A.4 and A.5 with Lemma 24 yields Theorem 27.

References

[1] S.N. Chandler-Wilde, R. Chonchaiya and M. Lindner: *Eigenvalue Problem meets Sierpinski Triangle: Computing the Spectrum of a Non-Self-Adjoint Random Operator*, Operators and Matrices, 5 (2011), 633-648.

[2] S.N. Chandler-Wilde, R. Chonchaiya and M. Lindner: *On the Spectra and Pseudospectra of a Class of non-self-adjoint Random Matrices and Operators*, Operators and Matrices, 7 (2013), 739-775.

[3] S.N. Chandler-Wilde and E.B. Davies: *Spectrum of a Feinberg-Zee Random Hopping Matrix*, Journal of Spectral Theory, 2 (2012), 147-179.

[4] E.B. Davies: *Spectral Theory of Pseudo-Ergodic Operators*, Commun. Math. Phys. 216 (2001), 687-704.

[5] E.B. Davies: *Linear Operators and their Spectra*, Cambridge Studies in Advanced Mathematics 106, Cambridge University Press, Cambridge, 2007.

[6] J. Feinberg and A. Zee: *Spectral Curves of Non-Hermitean Hamiltonians*, Nucl. Phys. B 552 (1999), 599-623.

[7] F.G. Frobenius: *Über Matrizen aus nicht negativen Elementen*, Sitzungsber. Preuß. Akad. Wiss. Phys.-Math. Kl. (1909), 456-477

[8] R. Hagen, S. Roch and B. Silbermann: *C^*-Algebras and Numerical Analysis*, Marcel Dekker, Inc., New York, Basel 2001.

[9] R. Hagger, M. Lindner and M. Seidel: *Essential Pseudospectra and Essential Norms of Band-Dominated Operators*, in preparation.

[10] D.E. Holz, H. Orland and A. Zee: *On the Remarkable Spectrum of a Non-Hermitian Random Matrix Model*, J. Phys. A Math. Gen 36 (2003), 3385-3400.

[11] M. Lindner, *Infinite Matrices and their Finite Sections*, Birkhäuser Verlag, Basel, Boston, Berlin, 2006.

[12] M. Lindner, *Fredholm Theory and Stable Approximation of Band Operators and Their Generalisations*, Habilitationsschrift, Chemnitz, 2009.

[13] M. Lindner, *A Note on the Spectrum of Bi-infinite Bi-diagonal Random Matrices*, Journal of Analysis and Applications 7 (2009), 269-278.

[14] M. Lindner and S. Roch, *Finite Sections of Random Jacobi Operators*, SIAM J. Numerical Analysis 50(1), 287-306 (2012).

[15] R. Szwarc, *Norm Estimates of Discrete Schrödinger operators*, Colloq. Math. 76 (1998), 153-160.