

# Dirichlet forms for singular diffusion on graphs

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## Abstract

We describe operators driving the time evolution of singular diffusion on finite graphs whose vertices are allowed to carry masses. The operators are defined by the method of quadratic forms on suitable Hilbert spaces. The model also covers quantum graphs and discrete Laplace operators.

MSC 2000: 47D06, 60J60, 47E05, 35Q99, 05C99

Keywords: gap diffusion, quantum graph, Dirichlet form,  $C_0$ -semigroup, positive, submarkovian

## Introduction

The present paper is a continuation and extension of [2]. We present suitable boundary or glueing conditions on graphs (quantum graphs) with singular second order differential operators on the edges. In particular, we describe those glueing conditions leading to positive and submarkovian  $C_0$ -semigroups.

The graph consists of finitely many bounded intervals, the edges, whose end points are connected with the vertices of the graph. On each of the edges  $e$  a finite Borel measure  $\mu_e$  is given, determining where particles may be located. The particles move according to “Brownian motion” but are slowed down or accelerated by the “speed measure”  $\mu_e$ . Further, each of the vertices  $v$  is provided with a weight  $\mu_v \geq 0$ , and particles may also be located at those vertices  $v$  with  $\mu_v > 0$ .

We refer to [2] for motivations for treating these topics. The extension with respect to [2] consists in two issues. On the one hand, the boundary conditions we describe are more general than proper glueing conditions. In fact, in the first access the graph structure does not intervene at all (Sections 2 and 3), and we only specify later the case of “local boundary conditions” (Section 5). On the other hand, we include the general case of vertices with masses, whereas in [2; Section 4] only a special case was treated. These results have been obtained in [7], and we mention [6], where also the case of vertices with masses is treated.

In Section 1 we recall some notation and facts from the one-dimensional case on an interval. In Section 2 we define the form in the Hilbert space  $\mathcal{H}_\Gamma$  on the graph which then defines the operator driving the evolution. We show that the

defined form  $\tau$  constitutes a form that is bounded below and closed. Let us point out that our definition of the form looks somewhat different from the one given in [2; Section 3]. In fact, looking at the definition of  $\tau$  in [2; Section 3], one realises that there is some interpretation needed in order to understand  $D(\tau)$  as a subset of the Hilbert space  $\mathcal{H}_\Gamma$ . This interpretation is made explicit in the present paper by the use of the mapping  $\iota$  introduced in Sections 1 and 2. In Section 3 we describe the operator  $H$  associated with the form  $\tau$  (Theorem 3.1). In Section 4 we indicate conditions for the  $C_0$ -semigroup  $(e^{-tH})_{t \geq 0}$  to be positive and submarkovian. In Section 5 we sketch the case of local boundary conditions.

## 1 One-dimensional prerequisites

In order to define the classical Dirichlet form we have to recall some notation and facts for a single interval  $[a, b] \subseteq \mathbb{R}$ , where  $a, b \in \mathbb{R}$ ,  $a < b$ . Let  $\mu$  be a finite Borel measure on  $[a, b]$ ,  $a, b \in \text{spt } \mu$ ,  $\mu(\{a, b\}) = 0$ . We define

$$\begin{aligned} C_\mu[a, b] &:= \{f \in C([a, b]); f \text{ affine linear on the components of } [a, b] \setminus \text{spt } \mu\}, \\ W_{2,\mu}^1(a, b) &:= W_2^1(a, b) \cap C_\mu[a, b]. \end{aligned}$$

For later use we recall the following inequalities. There exists a constant  $C > 0$  such that

$$\|f\|_\infty \leq C \left( \|f'\|_{L_2(a,b)}^2 + \|f\|_{L_2([a,b],\mu)}^2 \right)^{1/2} \quad (1.1)$$

for all  $f \in W_2^1(a, b) \cap C[a, b]$ , and for all  $r \in (0, b - a]$  one has

$$|f(a)| \leq r^{1/2} \|f'\|_{L_2(a,a+r)} + \|f\|_{L_2([a,a+r],\mu)} \mu([a, a+r])^{-1/2}, \quad (1.2)$$

and correspondingly for  $b$ ; cf. [2; Lemma 1.4 and Remark 3.2(b)].

Let  $\kappa: W_2^1(a, b) \cap C[a, b] \rightarrow L_2([a, b], \mu)$  be defined by  $\kappa f := f$ . Then it can be shown that  $R(\kappa) = R(\kappa|_{W_{2,\mu}^1(a,b)})$  (cf. [2; Lemma 1.2]), and that  $\kappa|_{W_{2,\mu}^1(a,b)}$  is injective (cf. [2; lower part of p. 639]). We define  $\iota := (\kappa|_{W_{2,\mu}^1(a,b)})^{-1}$ . Thus,  $\iota$  is an operator from  $L_2([a, b], \mu)$  to  $W_{2,\mu}^1(a, b)$ ,

$$D(\iota) = \left\{ f \in L_2([a, b], \mu); \text{ there exists } g \in W_2^1(a, b) \cap C[a, b] \text{ such that } g = f \text{ } \mu\text{-a.e.} \right\},$$

and  $\iota f$  is the unique element  $g \in W_{2,\mu}^1(a, b)$  such that  $g = f$   $\mu$ -a.e.

In order to describe the operator associated with the form defined in the following section we need some additional notions and facts concerning derivatives with respect to  $\mu$ .

If  $f \in L_{1,\text{loc}}(a, b)$ ,  $g \in L_1([a, b], \mu)$  are such that  $f' = g\mu$  (where  $f' = \partial f$  denotes the distributional derivative of  $f$ ), then we call  $g$  *distributional derivative of  $f$  with respect to  $\mu$* , and we write

$$\partial_\mu f := g.$$

Note that then necessarily  $f' = 0$  on  $[a, b] \setminus \text{spt } \mu$ , i.e.,  $f$  is constant on each of the components of  $[a, b] \setminus \text{spt } \mu$ . It is easy to see that this definition is equivalent to

$$f(x) = c + \int_{(a,x)} g(y) d\mu(y) \quad \text{a.e.}, \quad (1.3)$$

with some  $c \in \mathbb{K}$ . Thus, the function  $f$  has representatives of bounded variation and these have one-sided limits (not depending on the representative) at all points of  $[a, b]$ .

## 2 The form on the graph

Let  $\Gamma = (V, E, \gamma)$  be a finite directed graph. This means that  $V$  and  $E$  are finite sets,  $V \cap E = \emptyset$ ,  $V$  is the set of *vertices* (or *nodes*) of  $\Gamma$ ,  $E$  the set of *edges*, and  $\gamma = (\gamma_0, \gamma_1): E \rightarrow V \times V$  associates with each edge  $e$  a “starting vertex”  $\gamma_0(e)$ , and an “end vertex”  $\gamma_1(e)$ .

We assume that each edge  $e \in E$  corresponds to an interval  $[a_e, b_e] \subseteq \mathbb{R}$  (where  $a_e, b_e \in \mathbb{R}$ ,  $a_e < b_e$ ), and we assume that  $\mu_e$  is a finite Borel measure on  $[a_e, b_e]$  satisfying either  $\mu_e = 0$  or else  $a_e, b_e \in \text{spt } \mu_e$ ,  $\mu_e(\{a_e, b_e\}) = 0$ . We denote

$$E_0 := \{e \in E; \mu_e = 0\}, \quad E_1 := E \setminus E_0.$$

We further assume that, for each  $v \in V$ , we are given a weight  $\mu_v \geq 0$ , and we define

$$V_0 := \{v \in V; \mu_v = 0\}, \quad V_1 := V \setminus V_0.$$

**2.1 Remark.** The sets  $E_1$  and  $V_1$  encode the parts of the graph  $\Gamma$ , where a particle driven by the diffusion can be localised. In the present section we describe general glueing conditions which do not take into account the correspondence of the edges to the vertices. In the case  $E_1 = E$ ,  $V_1 = \emptyset$  and  $\mu_e$  the Lebesgue measure on  $[a_e, b_e]$ , the model will describe quantum graphs; cf. [3], [4], [5]. In the case  $E_1 = \emptyset$  we obtain (weighted) discrete diffusion on the vertices; cf. [1].

We are going to describe the self-adjoint operator driving the evolution in the Hilbert space

$$\mathcal{H}_\Gamma := \mathcal{H}_E \oplus \mathbb{K}^{V_1}$$

where on

$$\mathcal{H}_E := \bigoplus_{e \in E_1} L_2([a_e, b_e], \mu_e)$$

we use the scalar product

$$((f_e)_{e \in E_1} | (g_e)_{e \in E_1})_{\mathcal{H}_\Gamma} := \sum_{e \in E_1} (f_e | g_e)_{L_2([a_e, b_e], \mu_e)},$$

and on  $\mathbb{K}^{V_1}$  we use the scalar product

$$((f_v)_{v \in V_1} | (g_v)_{v \in V_1})_{\mathcal{H}_\Gamma} := \sum_{v \in V_1} f_v \overline{g_v} \mu_v$$

(for  $f = ((f_e)_{e \in E_1}, (f_v)_{v \in V_1}), g = ((g_e)_{e \in E_1}, (g_v)_{v \in V_1}) \in \mathcal{H}_\Gamma$ ).

In the following, the mapping  $\iota$  defined in Section 1 will be applied in the situation of the edges  $e \in E_1$ , and will then be denoted by  $\iota_e$ . We then define the operator  $\iota$  from  $\mathcal{H}_\Gamma$  to  $\prod_{e \in E_1} W_{2, \mu_e}^1(a_e, b_e) \times \mathbb{K}^{V_1}$ , by

$$\begin{aligned} D(\iota) &:= \{f \in \mathcal{H}_\Gamma; f_e \in D(\iota_e) \ (e \in E_1)\}, \\ (\iota f)_e &:= \iota_e f_e \quad (e \in E_1), \\ (\iota f)_v &:= f_v \quad (v \in V_1). \end{aligned}$$

We define the *trace mapping* (or *boundary value mapping*)  $\text{tr}: \prod_{e \in E_1} C[a_e, b_e] \times \mathbb{K}^{V_1} \rightarrow \mathbb{K}^{E'_1 \cup V_1}$ , where  $E'_1 := E_1 \times \{0, 1\}$ , by

$$\begin{aligned} \text{tr} f(e, j) &:= \begin{cases} f_e(a_e) & \text{if } e \in E_1, j = 0, \\ f_e(b_e) & \text{if } e \in E_1, j = 1, \end{cases} \\ \text{tr} f(v) &:= f_v \quad (v \in V_1). \end{aligned}$$

The space  $\mathbb{K}^{E'_1 \cup V_1}$  will be provided with the scalar product

$$(\xi | \eta) := \sum_{(e, j) \in E'_1} \xi(e, j) \overline{\eta(e, j)} + \sum_{v \in V_1} \xi(v) \overline{\eta(v)} \mu_v.$$

For the definition of the form we assume that  $X$  is a subspace of  $\mathbb{K}^{E'_1 \cup V_1}$  and that  $L$  is a self-adjoint operator in  $X$ . Then we define the form  $\tau$  by

$$\begin{aligned} D(\tau) &:= \{f \in D(\iota); \text{tr}(\iota f) \in X\}, \\ \tau(f, g) &:= \sum_{e \in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x) \overline{(\iota_e g_e)'(x)} dx + (L \text{tr}(\iota f) | \text{tr}(\iota g)). \end{aligned}$$

**2.2 Lemma.** *The form  $\tau$  defined above is symmetric.  $D(\tau)$  is dense if and only if*

$$\text{pr}_{V_1}(X) = \mathbb{K}^{V_1}, \tag{2.1}$$

where  $\text{pr}_{V_1}$  denotes the canonical projection  $\text{pr}_{V_1}: \mathbb{K}^{E'_1 \cup V_1} \rightarrow \mathbb{K}^{V_1}$ .

*Proof.* The symmetry of  $\tau$  is obvious.

Assume that  $D(\tau)$  is dense. The image of the dense set  $D(\tau)$  under the orthogonal projection

$$\text{pr}_2: \mathcal{H}_\Gamma \rightarrow \mathbb{K}^{V_1}$$

is dense in  $\mathbb{K}^{V_1}$ , and therefore is equal to  $\mathbb{K}^{V_1}$ . From the definition of  $D(\tau)$  it follows that  $\text{pr}_2(D(\tau))$  is contained in  $\text{pr}_{V_1}(X)$ , and therefore  $\text{pr}_{V_1}(X) = \mathbb{K}^{V_1}$ .

Now assume that (2.1) holds. For  $v \in V_1$  let  $\xi^v \in X$  be such that  $\xi^v(v) = 1$  and  $\xi^v(w) = 0$  for all  $w \in V_1 \setminus \{v\}$ . Let  $g^v \in D(\iota)$  be defined by  $\text{tr}(\iota g^v) = \xi^v$ , and  $g^v$  affine linear on the edges. The affine linear interpolation of the prescribed boundary values evidently yields an element of  $g^v \in D(\tau)$ .

Let  $f \in \mathcal{H}_\Gamma$ , and define

$$\tilde{f} := f - \sum_{v \in V_1} f_v g^v.$$

Then  $\tilde{f}_v = 0$  for all  $v \in V_1$ . Because  $C_c^1(a_e, b_e)$  is dense in  $L_2([a_e, b_e], \mu_e)$  ( $e \in E_1$ ), the function  $\tilde{f}$  can be approximated by functions in

$$D_c := \{f \in D(\tau); f_e \in C_c^1(a_e, b_e) (e \in E_1), f_v = 0 (v \in V_1)\}.$$

Therefore  $f$  can be approximated by functions in

$$D_c + \sum_{v \in V_1} f_v g^v \subseteq D(\tau). \quad \square$$

**2.3 Remark.** For the special case that  $X = \mathbb{K}^{E'_1 \cup V_1}$  and  $L = 0$  we denote the corresponding form by  $\tau_N$  (the index N indicating Neumann boundary conditions). The form  $\tau_N$  decomposes as the sum of the Neumann forms on each of the edges and the null form on  $\mathbb{K}^{V_1}$ . Therefore the closedness of  $\tau_N$  follows from the closedness in the one-dimensional cases; cf. [2; Section 1 and Remark 3.2].

**2.4 Theorem.** *The form  $\tau$  defined above is bounded below and closed.*

*Proof.* For  $f \in D(\tau)$  we obtain the estimate

$$\tau(f) = \tau_N(f) + (L \text{tr}(\iota f) | \text{tr}(\iota f)) \geq \tau_N(f) - \|L\| |\text{tr}(\iota f)|^2.$$

From the inequality (1.2) we obtain that the mapping  $f \mapsto \text{tr}(\iota f)|_{E'_1}$  is infinitesimally form small with respect to  $\tau_N$ . The remaining part of the trace,  $f \mapsto \text{tr}(\iota f)|_{V_1}$ , is bounded. These observations imply that  $\tau$  is bounded below and that the embedding  $D_{\tau_N} \ni f \mapsto \iota f \in (\prod_{e \in E_1} C[a_e, b_e] \times \mathbb{K}^{V_1}, \|\cdot\|_\infty)$  is continuous. (Here,  $D_{\tau_N}$  denotes  $D(\tau_N)$ , provided with the form norm.)

In order to obtain that  $\tau$  is closed it is sufficient to show that  $D(\tau)$  is a closed subset of  $D_{\tau_N}$ . This, however, is immediate from the continuity of the mapping  $D_{\tau_N} \ni f \mapsto \text{tr}(\iota f) \in \mathbb{K}^{E'_1 \cup V_1}$  (and the fact that  $X$  is a closed subspace of  $\mathbb{K}^{E'_1 \cup V_1}$ ).  $\square$

### 3 The operator $H$ associated with the form $\tau$

We assume that the notation and the hypotheses are as in the previous section, and that (2.1) holds.

Besides the trace mapping defined in the previous section we also need the *signed trace* (or *signed boundary values*)

$$\text{str}: \prod_{e \in E_1} BV(a_e, b_e) \rightarrow \mathbb{K}^{E_1} \subseteq \mathbb{K}^{E_1 \cup V_1}$$

(where  $BV(a_e, b_e)$  denotes the set of functions of bounded variation, with equivalence of functions coinciding a.e.), defined by

$$\text{str } f(e, j) := \begin{cases} f_e(a_e^+) & \text{if } e \in E_1, j = 0, \\ -f_e(b_e^-) & \text{if } e \in E_1, j = 1. \end{cases}$$

The inclusion  $\mathbb{K}^{E_1} \subseteq \mathbb{K}^{E_1 \cup V_1}$  is to be understood in the canonical sense; we want to be able to use  $\text{str } f$  also as an element of  $\mathbb{K}^{E_1 \cup V_1}$ .

For the description of the self-adjoint operator  $H$  associated with the form  $\tau$  we use a maximal operator  $\hat{H}$  for the differential part of the form. With the notation described in Section 1, we define

$$\begin{aligned} D(\hat{H}) &:= \left\{ f \in \prod_{e \in E_1} D(\iota_e); (\iota_e f_e)' \in L_1(a_e, b_e), \partial_{\mu_e}(\iota_e f_e)' \text{ exists,} \right. \\ &\quad \left. \partial_{\mu_e}(\iota_e f_e)' \in L_2([a_e, b_e], \mu_e) \ (e \in E_1) \right\} \\ \hat{H}f &:= (-\partial_{\mu_e}(\iota_e f_e)')_{e \in E_1} \quad (f \in D(\hat{H})). \end{aligned}$$

Thus, for  $f \in D(\hat{H})$ , the signed trace  $\text{str}((\iota_e f_e)')_{e \in E_1}$  exists, and it describes the “ingoing derivatives” from the endpoints of the intervals. It is to be understood that for  $(\iota_e f_e)'$  we choose representatives of bounded variation (which exist by the explanation given in Section 1), in order to be able to apply the signed trace mapping.

Let

$$X_0 := \{x \in X; \text{pr}_{V_1}(x) = 0\},$$

which could also be expressed as  $X_0 := X \cap \mathbb{K}^{E_1}$  (with our understanding of  $\mathbb{K}^{E_1}$  as a subspace of  $\mathbb{K}^{E_1 \cup V_1}$ ), and let  $Q_0$  be the orthogonal projection from  $\mathbb{K}^{E_1 \cup V_1}$  onto  $X_0$ . Also, for  $v \in V_1$ , let  $\xi^v \in X$  be such that  $\xi^v|_{V_1} = \mathbf{1}_{\{v\}}$  (see the proof of Lemma 2.2).

In the following, for  $f \in D(\iota)$  we will use the shorthand notation  $(\iota f)' := ((\iota_e f_e)')_{e \in E_1}$ .

**3.1 Theorem.** *The operator  $H$  associated with the form  $\tau$  is given by*

$$\begin{aligned} D(H) &= \left\{ f \in \mathcal{H}_\Gamma; (f_e)_{e \in E_1} \in D(\hat{H}), \text{tr}(\iota f) \in X, \right. \\ &\quad \left. Q_0 \text{str}(\iota f)' = Q_0 L \text{tr}(\iota f) \right\}, \\ ((Hf)_e)_{e \in E_1} &= \hat{H}(f_e)_{e \in E_1}, \\ (Hf)_v &= \frac{1}{\mu_v} (L \text{tr}(\iota f) - \text{str}(\iota f)' | \xi^v) \quad (v \in V_1). \end{aligned}$$

(For  $f \in D(H)$  and  $v \in V_1$ , the expression given for  $(Hf)_v$  does not depend on the choice of  $\xi^v$ .)

*Proof.* (i) A preliminary step: Let  $f \in D(\hat{H})$ ,  $g \in D(\tau)$ . For all  $e \in E_1$  one has

$$\int_{a_e}^{b_e} (\iota_e f_e)'(x) \overline{(\iota_e g_e)'(x)} dx = - \int_{(a_e, b_e)} \partial_{\mu_e} (\iota_e f_e)'(x) \overline{g_e(x)} d\mu_e(x) \\ + (\iota_e f_e)'(b_e^-) \overline{\iota_e g_e(b_e)} - (\iota_e f_e)'(a_e^+) \overline{\iota_e g_e(a_e)};$$

cf. [2; equ. (1.2)]. Summing this equation over all the edges in  $E_1$  we obtain

$$\sum_{e \in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x) \overline{(\iota_e g_e)'(x)} dx = (\hat{H}f | g)_{\mathcal{H}_E} - (\text{str}((\iota_e f_e)'_{e \in E_1} | \text{tr}(\iota g))_{\mathbb{K}^{E'_1}}).$$

(ii) Let  $f \in D(H)$ ,  $g \in D(\tau)$ . From  $D(H) \subseteq D(\tau)$  we conclude that  $\text{tr}(\iota f) \in X$ . As in [2; proof of Theorem 1.9] one obtains that  $(f_e)_{e \in E_1} \in D(\hat{H})$ ,  $\hat{H}(f_e)_{e \in E_1} = ((Hf)_e)_{e \in E_1}$ . Using part (i) above we obtain

$$(Hf | g)_{\mathcal{H}_\Gamma} \\ = - \sum_{e \in E_1} \int_{(a_e, b_e)} \partial_{\mu_e} (\iota_e f_e)'(x) \overline{g_e(x)} d\mu_e(x) + \sum_{v \in V_1} (Hf)_v \overline{g_v} \mu_v \\ = \sum_{e \in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x) \overline{(\iota_e g_e)'(x)} dx + (\text{str}(\iota f)' | \text{tr}(\iota g))_{\mathbb{K}^{E'_1}} + \sum_{v \in V_1} (Hf)_v \overline{g_v} \mu_v.$$

Because of

$$(Hf | g)_{\mathcal{H}_\Gamma} = \sum_{e \in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x) \overline{(\iota_e g_e)'(x)} dx + (L \text{tr}(\iota f) | \text{tr}(\iota g))$$

we therefore obtain

$$\sum_{v \in V_1} (Hf)_v \overline{g_v} \mu_v = (L \text{tr}(\iota f) - \text{str}(\iota f)' | \text{tr}(\iota g)). \quad (3.1)$$

For  $\xi \in X_0$  choose  $g \in D(\tau)$  satisfying  $\text{tr}(\iota g) = \xi$ , and  $g$  affine linear on the edges  $e \in E_1$ . Then equation (3.1) implies

$$0 = (L \text{tr}(\iota f) - \text{str}(\iota f)' | \xi).$$

This shows that  $Q_0 L \text{tr}(\iota f) = Q_0 \text{str}(\iota f)'$ .

Let  $v \in V_1$ , and choose  $g \in D(\tau)$  satisfying  $\text{tr}(\iota g) = \xi^v$ , and  $g$  affine linear on the edges  $e \in E_1$ . Then equation (3.1) yields

$$(Hf)_v \mu_v = \sum_{w \in V_1} (Hf)_w \overline{\xi^v(w)} \mu_w = (L \text{tr}(\iota f) - \text{str}(\iota f)' | \xi^v).$$

This shows the second part of the formula for  $Hf$ , and it also shows that  $(L \text{tr}(\iota f) - \text{str}(\iota f)' | \xi^v)$  is independent of the choice of  $\xi^v$ .

(iii) Now let  $\tilde{H}$  denote the operator indicated on the right hand side of the assertion, and let  $f \in D(\tilde{H})$ . Then  $f \in D(\tau)$ . Let  $g \in D(\tau)$ . Then  $\xi := \text{tr}(\iota g) - \sum_{v \in V_1} g_v \xi^v \in X_0$ , and therefore

$$(L \text{tr}(\iota f) - \text{str}(\iota f)' | \xi) = (Q_0(L \text{tr}(\iota f) - \text{str}(\iota f)' | \xi)) = 0.$$

Using part (i) as well as the previous equality we obtain

$$\begin{aligned} & (\tilde{H}f | g)_{\mathcal{H}_T} \\ &= (\hat{H}(f_e)_{e \in E_1} | (g_e)_{e \in E_1})_{\mathcal{H}_E} + \sum_{v \in V_1} \frac{1}{\mu_v} (L \text{tr}(\iota f) - \text{str}(\iota f)' | \xi^v) \overline{g_v} \mu_v \\ &= \sum_{e \in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x) \overline{(\iota_e g_e)'(x)} dx + (\text{str}(\iota f)' | \text{tr}(\iota g)) \\ & \quad + (L \text{tr}(\iota f) - \text{str}(\iota f)' | \text{tr}(\iota g)) \\ &= \tau(f, g). \end{aligned}$$

The definition of  $H$  then yields that  $f \in D(H)$  and  $Hf = \tilde{H}f$ .  $\square$

**3.2 Remark.** The case of a weight  $\mu_v > 0$  at a vertex leads to a case of Wentzell boundary condition at  $v$ . The expression of  $(Hf)_v$  in Theorem 3.1 generalises the expression obtained at a boundary point in the case of a single interval; cf. [8; Proposition 4.3].

## 4 Positivity and contractivity

In this section we indicate conditions for the  $C_0$ -semigroup  $(e^{-tH})_{t \geq 0}$  to be positive or submarkovian. We assume that the hypotheses are as in Section 2 and that (2.1) holds.

**4.1 Theorem.** (a) *Assume that  $X$  is a sublattice of  $\mathbb{K}^{E_1 \cup V_1}$  and that the semigroup  $(e^{-tL})_{t \geq 0}$  is positivity preserving. Then  $(e^{-tH})_{t \geq 0}$  is positivity preserving.*

(b) *Assume that  $X$  is a Stonean sublattice of  $\mathbb{K}^{E_1 \cup V_1}$  and that the semigroup  $(e^{-tL})_{t \geq 0}$  is a submarkovian semigroup. Then  $(e^{-tH})_{t \geq 0}$  is submarkovian.*

This result and its proof are completely analogous to [2; Theorem 3.5]; so we refrain from giving a complete proof but rather only mention the main ingredients. We refer to [2; Appendix] for the description of (Stonean) sublattices of  $\mathbb{K}^n$  and of positive (submarkovian) generators on these sublattices. The proof of Theorem 4.1 consists in an application of the Beurling-Deny criteria. So, in order to prove part (a), it is equivalent to prove that the normal contraction  $f \mapsto |f|$  acts on  $D(\tau)$ , and that  $\tau(|f|) \leq \tau(f)$  for all  $f \in D(\tau)$ . That the inequality works on the differential part is a one-dimensional issue which is taken care of in [2; Theorem 1.7]. For the trace part, the main observation is the equation



$\operatorname{tr} \iota|f| = |\operatorname{tr} \iota f|$ . This is less obvious than it might appear at the first glance since, in general, one does not have  $\iota|f| = |\iota f|$ . However, this equality holds on  $\operatorname{spt} \mu_e$ , and therefore on the end points of the intervals  $[a_e, b_e]$ , for all  $e \in E_1$ . The reasoning for part (b) is analogous.

## 5 Local boundary conditions

So far, the structure of the graph did not enter the considerations; in fact the function  $\gamma$  linking the edges to the vertices was not used at all. In order to explain what we understand by local boundary conditions, we need the following definitions.

For  $v \in V$ , the sets

$$E_{1,v,j} := \{e \in E_1; \gamma_j(e) = v\} \quad (j = 0, 1)$$

describe the sets of all edges having mass and starting or ending at  $v$ , respectively, and the set

$$E_{1,v} := (E_{1,v,0} \times \{0\}) \cup (E_{1,v,1} \times \{1\})$$

is the set of all edges having mass connected with  $v$  (and where loops starting and ending at  $v$  yield two contributions). Note that then  $E'_1 = \bigcup_{v \in V} E_{1,v}$ .

Recall that the boundary conditions are specified by the choice of a subspace  $X \subseteq \mathbb{K}^{E'_1 \cup V_1}$  and a self-adjoint operator  $L$  in  $X$ . The boundary conditions will be called *local* if for each  $v \in V$  there exists a subspace

$$X_v \subseteq \mathbb{K}^{E_{1,v}} \quad \text{if } v \in V_0, \quad X_v \subseteq \mathbb{K}^{E_{1,v} \cup \{v\}} \quad \text{if } v \in V_1,$$

and a selfadjoint operator  $L_v$  in  $X_v$ , such that

$$X = \bigoplus_{v \in V} X_v, \quad L = \bigoplus_{v \in V} L_v.$$

For  $v \in V$ , we define the “local trace mapping”

$$\operatorname{tr}_v: \prod_{e \in E_1} C[a_e, b_e] \times \mathbb{K}^{V_1} \rightarrow \begin{cases} \mathbb{K}^{E_{1,v}} & \text{if } v \in V_0, \\ \mathbb{K}^{E_{1,v} \cup \{v\}} & \text{if } v \in V_1 \end{cases}$$

by

$$\operatorname{tr}_v f := \begin{cases} \operatorname{tr} f|_{E_{1,v}} & \text{if } v \in V_0, \\ \operatorname{tr} f|_{E_{1,v} \cup \{v\}} & \text{if } v \in V_1. \end{cases}$$

Then for the form  $\tau$  we obtain

$$D(\tau) = \{f \in D(\iota); \operatorname{tr}_v(\iota f) \in X_v \ (v \in V)\},$$

$$\tau(f, g) = \sum_{e \in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x) \overline{(\iota_e g_e)'(x)} dx + \sum_{v \in V} (L_v \operatorname{tr}_v(\iota f) | \operatorname{tr}_v(\iota g)).$$

With

$$X_{v,0} := \begin{cases} X_v & \text{if } v \in V_0, \\ \{\xi \in X_v; \xi(v) = 0\} & \text{if } v \in V_1, \end{cases}$$

the condition (2.1) for  $D(\tau)$  to be dense then decomposes into

$$X_{v,0} \neq X_v \quad (v \in V_1),$$

or expressed differently, for all  $v \in V_1$  there exists  $\xi^v \in X_v$  such that  $\xi^v(v) = 1$ .

It is an easy task to translate the description of the associated operator  $H$ , given in Theorem 3.1, to the present case of local boundary conditions. The operator  $H$  associated with  $\tau$  is then given by

$$\begin{aligned} D(H) &= \{f \in \mathcal{H}_\Gamma; (f_e)_{e \in E_1} \in D(\hat{H}), \operatorname{tr}_v(\iota f) \in X_v, \\ &\quad Q_{v,0} \operatorname{str}_v(\iota f)' = Q_{v,0} L_v \operatorname{tr}_v(\iota f) \ (v \in V)\}, \\ ((Hf)_e)_{e \in E_1} &= \hat{H}(f_e)_{e \in E_1}, \\ (Hf)_v &= \frac{1}{\mu_v} (L_v \operatorname{tr}_v(\iota f) - \operatorname{str}_v(\iota f)' | \xi^v) \quad (v \in V_1). \end{aligned}$$

Here, for  $v \in V$  the mapping  $\operatorname{str}_v: \prod_{e \in E_1} BV(a_e, b_e) \rightarrow \mathbb{K}^{E_1, v}$  is defined by  $\operatorname{str}_v f := (\operatorname{str} f)|_{E_1, v}$ , and  $Q_{v,0}$  is the orthogonal projection onto  $X_{v,0}$  in  $\mathbb{K}^{E_1, v}$ , for  $v \in V_0$ , or in  $\mathbb{K}^{E_1, v \cup \{v\}}$ , for  $v \in V_1$ . We will not put down further details here. Similarly, the conditions for  $(e^{-tH})_{t \geq 0}$  to be positive and submarkovian, Theorem 4.1, can be spelled out in terms of the spaces  $X_v$  and the operators  $L_v$ . The statements are then analogous to [2; Theorem 3.5], where the case that  $E = E_1$  and  $V = V_0$  is treated.

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