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Enhancing Eigenvector Approximations of Huge Gyroscopic Eigenproblems from AMLS with Subspace Iteration

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***Abstract*: The Automated Multilevel Sub-structuring (AMLS) method is a powerful technique for computing a large number of eigenpairs with moderate accuracy for huge definite eigenvalue problems in structural analysis. It also turned out to be a useful tool to construct a suitable ansatz space for orthogonal projection methods for gyroscopic eigenvalue problems. Numerical examples demonstrate that the approximations to eigenvalues computed with AMLS are often of limited but sufficient accuracy, whereas the modal errors of the corresponding eigenvectors are still quite large. This paper takes advantage of information gained from AMLS to improve the obtained eigenpairs via a small number of subspace iteration steps.**

Keywords: eigenvalue, automated multilevel sub-structuring, subspace iteration, gyroscopic eigenproblem

1. INTRODUCTION

Simulation of acoustic properties has gained increasing importance in the engineering design process, in particular in automobile industries. Significant progress in reducing traffic noise has been achieved with respect to the sound radiation of the engines of vehicles. Hence, nowadays the noise radiated from rolling car tires, excited by road surface roughness and tread pattern impact, has become the major source of noise at speeds above 40 km/h for passenger cars and 60 km/h for trucks. Therefore, much effort has been directed into the development of methods allowing for a simulation of the effect of tire-road surface interaction.

According to [6,7] the simulation of the tire/road noise is performed in three subsequent steps. First, in a nonlinear steady state rolling analysis the distortion of the tire in contact with the road surface is computed using finite elements in an Arbitrary-Lagrangian-Eulerian (ALE) approach. Next, the transient vibrations governed by the eigenpairs of a gyroscopic eigenvalue problem

$$Q(\omega)x := Kx + \omega iGx - \omega^2 Mx = 0 \qquad (1)$$

are assumed to be superimposed onto the nonlinear deflections. Finally, the acoustic analysis is carried out solving Helmholtz's equation on the exterior of the domain occupied by the tire where the normal velocities at the wheel surface, extracted from the vibration analysis, are taken as boundary conditions.

In this paper we consider only the second step, i.e. the numerical solution of the eigenvalue problem (1) where K is the stiffness matrix modified by the inertia forces due to the stationary rolling, M is the mass matrix, and G is the gyroscopic matrix stemming from the Coriolis force. Clearly, K and M are symmetric and positive definite, and G is skew-symmetric. The eigenvalues ω (which are influenced by the rotational speed of the tire) are real, whereas the eigenvectors are complex, and have to be interpreted as traveling waves on the surface of the tire rather than standing vibrations.

Due to the complicated interior structure of a belted tire, the matrices K, M and G of a sufficiently accurate finite element model are very large and sparse. Moreover, for the acoustic analysis many eigenpairs (up to 2,000 Hz) are needed, when determining the initial conditions for the Helmholtz equation for the third step, which are computed in a Fourier analysis of the tire excitations by the roughness of the road surface.

A common approach for solving the quadratic eigenvalue problem is linearization, i.e. to transform (1) into an equivalent linear eigenvalue problem

$$Aq := \begin{bmatrix} iG & K \\ K & O \end{bmatrix} \begin{bmatrix} \omega x \\ x \end{bmatrix} = \omega \begin{bmatrix} M & O \\ O & K \end{bmatrix} \begin{bmatrix} \omega x \\ x \end{bmatrix} =: \omega Bq \qquad (2)$$

and to apply the shift-and-invert Lanczos method as implemented in the software package ARPACK [22]. Then in every iteration step ARPACK interrupts and in reverse communication the user has to supply the solution of a linear complex valued system $(A-\sigma B)z = Bc$ for some shift σ and right hand side c. Although the special structure of the matrices in (2) allows for an efficient solution of these systems this approach requires an excessive amount of storage and computing time (cf.[6]). Similar problems occur when applying structure preserving linearizations like SHIRA (*skew-Hamiltonian, isotropic implicitly restarted Arnoldi method*)[23]. More efficient for solving the gyroscopic eigenvalue problem (1) than the implicitly restarted Lanczos method is the *Automated Multi-Level Sub-structuring* (AMLS) method.

The (AMLS) method, proposed by Bennighof and co-authors[1,2,3,4,5,18], is an efficient condensation method for computing hundreds and thousands of eigenmodes and frequency responses for large and complex structures. The standard AMLS has been designed for linear symmetric eigenvalues problems and has been successfully applied to many engineering problems in recent years including vibro-acoustic analysis in automotive industry[20], ship vibrations[21], electromagnetic problems[24,29], and has been

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generalized to gyroscopic problems[13], and vibrations of fluid-solid structures[26]. Details of implementations are contained in [11,14,18,31].

AMLS was developed in the nineties of the last century for frequency response analysis. From October 2001, the computer code developed by Bennighof and co-workers was commercialized and distributed by the German company CDH GmbH, as an add-on module for Nastran. Since then, the software has been used by the automotive industry to solve large vibro-acoustic problems.

The commercial success of CDH/AMLS also stimulated the development of the method by other software companies. In 2004, MSC.Software offered an implementation of AMLS as part of MSC.Nastran. This implemention is called Matrix Domain Automated Component Modes Synthesis (MDACMS). In 2006, Dassault Systemes introduced a version of AMLS under the name ABAQUS/AMS, which is available as an add-on module for ABAQUS from version 6.6. Now, AMLS is employed widely in commercial codes to solve dynamic problems associated with large models specially in the field of vibration and acoustics of vehicles.

Compared to Krylov subspace type approaches AMLS reduces computational resources in terms of computing time and hardware requirements to determine a large number of eigenpairs at the lower end of the spectrum of a huge eigenvalue problem with high modal density. An evaluation and the comparison to the block Lanczos method for a broadband vibro-acoustic analysis of a passenger car body is contained in [20] demonstrating that AMLS enabled a reduction in runtime from several days on a supercomputer to a few hours using an off-the-shelf workstation.

It is important to note that AMLS usually provides approximate solutions which are less accurate than the ones obtained with Krylov type methods. However, in many applications, the underlying algebraic eigenvalue problem is a finite element model of the original continuous problem, and so the level of accuracy required for its numerical solution is no more than what is furnished by the FE model. Numerical examples demonstrate that the approximations to eigenvalues computed with AMLS are often of this limited but sufficient accuracy, whereas the modal errors of eigenvectors are usually still quite large. In many engineering areas including frequency response analysis in automobile industry AMLS provides sufficiently accurate results. However, in the sense of structural dynamics they are not satisfactory because the precision of extracted eigenvector approximations are too low to meet the requirements in strain and stress computations.

It is very costly to enhance the accuracy of the eigenvector approximations with AMLS. From the mathematical point of view, AMLS is a projection method where the large problem under consideration is projected to a subspace V spanned by all eigenmodes of the undamped clamped sub-structures on several levels the eigenfrequencies of which do not exceed a predetermined cut-off frequency ω_{cutoff}. Given the sub-structuring and ω_{cutoff} AMLS constructs the subspace V and the projected problem and (differently from iterative projection methods like the Lanczos of the Jacobi-Davidson method) there is no way to expand V reusing the information from the AMLS run. One has to rerun AMLS with a finer sub-structuring and/or a higher cut-off frequency.

In [31] we proposed a combination of AMLS with subspace iteration taking advantage of the block structure of the transformed stiffness matrix, and we generalized this approach to the gyroscopic eigenproblem (1) in [30]. In this paper we improve this approach taking advantage of symmetry properties of the linearization (2) which essentially divides the computing time in halves.

Our paper is organized as follows. In Chapter 2 we briefly summarize the AMLS method for linear positive definite eigenvalue problems, and Chapter 3 recapitulates AMLS for gyroscopic problems. Chapter 4 combines AMLS with subspace iteration in order to improve the eigenvalue and eigenvector approximations, and Chapter 5 demonstrates the efficiency of this approach for a huge finite element model of a rotating tire.

2. AMLS FOR LINEAR EIGENVALUE PROBLEMS

In this section we summarize the AMLS method for computing eigenvalues and corresponding eigenvectors of a linear eigenvalue problem

$$Kx = \lambda Mx \tag{3}$$

in a frequency range of interest. It was developed by Bennighof and co-workers over the last twenty years [1,2,3,4,5,18], who applied it to solve frequency response problems involving large and complex models. Usually (3) is a finite element model of some problem, where the stiffness matrix $K \in R^{n\times n}$ and the mass matrix $M \in R^{n\times n}$ are symmetric and M is positive definite. AMLS is a Rayleigh-Ritz method where the ansatz space is constructed making use of a multi-level domain decomposition.

Similarly as in the component mode synthesis method (CMS)[17,8] the structure is partitioned into a small number of sub-structures based on the sparsity pattern of the system matrices, but more generally than in CMS these sub-structures in turn are sub-structured on a number of levels yielding a tree topology for the sub-structures. Fig. 1 shows the graph of an example where each parent sub-structure has at most two children sub-structures. Other sub-structure trees are possible and are in use.

We stress the fact that sub-structuring does not necessarily mean that it is obtained by a domain decomposition of a real structure, but it is understood in a purely algebraic sense. Although sub-structuring by hand may yield much smaller interfaces than the ones obtained by automatic partitioners[12], the dissection of the matrices can be derived by applying a graph partitioner like CHACO[15] or METIS[19] to the undirected graph corresponding to the nonzero pattern of the matrices under consideration. These programs have been designed to construct fill-reducing orderings of sparse matrices for efficient factorization but have shown to be a beneficial basis of sub-structuring methods as well. In either case, because of its pictographic nomenclature we will use terms

like sub-structure or eigenmode from frequency response problems when describing the AMLS method.

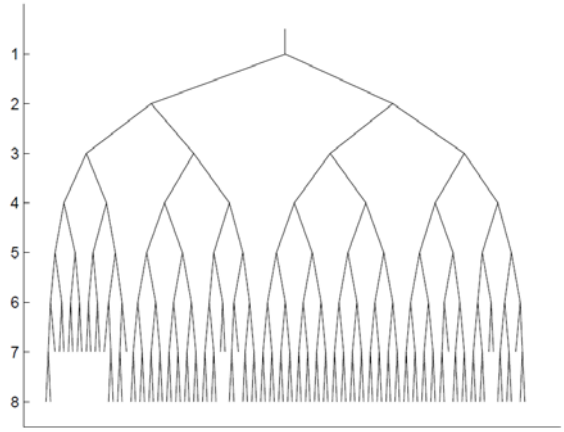

Figure 1 Sub-structure tree

AMLS consists of two ingredients. First, based on the sub-structuring the stiffness matrix K is transformed to block diagonal form by Gaussian elimination, and secondly, the dimension is reduced substantially by modal condensation of the sub-structures.

We describe a typical reduction step of AMLS. After reordering the degrees of freedom (this is assumed only for explanation; in an implementation the reordering is incorporated implicitly) the current reduced problem obtains the following form

$$\begin{bmatrix} K_p & O & O \\ O & K_c & K_{cr} \\ O & K_{rc} & K_r \end{bmatrix} \begin{bmatrix} x_p \\ x_c \\ x_r \end{bmatrix} = \lambda \begin{bmatrix} M_p & M_{pc} & M_{pr} \\ M_{cp} & M_c & M_{cr} \\ M_{rp} & M_{rc} & M_r \end{bmatrix} \begin{bmatrix} x_p \\ x_c \\ x_r \end{bmatrix} \quad (4)$$

where x_p denotes the degrees of freedom which already obtained in previous reduction steps, x_c are the degrees of freedom which are treated in the current step of AMLS, and x_r collects the ones to be handled in remaining steps of the algorithm. Notice, that x_p is empty in the first step, and x_c corresponds to unknowns of one substructure (in the first step of a leave of the substructure tree).So, the dimensions of K_c and M_c are usually quite small.
Applying the variable transformation

$$\begin{bmatrix} x_p \\ x_c \\ x_r \end{bmatrix} = \begin{bmatrix} I & O & O \\ O & I & -K_c^{-1}K_{cr} \\ O & O & I \end{bmatrix} \begin{bmatrix} x_p \\ \tilde{x}_c \\ x_r \end{bmatrix} =: U_c \tilde{x} \quad (5)$$

and multiplying with U_c^T from the left to retain the symmetry of the eigenvalue problem, the degrees of freedom of the current sub-structure are decoupled from the remaining ones in the stiffness matrix. Thus (4) obtains the following form:

$$\begin{bmatrix} K_p & O & O \\ O & K_c & O \\ O & O & \tilde{K}_r \end{bmatrix} \begin{bmatrix} x_p \\ \tilde{x}_c \\ x_r \end{bmatrix} = \lambda \begin{bmatrix} M_p & M_{pc} & \tilde{M}_{pr} \\ M_{cp} & M_c & \tilde{M}_{cr} \\ \tilde{M}_{rp} & \tilde{M}_{rc} & \tilde{M}_r \end{bmatrix} \begin{bmatrix} x_p \\ \tilde{x}_c \\ x_r \end{bmatrix}, \quad (6)$$

Where

$$\tilde{K}_r = K_r - K_{rc}K_c^{-1}K_{cr}$$

$$\tilde{M}_r = M_r - M_{rc}K_c^{-1}K_{cr} - K_{rc}K_c^{-1}M_{cr} + K_{rc}K_c^{-1}M_cK_c^{-1}K_{cr}$$

$$\tilde{M}_{pr} = M_{pr} - M_{pc}K_c^{-1}K_{cr} = \tilde{M}_{rp}^T$$

$$\tilde{M}_{cr} = M_{cr} - M_cK_c^{-1}K_{cr} = \tilde{M}_{rc}^T.$$

Recall that most of the columns of K_{cr} are null vectors. Only those columns of K_{cr} contain non-zero components the corresponding degree of freedom of which lies on the boundary of the current sub-structure. Hence, $K_c^{-1}K_{cr}$ can be computed very efficiently since only a very small number of small linear systems has to be solved.

Next the dimension of the problem is reduced by modal condensation of the current level. To this end the eigenvalue problem

$$K_c\phi_j = \omega_j M_c\phi_c, \quad \phi_c^T M_c\phi_j = \delta_{ij} \quad (7)$$

is solved, and problem (6) is projected to the space spanned by the columns of $Z_c = \text{tridiag}[I_p, P_c, I_r]$, where $\dim I_p = \dim .K_p$, $\dim I_r = \dim K_r$ and the columns of P_c are the eigenmodes of problem (7) corresponding to eigenvalues ω_j not exceeding the cut-off frequency ω_{cutoff}.

Again this step is very inexpensive because of the small dimension of K_c and M_c.

After we have passed through all sub-structures we finally arrive at the reduced eigenvalue problem

$$\tilde{K}\tilde{x} = \lambda\tilde{M}\tilde{x}, \quad (8)$$

where $\tilde{K}$ is a diagonal matrix.

Obviously, the modal reduction of problem (6) does not modify the block matrices $\tilde{K}_r$ and $\tilde{M}_r$ but only those blocks of both matrices which are handled currently or have been modified previously. Hence, we arrive at the same reduced problem if we first apply the variable transformations with (5) on all sub-structures, and then in a second step all modal reductions. Hence, if $U := U_1U_2\ldots U_m$ denotes the product of all transformations U_c in (5), and if $Z := Z_1Z_2\ldots Z_m$ denotes the product of all reduction matrices Z_c, then

$$\tilde{K} = Z^T\hat{K}Z = Z^TU^TKUZ \quad \text{and} \quad \tilde{M} = Z^T\hat{M}T = Z^TU^TMUZ.$$

Since all transformations in the first step are congruence transformations of the pencil (K, M) the eigenvalue problem (3) is equivalent to

$$\hat{K}\hat{x} = \lambda\hat{M}\hat{x}, \quad \hat{K} := U^TKU, \hat{M} := U^TMU, U := U_1U_2\ldots U_m \quad (9)$$

and $\hat{K}$ is a block diagonal matrix with very small blocks on its diagonal, whereas the block matrix $\hat{M}$ will contain very many dense sub-matrices. Hence, along with the AMLS reduction the transformed matrix $\hat{K}$ can be computed and stored cheaply, whereas the storage of $\hat{M}$ has to be avoided because it would require a huge amount of storage. We will take advantage of this observation in Section 4.

The description of AMLS above is valid for any order of the sub-structures, but handling the sub-structures in an appropriate ordering is most important to ensure computational efficiency. The transformed matrix (6) demonstrates that the decoupling of a sub-structure modifies only the matrices of sub-structures which are connected to it. Hence, to keep the storage needed for the mass matrix as

small as possible one should start the elimination at the leaves of the partition tree, and as soon as all sub-structures which are connected to an interface on the superior level have been reduced the interface should be decoupled from all other sub-structures in the stiffness matrix and should be reduced as well.

More generally, the symmetric Gaussian eliminations and the condensations are performed moving through the partition tree on a postorder traversal, i.e. beginning on the finest level visiting all tree nodes where a substructure is decoupled and reduced as soon as all of its children have been reduced. In this interleaving way one avoids the storage of large dense sub-matrices of the transformed mass matrix. We already mentioned that for any selection of the ordering the reduced matrix $\tilde{K}$ becomes diagonal, the mass matrix is reduced to a generalized arrow-head form shown in Fig. 2. Details of an implementation of AMLS are contained in [11,14,18,31].

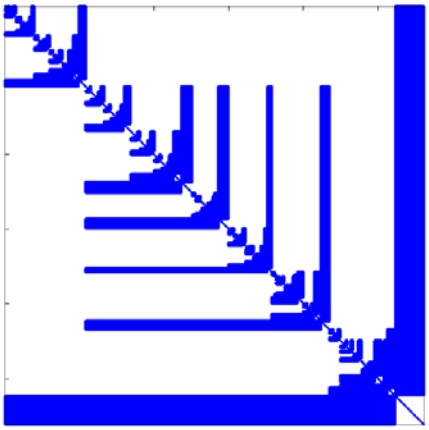

Figure 2 Reduced mass matrix

3. AMLS FOR GYROSCOPIC EIGENVALUE PROBLEMS

The gyroscopic eigenvalue problem (1) is equivalent to its Hermitian linearization

$$\begin{bmatrix} iG & K \\ K & O \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \omega \begin{bmatrix} M & O \\ O & K \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix}, \quad y = \omega x \qquad (10)$$

to which AMLS does not apply directly since the matrix on the left hand side is not positive definite, and for the transformed problem with $\mu := \omega^{-1}$ AMLS yields approximations to eigenpairs corresponding to the smallest eigenvalues μ_j, i.e. to the largest eigenvalues of (1) in modulus which are everything else of note.

Since the influence of the gyroscopic matrix G on the eigenvectors of (1) is usually not very high compared to the mass and stiffness matrices, it is reasonable to neglect the linear term in (1) when constructing the ansatz space for the Rayleigh-Ritz projection, i.e. when defining the sub-structuring, computing the transformations of K to block diagonal form, and employing the modal reductions corresponding to the sub-structures.

Hence, the AMLS reduction is applied to the pencil (K, M) as for a vibration problem in frequency response analysis, and the resulting congruence transformations and modal reductions are also applied to the skew-symmetric matrix G.

Thus, one obtains a reduced model

$$\tilde{K}\tilde{x} + i\omega\tilde{G}\tilde{x} - \omega^2\tilde{M}\tilde{x} = 0, \qquad (11)$$

where the reduced stiffness matrix $\tilde{K} = Z^T U^T K U Z$ and mass matrix $\tilde{M}$ are determined in the AMLS reduction as explained in Section 2, and $\tilde{G} = Z^T U^T G U Z$ is the projected gyroscopic matrix which can be evaluated along with $\tilde{K}$ and $\tilde{M}$ within the AMLS process for (K, M).

Hence, the reduced problem (11) has the same structure as the original gyroscopic eigenvalue problem (1), but it is of much smaller dimension and can therefore be solved by the Lanczos method using a linearization like (10) or by the nonlinear Arnoldi method[28] or even by a dense linear eigensolver.

It is important to note that differently from the Lanczos method all transformations in AMLS are real, and therefore the reduction can be performed in real arithmetic. Complex arithmetic is only required when solving the projected problem (11).

The paper [13] contains an example demonstrating the efficiency of this approach. A finite element model of a deformable wheel rolling on a rigid plane surface of dimension approximately 125,000 was reduced by AMLS to a gyroscopic problem of dimension 2,635 on a personal computer (namely a Pentium 4 processor with 3.0 GHz and 1 GB storage) requiring a CPU time of 976 seconds. Solving its linearization

$$\begin{bmatrix} i\tilde{G} & \tilde{K} \\ \tilde{K} & O \end{bmatrix}\begin{bmatrix} \tilde{y} \\ \tilde{x} \end{bmatrix} = \omega \begin{bmatrix} \tilde{M} & O \\ O & \tilde{K} \end{bmatrix}\begin{bmatrix} \tilde{y} \\ \tilde{x} \end{bmatrix}, \quad \tilde{y} = \omega\tilde{x} \qquad (12)$$

with the MATLAB function **eigs** (i.e. by ARPACK) needed another 124 seconds. Thus approximate eigenvalues for the smallest 180 eigenvalues (up to 2,000 Hz) were obtained the relative errors of which were all less than 0.65% . We will come back to this example in Section 5.

4. AMLS IMPROVED WITH SUBSPACE ITERATION

AMLS is a one shot projection method, i.e. after having chosen a cut-off frequency the method produces a fixed subspace $\mathrm{V} := \mathrm{span}\{V\}$, $V := UZ$ and the corresponding projected eigenproblem. Differently from iterative projections methods such as Krylov subspace or Jacobi-Davidson methods there is no way to expand the subspace V further reusing the projected problem if the computed approximate eigenpairs turn out to be not accurate enough. One has to repeat the reduction with a refined sub-structuring and/or a higher cut-off frequency.

Alternatively, one can improve the subspace V obtained with AMLS by subspace iteration. We explain the approach for the linear problem $Kx = \lambda Mx$ before we discuss its variant for the gyroscopic problem.

Let the columns of $V_0 \in R^{n\times p}$ form an approximate basis of the invariant subspace of the pencil (K, M)

corresponding to the wanted eigenvalues. Then one step of subspace iteration requires to solve a linear system $(K-\sigma M)V_1 = MV_0$ for V_1 where σ is some shift close to the wanted eigenvalues. However, for huge matrices K and M a factorization of $K-\sigma M$ and a solution of this system is very costly.

Alternatively, we may apply subspace iteration to the transformed problem

$$\hat{K}z := U^T KUz = \lambda U^T MUz =: \hat{M}z, \qquad (13)$$

where $U = U_1,\ldots,U_m$ is the matrix constructed in the AMLS process that transforms K to block diagonal form.

Due to the interleaving implementation of AMLS the matrices $\hat{K}$ and $\hat{M}$ are usually not stored when computing the reduced model, but in principle this could be easily done. The matrix $\hat{K}$ then obtains block diagonal form with moderate block sizes, but owing to fill in during the elimination process $\hat{M}$ will contain many dense sub-matrices requiring a huge amount of storage. So, this approach is also not efficient.

The way out is to combine the benefits of both approaches, i.e. to apply subspace iteration to the transformed system (13), but to evaluate $\hat{M}\hat{V}$ taking advantage of the transformation matrix U and the sparse structure of the original mass matrix M. This procedure was proposed in [31] for definite linear eigenvalue problems. In the following we modify this approach for improving the AMLS approximation of a gyroscopic problem (1).

We now assume that the matrices of the gyroscopic eigenvalue problem (1) are complex, i.e. $K \in C^{p\times p}$ and $M \in C^{p\times p}$ are Hermitian and positive definite and $G \in C^{p\times p}$ is skew-Hermitian (the matrices of the original problem are real, but approximations to eigenvectors will be complex, and so are the matrices of a projected problem (11) .

Then the following facts are well known [10]: problem (1) has 2*p* real eigenvalues, *p* of which are negative and *p* are positive. For real matrices *K*, *M* and *G* these eigenvalues come in pairs $(\lambda_j, -\lambda_j)$, $j = 1,\ldots,p$. In the general complex case positive and negative eigenvalues are of the same magnitude in modulus if *G* is small compared to *K* and *M*. The eigenvectors corresponding to the positive eigenvalues form a basis of C^p, and the same holds true for the eigenvectors corresponding to the negative eigenvalues.

If the subspace iteration is applied to the linearized eigenvalue problem (10) with shift $\sigma = 0$ (this is the only way to take advantage of the transformed block diagonal matrix $\hat{K}$) with an initial basis $X \in R^{2n\times p}$, then one obtains convergence to an invariant subspace corresponding to eigenvalues, *p*/2 of which are negative and *p*/2 are positive.

Hence, if X is obtained from an AMLS reduction and subsequent solution of the projected problem (11) then its columns are all approximations to eigenvectors corresponding to positive eigenvalues, and therefore convergence to the invariant subspace as described in the last paragraph will be very slow. This observation suggests to apply the subspace iteration in the following way [30]:

Let $V \in R^{n\times p}$ be the matrix of eigenvector approximations of $Kx = \lambda Mx$ obtained from AMLS, and let $\Lambda \in R^{p\times p}$ be the diagonal matrix containing the approximations to the *p* smallest positive eigenvalues. Then the following choice of the initial basis for the subspace iteration for (10) contains also good approximations to eigenvectors corresponding to the *p* largest negative eigenvalues, and we therefore can expect fast convergence:

$$X := \begin{bmatrix} V\Lambda^{1/2} & -V\Lambda^{1/2} \\ V & V \end{bmatrix} =: \begin{bmatrix} P^{(0)} \\ Q^{(0)} \end{bmatrix}. \qquad (14)$$

Then the *k*-th step of subspace iteration

$$\begin{bmatrix} iG & K \\ K & O \end{bmatrix}\begin{bmatrix} P^{(k)} \\ Q^{(k)} \end{bmatrix} = \begin{bmatrix} M & O \\ O & K \end{bmatrix}\begin{bmatrix} P^{(k-1)} \\ Q^{(k-1)} \end{bmatrix}$$

reads as follows:

$$P^{(k)} = Q^{(k-1)},\ KQ^{(k)} = MP^{(k-1)} - iGQ^{(k-1)}. \qquad (15)$$

After n_k steps of subspace iteration one gets $\breve{P} := P^{(n_k)}$ and $\breve{Q} := Q^{(n_k)}$ and the projected eigenvalue problem has the form

$$\begin{bmatrix} \breve{P}^H & \breve{Q}^H \end{bmatrix}\begin{bmatrix} iG & K \\ K & O \end{bmatrix}\begin{bmatrix} \breve{P} \\ \breve{Q} \end{bmatrix} z = \omega \begin{bmatrix} \breve{P}^H & \breve{Q}^H \end{bmatrix}\begin{bmatrix} M & O \\ O & K \end{bmatrix}\begin{bmatrix} \breve{P} \\ \breve{Q} \end{bmatrix} z$$

which is equivalent to

$$\breve{K}z := (i\breve{P}^H G\breve{P} + \breve{P}^H K\breve{Q} + \breve{Q}^H K\breve{P})z = \omega(\breve{P}^H M\breve{P} + \breve{Q}^H K\breve{Q})z =: \omega\breve{M}z.$$

To avoid the use of the transformed matrix $\hat{M} = U^T MU$ which is not computed in the AMLS method and which is much too memory-consuming we determine the right hand side of the linear system in (15) in original variables as $R = MP^{(k-1)} - iGP^{(k)}$. But to take advantage of the block structure of the transformed stiffness matrix $\hat{K} = U^T KU$ we solve the transformed system $\hat{K}\hat{Q}^{(k)} = \hat{R}$. Hence, in every iteration step two forward transformations $\hat{Q}^{(k)} := U^T Q^{(k)}$ and $\hat{R} = U^T R$ and one backward transformation $Q^{(k)} = U\hat{Q}^{(k)}$ are required. Details on the implementation of the forward and backward transformations are contained in [31].

Algorithm 1 contains a pseudo code of the resulting method. It is interesting to note that usually a small number of subspace iteration steps improves the approximations sufficiently, which also guarantees the stability of iterations (i.e. no orthogonalization between the individual iterations is necessary) with only one Rayleigh-Ritz analysis of $\breve{K}z = \omega\breve{M}z$.

However, this approach does not use the special structure of the matrices in problem (10) to its full capacity. If for some k the bases satisfy the following symmetry properties (cf. the initial bases(14))

$$P^{(k-1)} = [P_{k-1}, -\overline{P_{k-1}}] \quad \text{and} \quad Q^{(k-1)} = [Q_{k-1}, \overline{Q_{k-1}}] \qquad (16)$$

then it is easily verified that

$$P^{(k)} = [P_k, \overline{P_k}] \quad \text{and} \quad Q^{(k)} = [Q_k, -\overline{Q_k}] \tag{17}$$

ALGORITHM 1
SUBSPACE ITERATION WITH AMLS.

Require: Diagonal matrix Λ containing eigenvalue approximations from AMLS, transformed eigenvectors $\hat{V}$, the transformed stiffness matrix $\hat{K}$, the transformation matrix U from AMLS, and the maximum iteration number n_k
1: initialize the iteration matrices $\hat{Q}^{(0)} = [\hat{V}, \hat{V}]$ and $\hat{P}^{(0)} = [\hat{V}\Lambda^{1/2}, -\hat{V}\Lambda^{1/2}]$
2: transform backward $P^{(0)} = U\hat{P}^{(0)}$
3: **For** $k = 1, 2, \ldots, n_k$ **do**
4: transform backward $Q^{(k-1)} = U\hat{Q}^{(k-1)}$
5: compute $R = MP^{(k-1)} - iGQ^{(k-1)}$
6:transform forward $\hat{R} = U^T R$
7: $P^{(k)} = Q^{(k-1)}$
8: solve for $\hat{Q}^{(k)}$: $\hat{K}\hat{Q}^{(k)} = \hat{R}$
9: **end for**
10: $T = \hat{R}^H \hat{Q}^{(n_k)}$
11: projected mass matrix $\breve{M} = (P^{(n_k)})^H M P^{(n_k)} + T$
12: reload R and compute $S = R^H P^{(n_k)}$
13:projected stiffness matrix $\breve{K} = i(P^{(n_k)})^H G P^{(n_k)} + S + S^H$.
14: solve projected problem $\breve{K}Z = \breve{M}Z\breve{\Lambda}$
15: sort out positive eigenvalue $\breve{\Lambda}_+$ and corresponding eigenvectors Z_+
16: compute improved eigenvectors $V^{(n_k)} = U\hat{Q}^{n_k} Z_+$.

and

$$P^{(k+1)} = [P_{k+1}, -\overline{P_{k+1}}] \quad \text{and} \quad Q^{(k+1)} = [Q_{k+1}, \overline{Q_{k+1}}]. \tag{18}$$

Hence, the symmetry is preserved in every other step.

For the matrix $R^{(k-1)} := MP^{(k-1)} - iGQ^{(k-1)}$ in step 5 of Algorithm 1 we obtain

$$R^{(k-1)} = [R_{k-1}, -\overline{R_{k-1}}] \quad \text{and} \quad R^{(k)} = [R_k, \overline{R_k}]. \tag{19}$$

ALGORITHM 2
SUBSPACE ITERATION WITH AMLS UTILIZING SYMMETRY.

Require: Diagonal matrix Λ containing eigenvalue approximations from AMLS, transformed eigenvectors $\hat{V}$, the transformed stiffness matrix $\hat{K}$ the transformation matrix U from AMLS, and the maximum iteration number n_k
1: initialize the iteration matrices $\hat{Q}_0 = \hat{V}$ and $\hat{P}_0 = \hat{V}\Lambda^{1/2}$
2: transform backward $P_0 = U\hat{P}_0$
3: **For** $k = 1, 2, \ldots, n_k$ **do**
4: transform backward $Q_{k-1} = U\hat{Q}_{k-1}$
5: compute $R = MP_{k-1} - iGQ_{k-1}$
6:transform forward $\hat{R} = U^T R$
7: $P_k = Q_{k-1}$
8: solve for $\hat{Q}_k$: $\hat{K}\hat{Q}_k = \hat{R}$
9: **end for**
10: $T = [\hat{R}, -\hat{R}]^H [\hat{Q}_{n_c}, -\hat{Q}_{n_c}]$
11: $\breve{M} = [P_{n_c}, \overline{P_{n_c}}]^H M [P_{n_c}, \overline{P_{n_c}}] + T$
12: reload R and compute $S = [R, -\overline{R}]^H [P_{n_c}, \overline{P_{n_c}}]$
13: $\breve{K} = i[P_{n_c}, \overline{P_{n_c}}]^H G [P_{n_c}, \overline{P_{n_c}}] + S + S^H$.
14: solve projected problem $\breve{K}Z = \breve{M}Z\breve{\Lambda}$
15: sort out positive eigenvalue $\breve{\Lambda}_+$ and corresponding eigenvectors Z_+
16: compute improved eigenvectors $V_{n_k} = U[\hat{Q}_{n_k}, -\hat{Q}_{n_k}]Z_+$.

Hence, Algorithm 1 can be modified such that it iterates only on matrices $P_k, Q_k, R_k \in C^{n \times p}$ instead of $P^{(k)}, Q^{(k)}, R^{(k)} \in C^{n \times 2p}$, and uses the symmetry properties above to determine the projected problem. This essentially cuts the arithmetical cost into halves.

In Algorithm 2 we use $P^{(n_k)} = [P_{n_k}, \overline{P_{n_k}}]$ and $Q^{(k)} = [Q_{n_k}, -\overline{Q_{n_k}}]$ which corresponds to an odd number of iteration steps. If n_k is even, then the symmetry (17) is arrived after n_k steps for the initial matrices $P^{(0)} = [V\Lambda^{1/2}, V\Lambda^{1/2}]$ and $Q^{(0)} = [V, -V]$.

5. NUMERICAL RESULTS

We consider a finite element model of a deformable wheel rolling on a rigid plane surface which is obtained by an Arbitrary Lagrangian Eulerian (ALE) formulation according to the derivation and presentation in [6]. Our model of a rotating tire consists of 39,204 brick elements with 124,992 degrees of freedom and accounts for 20 different material groups (cf. Figure 3). The speed is assumed to be 60 km/h. Our aim is to determine approximations to the smallest 200 eigenvalues and corresponding eigenvectors.

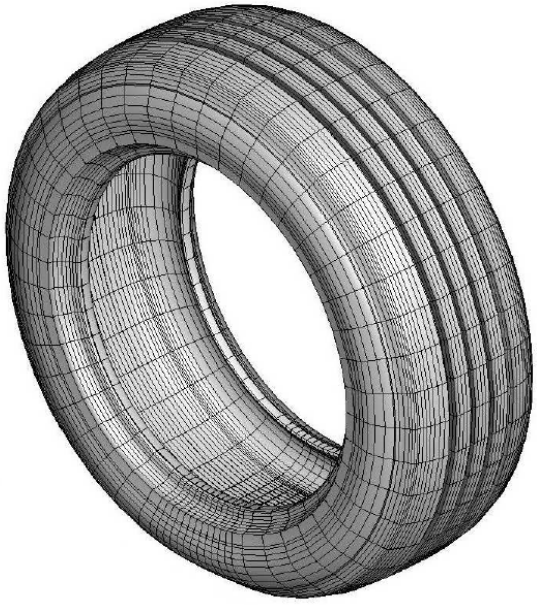

Figure 3 Continental AG 205/55R16-91H tire

Applying the shift-and-invert Lanczos method to the linearization (2) of the gyroscopic eigenproblem (1) requires an LU factorization of $Q(\omega)$ for every shift ω, which is a complex matrix. Determining the factorization by SuperLU[9] requires 6.04 GByte of memory and a CPU time of 3910 seconds on one PA-RISC(750 MHz) processor of an HP superdome.

Applying the nonlinear Arnoldi method[27] the preconditioners can be chosen as real matrices $K - \omega^2 M$, the LU factorization of which requires 2.7 GByte storage

and 1,940 seconds with SuperLU, and 2.86 GByte storage and 1,080 seconds with the multi frontal solver MA57 of HSL[16]. Since the LU factorization has to be updated several times a total CPU time of more than 12 hours results on one processor of the superdome.

AMLS demands much less storage and the problem under consideration can be solved on a workstation. The numerical tests were performed on a 64-bit HP workstation with an Intel Xeon CPU(3.20 GHz, 2 cores) and 24GB memory. AMLS and the two subspace iteration algorithms were implemented with Matlab R2009a.

The AMLS method addressing the linear eigenvalue problem $Kx = \lambda Mx$ costs 881.2 seconds for the AMLS reduction and 270.4 seconds for solving the projected linear eigenvalue problem of dimension 2,263 by **eig**.

Following Bathe's recommendation to use $\min\{2p, p+8\}$ initial vectors in subspace iteration if p eigenpairs are wanted we initialized the subspace iteration methods with the lowest 208 positive eigenvalues and corresponding eigenvectors.

Executing the subspace iteration for problem (10) requires the solution of a linear system

$$KQ_k = MP_{k-1} - iGQ_{k-1}$$

for Q_k in every iteration step. Solving this problem directly with PARDISO[25] without utilizing the block diagonal structure of $\breve{K}$ needs 1235.8 seconds.

Table 1 shows the computing times for the combination of subspace iteration and AMLS with (Algo.2) and without (Algo.1) taking advantage of the special structure of problem (10). It demonstrates clearly the superiority of our approach upon the direct application of subspace iteration, and that the computing time is essentially reduced by Algorithm 2.

TABLE 1. COMPUTATION TIME WITH 208 ITERATION VECTORS

Computational Steps	Algo. 2(s)	Algo. 1(s)
Initial eigenvectors from AMLS	2.4	2.4
1 Iteration	67.8	148.9
2 Iterations	141.3	304.9
3 Iterations	216.8	462.1
4 Iterations	289.4	621.8
Compute $\breve{K}$ and $\breve{M}$ if $n_k = 1$	6.2	9.7
Compute $\breve{K}$ and $\breve{M}$ if $n_k > 1$	8.7	19.6
Solve $\breve{K}Z = \breve{M}Z\breve{\Lambda}$ by **eig**	2.6	2.6
Solve $\breve{K}Z = \breve{M}Z\breve{\Lambda}$ by **eig**	2.6	2.6
Compute final eigenvectors	25.6	25.6

The relative errors of eigenvalues computed by both methods with four iteration steps are given in Fig.4.

To evaluate the accuracy of eigenpairs we use modal errors

$$\eta_g = \frac{\| Kx + i\omega Gx - \omega^2 Mx\|}{\| \omega^2 Mx\|}. \qquad (20)$$

Fig.5 shows the reduction of modal errors for four iterations. It is interesting to note that essential improvements are obtained only every other iteration step.

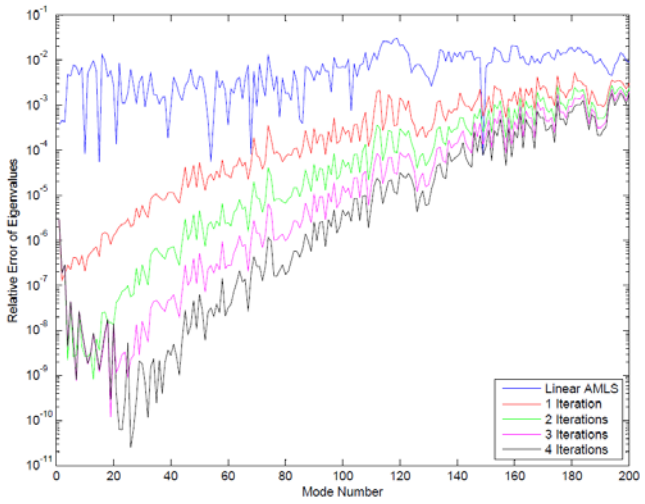

Figure 4 Relative errors of eigenvalues computed by subspace iteration with AMLS utilizing 208 iteration vectors

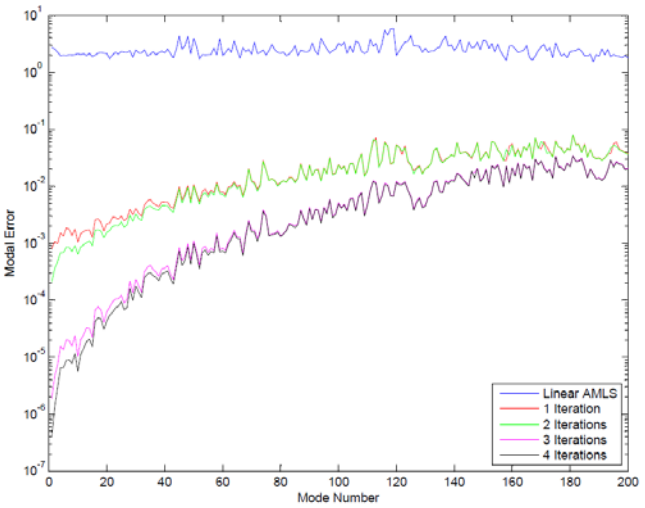

Figure 5 Modal errors computed by subspace iteration with AMLS utilizing 208 iteration vectors

6. CONCLUSIONS

AMLS is a powerful tool for computing a large number of eigenvalues of huge eigenvalue problems. Numerical examples demonstrate that the approximations to eigenvalues computed with AMLS are often of limited but sufficient accuracy, whereas the modal errors of eigenvectors are usually quite large. In this paper we propose a combination of AMLS with subspace iteration which takes advantage of the block structure of the transformed stiffness matrix but does not use the transformed mass matrix (which would contain a large number of dense sub-matrices and which is not computed in the AMLS process). A similar approach was already proposed in [31] for definite eigenvalue problems in structural analysis, and it could be also modified for improving eigenvector approximations obtained by AMLS for unsymmetric eigenvalue problems in fluid-solid vibrations [26].

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