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SOLVING HUGE GYROSCOPIC EIGENPROBLEMS WITH AMLS AND SUBSPACE ITERATION

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ABSTRACT

The Automated Multilevel Sub-structuring (AMLS) method is a powerful technique for computing a large number of eigenpairs with moderate accuracy for huge definite eigenvalue problems in structural analysis. It also turned out to be a useful tool to construct a suitable ansatz space for orthogonal projection methods for gyroscopic problems. This paper takes advantage of information gained from AMLS to improve the obtained eigenpairs via a small number of subspace iteration steps.

1 Introduction

Simulation of acoustic properties has gained increasing importance in the engineering design process, in particular in automobile industries. Significant progress in reducing traffic noise has been achieved with respect to the sound radiation of the engines of vehicles. Hence, nowadays the noise radiated from rolling car tires, excited by road surface roughness and tread pattern impact, has become the major source of noise at speeds above 40 km/h for passenger cars and 60 km/h for trucks. Therefore, much effort has been directed into the development of methods allowing for a simulation of the effect of tire-road surface interaction.

According to [1, 2] the simulation of the tire/road noise is performed in three subsequent steps. First, in a nonlinear steady state rolling analysis the distortion of the tire in contact with the road surface is computed using finite elements in an Arbitrary–Lagrangian–Eulerian (ALE) approach. Next, the transient vibrations governed by the eigenpairs of a gyroscopic eigenvalue problem

$$Q(\omega)x := Kx + \omega iGx - \omega^2 Mx = 0. \tag{1}$$

are assumed to be superimposed onto the nonlinear deflections. Finally, the acoustic analysis is carried out solving Helmholtz's equation on the exterior of the domain occupied by the tire where the normal velocities at the wheel surface, extracted from the vibration analysis, are taken as boundary conditions.

In this paper we consider only the second step, i.e. the numerical solution of the eigenproblem (1) where K is the stiffness matrix modified by the inertia forces due to the stationary rolling, M is the mass matrix, and G is the gyroscopic matrix stemming from the Coriolis force. Clearly, K and M are symmetric and positive definite, and G is skew–symmetric. The eigenvalues ω (which are influenced by the rotational speed of the tire) are real, whereas the eigenvectors are complex, and have to be interpreted as traveling waves on the surface of the tire rather than standing vibrations.

Due to the complicated interior structure of a belted tire the matrices K, M and G of a sufficiently accurate finite element model are very large and sparse. Moreover, for the acoustic analysis many eigenpairs (up to 2000 Hz) are needed, when determining the initial conditions for the Helmholtz equation for the third step, which are computed in a Fourier analysis of the tire excitations by the roughness of the road surface.

A common approach for solving the quadratic eigenvalue problem is linearization, i.e. to transform (1) into an equivalent

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linear eigenvalue problem

$$Aq := \begin{bmatrix} iG & K \\ K & O \end{bmatrix} \begin{bmatrix} \omega x \\ x \end{bmatrix} = \omega \begin{bmatrix} M & O \\ O & K \end{bmatrix} \begin{bmatrix} \omega x \\ x \end{bmatrix} =: \omega Bq \quad (2)$$

and to apply the shift-and-invert Lanczos method as implemented in the software package ARPACK [3]. Then in every iteration step ARPACK interrupts and in reverse communication the user has to supply the solutions of a linear complex valued system $(A - \sigma B)z = Bc$ for some shift σ and right hand side c. Although the special structure of the matrices in (2) allows for an efficient solution of these systems this approach requires an excessive amount of storage and computing time (cf. [1]). More efficient for solving the gyroscopic eigenvalue problem (1) than the implicitly restarted Lanczos method is the *Automated Multi-Level Sub-structuring* (AMLS) method.

The (AMLS) method, proposed by Bennighof and co-authors [4, 5], is an efficient condensation method for computing hundreds and thousands of eigenmodes and frequency responses for large and complex structures. The standard AMLS has been designed for linear symmetric eigenvalues problems and has been successfully applied to many engineering problems in recent years including vibro-acoustic analysis in automotive industry [6], ship vibrations [7], electromagnetic problems [8, 9], and has been generalized to gyroscopic problems [10], and vibrations of fluid–solid structures [11]. Details of implementations are contained in [12–15].

Compared to Krylov subspace type approaches AMLS reduces computational resources in terms of computing time and hardware requirements to determine a large number of eigenpairs at the lower end of the spectrum of a huge eigenvalue problem with high modal density. An evaluation and the comparison to the block Lanczos method for a broadband vibro-acoustic analysis of a passenger car body is contained in [6] demonstrating that AMLS enabled a reduction in runtime from several days on a supercomputer to a few hours using an off-the-shelf workstation.

It is important to note that AMLS usually provides approximate solutions which are less accurate than the ones obtained with Krylov type methods. However, in many applications, the underlying algebraic eigenvalue problem is a finite element model of the original continuous problem, and so the level of accuracy required for its numerical solution is no more than what is furnished by the FE model. Numerical examples demonstrate that the approximations to eigenvalues computed with AMLS are often of this limited but sufficient accuracy, whereas the modal errors of eigenvectors are usually still quite large.

In [15] we proposed a combination of AMLS with subspace iteration taking advantage of the block structure of the transformed stiffness matrix, and we generalized this approach to the gyroscopic eigenproblem (1) in [16]. In this paper we improve this approach taking advantage of symmetry properties of the linearization (2) which essentially divides the computing time in halves.

Our paper is organized as follows. In Chapter 2 we briefly summarize the AMLS method for linear positive definite eigenvalue problems, and Chapter 3 recapitulates AMLS for gyroscopic problems. Chapter 4 combines AMLS with subspace iteration in order to improve the eigenvalue and eigenvector approximations, and Chapter 5 demonstrates the efficiency of this approach for a huge finite element model of a rotating tire.

2 AMLS for linear eigenvalue problems

In this subsection we summarize the AMLS method for computing eigenvalues and corresponding eigenvectors of a linear eigenvalue problem

$$Kx = \lambda Mx \quad (3)$$

in a frequency range of interest. Usually (3) is a finite element model of some problem, where the stiffness matrix $K \in R^{n\times n}$ and the mass matrix $M \in R^{n\times n}$ are symmetric and M is positive definite. AMLS is a Rayleigh–Ritz method where the ansatz space is constructed making use of a multi–level domain decomposition.

Similarly as in the component mode synthesis method (CMS) [17] the structure is partitioned into a small number of substructures based on the sparsity pattern of the system matrices, but more generally than in CMS these substructures in turn are sub-structured on a number of levels yielding a tree topology for the substructures.

AMLS consists of two ingredients. First, based on the substructuring the stiffness matrix K is transformed to block diagonal form by Gaussian elimination, and secondly, the dimension is reduced substantially by modal condensation of the substructures.

If K_{ss} is a sub-matrix of K corresponding to a particular substructure, then after reordering rows and columns in (3) the pencil obtains the form

$$\left(\begin{bmatrix} K_{ss} & K_{sr} \\ K_{rs} & K_{rr} \end{bmatrix}, \begin{bmatrix} M_{ss} & M_{sr} \\ M_{rs} & M_{rr} \end{bmatrix} \right),$$

and with block Gaussian elimination, i.e. post- and premultiplying this pencil with

$$U_s = \begin{bmatrix} I & -K_{ss}^{-1}K_{sr} \\ O & I \end{bmatrix}$$

and U_s^T, respectively, K_{ss} is decoupled, and the pencil obtains the

following form

$$\left(U_s^T K U_s, U_s^T M U_s\right) = \left(\begin{bmatrix} K_{ss} & O \\ O & \tilde{K}_{rr} \end{bmatrix}, \begin{bmatrix} M_{ss} & \tilde{M}_{sr} \\ \tilde{M}_{sr}^T & \tilde{M}_{rr} \end{bmatrix} \right)$$

where

$$\begin{aligned} \tilde{K}_{rr} &= K_{rr} - K_{rs}K_{ss}^{-1}K_{sr}, \\ \tilde{M}_{rr} &= M_{rr} - M_{rs}K_{ss}^{-1}K_{si} - K_{rs}K_{ss}^{-1}M_{sr} + K_{rs}K_{ss}^{-1}M_{ss}K_{ss}^{-1}K_{sr} \\ \tilde{M}_{sr} &= M_{sr} - M_{ss}K_{ss}^{-1}K_{sr} = \tilde{M}_{rs}^T. \end{aligned}$$

Repeating the block elimination for all substructures $1,\ldots,m$ we get

$$\tilde{K} = U^T K U,\ \tilde{M} = U^T M U \text{ with } U = U_1 U_2 \ldots U_m$$

where the transformed stiffness matrix $\tilde{K}$ has block diagonal form. Notice that in an implementation of AMLS the reordering of the matrices is incorporated implicitly.

To reduce the dimension of the eigenproblem we determine for every substructure (after decoupling it from the remaining degrees of freedom in the stiffness matrix as above, and neglecting connections to other substructures in the mass matrix) all eigenvalues λ_{sj} not exceeding a cut off frequency λ_{cutoff} and corresponding eigenvectors z_{sj}, $j = 1,\ldots,m_s$. Then with $Z_s = [z_{s1},\ldots,z_{sm_s}]$ and the global block diagonal projection matrix $Z = \text{diag}\{Z_1,\ldots,Z_m\}$ we finally get the reduced eigenvalue problem

$$K_c x_c = \lambda M_c x_c \tag{4}$$

where

$$K_c = Z^T \tilde{K} Z = Z^T U^T K U Z$$

is a diagonal matrix and

$$M_c = Z^T \tilde{M} Z = Z^T U^T M U Z$$

has generalized block arrowhead form.

It is important to note that in an implementation the block Gaussian eliminations and the condensations are performed in an interleaving way to avoid the storage of large dense sub-matrices of the transformed mass matrix which would occur in the course of the block elimination: as soon as a sub-matrix pencil $(\tilde{K}_{ss}, \tilde{M}_{ss})$ has been formed, the eigenproblem $\tilde{K}_{ss} Z_s = \tilde{M}_{ss} Z_s \Lambda_s$ is solved and the corresponding projection is executed. Details of an implementation of AMLS are contained in [12–15].

3 AMLS for gyroscopic eigenvalue problems

The gyroscopic eigenvalue problem (1) is equivalent to its Hermitian linearization

$$\begin{bmatrix} iG & K \\ K & O \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \omega \begin{bmatrix} M & O \\ O & K \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}, \quad y = \omega x \tag{5}$$

to which AMLS does not apply directly since the matrix on the left hand side is not positive definite, and for the transformed problem with $\mu := \omega^{-1}$ AMLS yields approximations to eigenpairs corresponding to the smallest eigenvalues μ_j, i.e. to the largest eigenvalues of (1) in modulus which are everything else of note.

Since the influence of the gyroscopic matrix G on the eigenvectors of (1) is usually not very high compared to the mass and stiffness matrices, it is reasonable to neglect the linear term in (1) when constructing the ansatz space for the Rayleigh–Ritz projection, i.e. when defining the substructuring, computing the transformations of K to block diagonal form, and employing the modal reductions corresponding to the substructures.

Hence, the AMLS reduction is applied to the pencil (K,M) as for a vibration problem in frequency response analysis, and the resulting congruence transformations and modal reductions are also applied to the skew-symmetric matrix G.

Thus, one obtains a reduced model

$$K_c y + i\omega G_c y - \omega^2 M_c y = 0, \tag{6}$$

where the reduced stiffness matrix $K_c = Z^T U^T K U Z$ and mass matrix M_c are determined in the AMLS reduction as explained in Section 2, and $G_c = Z^T U^T G U Z$ is the projected gyroscopic matrix which can be evaluated along with K_c and M_c within the AMLS process for (K,M).

Hence, the reduced problem (6) has the same structure as (1), but it is of much smaller dimension and can therefore be solved by the Lanczos method using a linearization like (5) or by the nonlinear Arnoldi method or even by a dense linear eigensolver.

It is important to note, that differently from the Lanczos method all transformations in AMLS are real, and therefore the reduction can be performed in real arithmetic. Complex arithmetic is only required when solving the projected problem (6).

The paper [10] contains an example demonstrating the efficiency of this approach. A finite element model of a deformable wheel rolling on a rigid plane surface of dimension approximately 125000 was reduced by AMLS to a gyroscopic problem of dimension 2635 an a personal computer (namely a Pentium 4 processor with 3.0 GHz and 1 GB storage) requiring a CPU time of 976 seconds. Solving its linearization (5) with the MATLAB function `eigs` (i.e. by ARPACK) needed another 124 seconds.

Thus approximate eigenvalues for the smallest 180 eigenvalues (up to 2000 Hz) were obtained the relative errors of which were all less than 0.65%. We will come back to this example in Section 5.

4 AMLS improved with subspace iteration

AMLS is a one shot projection method, i.e. after having chosen a cut-off frequency the method produces a fixed subspace $\mathcal{V} := \text{span}\{V\}$, $V := UZ$ and the corresponding projected eigenproblem. Differently from iterative projections methods such as Krylov subspace or Jacobi–Davidson methods there is no way to expand the subspace $\mathcal{V}$ further reusing the projected problem if the computed approximate eigenpairs turn out to be not accurate enough. One has to repeat the reduction with a higher cut-off frequency.

Alternatively, one can improve the subspace $\mathcal{V}$ obtained with AMLS by subspace iteration. We explain the approach for the linear problem $Kx = \lambda Mx$ before we discuss its variant for the gyroscopic problem.

Let the columns of $V_0 \in R^{n\times p}$ form an approximate basis of the invariant subspace of the pencil (K,M) corresponding to the wanted eigenvalues. Then one step of subspace iteration requires to solve a linear system $(K-\sigma M)V_1 = MV_0$ for V_1 where σ is some shift close to the wanted eigenvalues. However, for huge matrices K and M a factorization of $K-\sigma M$ and a solution of this system is very costly.

Alternatively, we may apply subspace iteration to the transformed problem

$$\tilde{K}z := U^T KUz = \lambda U^T MUz =: \tilde{M}z, \tag{7}$$

where $U = U_1,\ldots,U_m$ is the matrix constructed in the AMLS process that transforms K to block diagonal form. Due to the interleaving implementation of AMLS the matrices $\tilde{K}$ and $\tilde{M}$ are usually not stored when computing the reduced model, but in principle this could be easily done. The matrix $\tilde{K}$ then obtains block diagonal form with moderate block sizes, but owing to fill in during the elimination process $\tilde{M}$ will contain many dense submatrices requiring a huge amount of storage. So, this approach is also not efficient.

The way out is to combine the benefits of both approaches, i.e. to apply subspace iteration to the transformed system (7), but to evaluate $\tilde{M}\tilde{V}$ taking advantage of the transformation matrix U and the sparse structure of the original mass matrix M. This procedure was proposed in [15] for definite linear eigenvalue problems. In the following we modify this approach for improving the AMLS approximation of a gyroscopic problem (1).

We now assume that the matrices of the gyroscopic eigenvalue problem (1) are complex, i.e. $K \in C^{p\times p}$ and $M \in C^{p\times p}$ are Hermitian and positive definite and $G \in C^{p\times p}$ is skew–Hermitian (the matrices of the original problem are real, but approximations to eigenvectors will be complex, and so are the matrices of a projected problem (6)).

Then the following facts are well known (cf. [18]): problem (1) has $2p$ real eigenvalues, p of which are negative and p are positive. For real matrices K, M and G these eigenvalues come in pairs $(\lambda_j,-\lambda_j)$, $j = 1,\ldots,p$. In the general complex case positive and negative eigenvalues are of the same magnitude in modulus if G is small compared to K and M. The eigenvectors corresponding to the positive eigenvalues form a basis of C^p, and the same holds true for the eigenvectors corresponding to the negative eigenvalues.

If the subspace iteration is applied to the linearized eigenvalue problem (5) with shift $\sigma = 0$ (this is the only way to take advantage of the transformed block diagonal matrix $\tilde{K}$) with an initial basis $X \in R^{2n\times p}$, then one obtains convergence to an invariant subspace corresponding to eigenvalues, $p/2$ of which are negative and $p/2$ are positive.

Hence, if X is obtained from an AMLS reduction and subsequent solution of the projected problem (6) then its columns are all approximations to eigenvectors corresponding to positive eigenvalues, and therefore convergence to the invariant subspace as described in the last paragraph will be very slow. This observation suggests to apply the subspace iteration in the following way [16]:

Let $V \in R^{n\times p}$ be the matrix of eigenvector approximations of $Kx = \lambda Mx$ obtained from AMLS, and let $\Lambda \in R^{p\times p}$ be the diagonal matrix containing the approximations to the p smallest positive eigenvalues. Then the following choice of the initial basis for the subspace iteration for (5) contains also good approximations to eigenvectors corresponding to the p largest negative eigenvalues, and we therefore can expect fast convergence:

$$X := \begin{bmatrix} V\Lambda^{1/2} & -V\Lambda^{1/2} \\ V & V \end{bmatrix} =: \begin{bmatrix} P^{(0)} \\ Q^{(0)} \end{bmatrix}. \tag{8}$$

Then the kth step of subspace iteration

$$\begin{bmatrix} iG & K \\ K & O \end{bmatrix} \begin{bmatrix} P^{(k)} \\ Q^{(k)} \end{bmatrix} = \begin{bmatrix} M & O \\ O & K \end{bmatrix} \begin{bmatrix} P^{(k-1)} \\ Q^{(k-1)} \end{bmatrix}$$

reads as follows:

$$P^{(k)} = Q^{(k-1)},\ KQ^{(k)} = MP^{(k-1)} - iGQ^{((k-1))}. \tag{9}$$

After n_k steps of subspace iteration one gets $\hat{P} := P^{(n_k)}$ and $\hat{Q} := Q^{(n_k)}$ and the projected eigenvalue problem has the form

$$\begin{bmatrix} \hat{P}^H & \hat{Q}^H \end{bmatrix} \begin{bmatrix} iG & K \\ K & O \end{bmatrix} \begin{bmatrix} \hat{P} \\ \hat{Q} \end{bmatrix} z = \omega \begin{bmatrix} \hat{P}^H & \hat{Q}^H \end{bmatrix} \begin{bmatrix} M & O \\ O & K \end{bmatrix} \begin{bmatrix} \hat{P} \\ \hat{Q} \end{bmatrix} z$$

which is equivalent to

$$\hat{K}z := (i\hat{P}^H G\hat{P} + \hat{P}^H K\hat{Q} + \hat{Q}^H K\hat{P})z = \omega(\hat{P}^H M\hat{P} + \hat{Q}^H K\hat{Q})z =: \omega\hat{M}z.$$

To avoid the use of the transformed matrix $\tilde{M} = U^T MU$ which is not computed in the AMLS method and which is much too memory–consuming we determine the right hand side of the linear system in (9) in original variables as $R = MP^{(k-1)} - iGP^{(k)}$. But to take advantage of the block structure of the transformed stiffness matrix $\tilde{K} = U^T KU$ we solve the transformed system $\tilde{K}\tilde{Q}^{(k)} = \tilde{R}$. Hence, in every iteration step two forward transformations $\tilde{Q}^{(k)} := U^T Q^{(k)}$ and $\tilde{R} = U^T R$ and one backward transformation $Q^{(k)} = U\tilde{Q}^{(k)}$ are required. Details on the implementation of the forward and backward transformations are contained in [15].

Algorithm 1 contains a pseudocode of the resulting method. It is interesting to note that usually a small number of subspace iteration steps improves the approximations sufficiently, which also guarantees the stability of iterations (i.e. no orthogonalization between the individual iterations is necessary) with only one Rayleigh-Ritz analysis of $\hat{K}z = \omega\hat{M}z$.

Algorithm 1 Subspace iteration with AMLS.

Require: Diagonal matrix Λ containing eigenvalue approximations from AMLS, transformed eigenvectors $\tilde{V}$, the transformed stiffness matrix $\tilde{K}$ the transformation matrix U from AMLS, and the maximum iteration number n_k

1: initialize the iteration matrices $\tilde{Q}^{(0)} = [\tilde{V}, \tilde{V}]$ and $\tilde{P}^{(0)} = [\tilde{V}\Lambda^{1/2}, -\tilde{V}\Lambda^{1/2}]$
2: transform backward $P^{(0)} = U\tilde{P}^{(0)}$
3: **for** $k = 1, 2, \ldots, n_k$ **do**
4: transform backward $Q^{(k-1)} = U\tilde{Q}^{(k-1)}$
5: compute $R = MP^{(k-1)} - iGQ^{(k-1)}$
6: transform forward $\tilde{R} = U^T R$
7: $P^{(k)} = Q^{(k-1)}$
8: solve for $\tilde{Q}^{(k)}$: $\tilde{K}\tilde{Q}^{(k)} = \tilde{R}$
9: **end for**
10: $T = \tilde{R}^H \tilde{Q}^{(n_k)}$
11: projected mass matrix $\hat{M} = (P^{(n_k)})^H MP^{(n_k)} + T$
12: reload R and compute $S = R^H P^{(n_k)}$
13: projected stiffness matrix $\hat{K} = i(P^{(n_k)})^H GP^{(n_k)} + S + S^H$.
14: solve projected problem $\hat{K}Z = \hat{M}Z\hat{\Lambda}$
15: sort out positive eigenvalue $\hat{\Lambda}_+$ and corresponding eigenvectors Z_+
16: compute improved eigenvectors $V^{(n_k)} = \hat{Q}Z_+$.

However, this approach does not use the special structure of the matrices in problem (5) to its full capacity. If for some k the bases satisfy the following symmetry properties (cf. the initial bases (8))

$$P^{(k-1)} = [P_{k-1}, -\overline{P_{k-1}}] \quad \text{and} \quad Q^{(k-1)} = [Q_{k-1}, \overline{Q_{k-1}}] \tag{10}$$

then it is easily verified that

$$P^{(k)} = [P_k, \overline{P_k}] \quad \text{and} \quad Q^{(k)} = [Q_k, -\overline{Q_k}] \tag{11}$$

and

$$P^{(k+1)} = [P_{k+1}, -\overline{P_{k+1}}] \quad \text{and} \quad Q^{(k+1)} = [Q_{k+1}, \overline{Q_{k+1}}]. \tag{12}$$

Hence, the symmetry is preserved in every other step.

For the matrix $R^{(k-1)} := MP^{(k-1)} - iGQ^{(k-1)}$ in step 5 of Algorithm 1 we obtain

$$R^{(k-1)} = [R_{k-1}, -\overline{R_{k-1}}] \quad \text{and} \quad R^{(k)} = [R_k, \overline{R_k}]. \tag{13}$$

Hence, Algorithm 1 can be modified such that it iterates only on matrices $P_k, Q_k, R_k \in C^{n\times p}$ instead of $P^{(k)}, Q^{(k)}, R^{(k)} \in C^{n\times 2p}$, and uses the symmetry properties above to determine the projected problem. This essentially cuts the arithmetical cost into halves.

In Algorithm 2 we use $P^{(n_k)} = [P_{n_k}, \overline{P_{n_k}}]$ and $Q^{(k)} = [Q_{n_k}, -\overline{Q_{n_k}}]$ which corresponds to an odd number of iteration steps. If n_k is even, then the symmetry (11) is arrived after n_k steps for the initial matrices $P^{(0)} = [V\Lambda^{1/2}, V\Lambda^{1/2}]$ and $Q^{(0)} = [V, -V]$.

5 Numerical results

We consider a finite element model of a deformable wheel rolling on a rigid plane surface which is obtained by an Arbitrary Lagrangian Eulerian (ALE) formulation according to the derivation and presentation in [1]. Our model of a rotating tire consists of 39204 brick elements with 124992 degrees of freedom and accounts for 20 different material groups. The speed is assumed to be 60 km/h. Our aim is to determine approximations to the smallest 200 eigenvalues and corresponding eigenvectors.

The numerical tests were performed on a 64-bit HP workstation with an Intel Xeon CPU (3.20 GHz, 2 cores) and 24GB memory. AMLS and the two subspace iteration algorithms were implemented with Matlab R2009a.

The AMLS method addressing the linear eigenvalue problem $Kx = \lambda Mx$ costs 881.2 seconds for the AMLS projection and 270.4 seconds for solving the projected linear eigenvalue problem of dimension 2263 by `eig`.

Algorithm 2 Subspace iteration with AMLS utilizing symmetry.

Require: Diagonal matrix $\tilde{\Lambda}$ containing eigenvalue approximations from AMLS, transformed eigenvectors $\tilde{V}$, the transformed stiffness matrix $\tilde{K}$ the transformation matrix U from AMLS, and the maximum iteration number n_k

initialize the iteration matrices $\tilde{Q}_0 = \tilde{V}$ and $\tilde{P}_0 = \tilde{V}\tilde{\Lambda}^{1/2}$
transform backward $P_0 = U\tilde{P}_0$
for $k = 1, 2, \ldots, n_k$ **do**
    transform backward $Q_{k-1} = U\tilde{Q}_{k-1}$
    compute $R = MP_{k-1} - iGQ_{k-1}$
    transform forward $\tilde{R} = U^T R$
    $P_k = Q_{k-1}$
    solve for $\tilde{Q}_k$: $\tilde{K}\tilde{Q}_k = \tilde{R}$
end for
$T = [\tilde{R}, -\overline{\tilde{R}}]^H [\tilde{Q}_{n_k}, -\overline{\tilde{Q}_{n_k}}]$
projected mass matrix $\hat{M} = [P_{n_k}, \overline{P_{n_k}}]^H M [P_{n_k}, \overline{P_{n_k}}] + T$
reload R and compute $S = [R, -\overline{R}]^H [P_{n_k}, \overline{P_{n_k}}]$
projected stiffness matrix $\hat{K} = i[P_{n_k}, \overline{P_{n_k}}]^H G[Pn_k, \overline{P_{n_k}}] + S + S^H$.
solve projected problem $\hat{K}Z = \hat{M}Z\hat{\Lambda}$
sort out positive eigenvalue $\hat{\Lambda}_+$ and corresponding eigenvectors Z_+
compute improved eigenvectors $V^{(n_k)} = [Q_{n_k}, -\overline{Q_{n_k}}]Z_+$.

Following Bathe's recommendation to use $\min\{2p, p+8\}$ initial vectors in subspace iteration if p eigenpairs are wanted we initialized the subspace iteration methods with the lowest 208 positive eigenvalues and corresponding eigenvectors.

Executing the subspace iteration for problem (5) requires the solution of a linear system

$$KQ_k = MP_{k-1} - iGQ_{k-1}$$

for Q_k in every iteration step. Solving this problem directly without utilizing the block diagonal structure of $\tilde{K}$ needs 1235.8 seconds.

Table 1 shows the computing times for the combination of subspace iteration and AMLS with (Algo. 2) and without (Algo. 1) taking advantage of the special structure of problem (5). It demonstrates clearly the superiority of our approach upon the direct application of subspace iteration, and that the computing time is essentially reduced by Algorithm 2.

The relative errors of eigenvalues computed by both methods with four iteration steps are given in Fig.1.

To evaluate the accuracy of eigenpairs we use modal errors

$$\varepsilon_g = \frac{\|Kx + i\omega Gx - \omega^2 Mx\|}{\|\omega^2 Mx\|}. \qquad (14)$$

Fig.2 shows the reduction of modal errors for four iterations. It is interesting to note that essential improvements are obtained only every other iteration step.

TABLE 1. Computation time with 208 iteration vectors

Computational Steps	Algo. 2(s)	Algo. 1(s)
Initial eigenvectors from AMLS	2.4	2.4
1 Iteration	67.8	148.9
2 Iterations	141.3	304.9
3 Iterations	216.8	462.1
4 Iterations	289.4	621.8
Compute $\hat{K}$ and $\hat{M}$ if $n_k = 1$	6.2	9.7
Compute $\hat{K}$ and $\hat{M}$ if $n_k > 1$	8.7	19.6
Solve $\hat{K}Z = \hat{M}Z\hat{\Lambda}$ by `eig`	2.6	2.6
Compute final eigenvectors	25.6	25.6

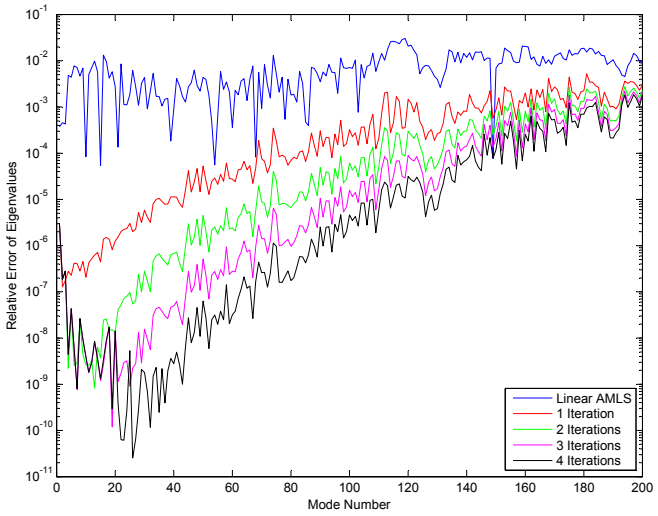

FIGURE 1. Relative errors of eigenvalues computed by subspace iteration with AMLS utilizing 208 iteration vectors

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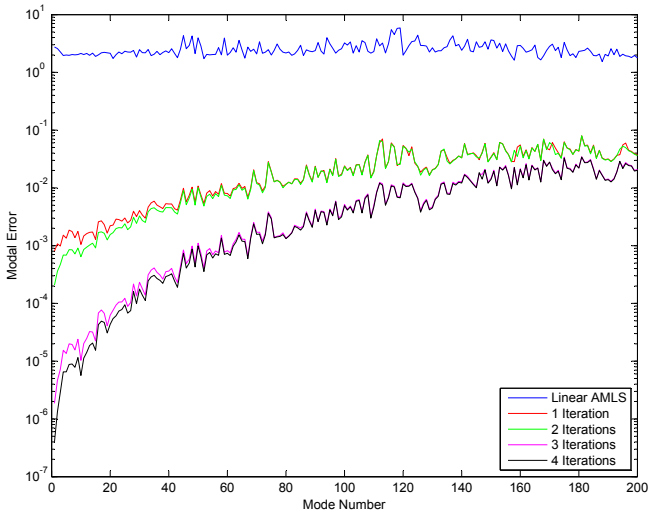

FIGURE 2. Modal errors computed by subspace iteration with AMLS utilizing 208 iteration vectors

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