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FREE VIBRATIONS OF FLUID–SOLID STRUCTURES WITH STRONG COUPLING

Heinrich Voss*
Institute of Numerical Simulation
Hamburg University of Technology
D-21073 Hamburg, Germany
Email: voss@tuhh.de

Markus Stammberger
Institute of Numerical Simulation
Hamburg University of Technology
D-21073 Hamburg, Germany
Email: markus.stammberger@tuhh.de

ABSTRACT

Free vibrations of fluid-solid structures are governed by unsymmetric eigenvalue problems. A common approach which works fine for weakly coupled systems is to project the problem to a space spanned by modes of the uncoupled system. For strongly coupled systems however the approximation properties are not satisfactory. This paper reports on a framework for taking advantage of the structure of the unsymmetric eigenvalue problem allowing for a variational characterization of its eigenvalues, and structure preserving iterative projection methods. We further cover an adjusted automated multi-level sub-structuring method for huge fluid-solid structures.

INTRODUCTION

In this paper we consider free vibrations of an elastic structure which is completely filled with a homogeneous, inviscid and compressible fluid (liquid or gas). Different formulations have been proposed to model this problem ([1–4], e.g.), the most obvious of which describes the structure by its relative displacement field and the fluid by its pressure [4, 5].

This approach yields the following eigenvalue problem:

$$\begin{aligned} \operatorname{div}[\sigma(u)] + \omega^2 \rho_s u &= 0 \text{ in } \Omega_s, \\ \Delta p + \tfrac{\omega^2}{c^2} p &= 0 \text{ in } \Omega_f, \\ \sigma(u)\cdot n - pn &= 0 \text{ on } \Gamma_I, \\ \nabla p \cdot n + \omega^2 \rho_f u \cdot n &= 0 \text{ on } \Gamma_I, \\ u &= 0 \text{ on } \Gamma_O, \end{aligned} \tag{1}$$

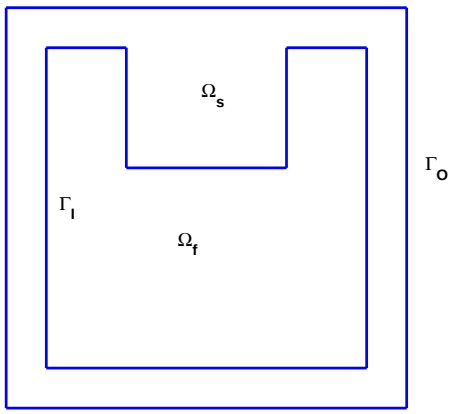

FIGURE 1. FLUID-SOLID STRUCTURE.

where Ω_s and Ω_f denotes the region occupied by the structure and the fluid, respectively, Γ_O the outer boundary of the structure, Γ_I the interface between the fluid and the structure, and n the outward pointing normal of Ω_s on Γ_I. u refers to the solid displacement, p to the fluid pressure, ω indicates the eigenfrequency, $\sigma(u)$ the linearized stress tensor, and ρ_s and ρ_f the density of the solid and the fluid, respectively.

Discretizing by finite elements yields the unsymmetric matrix eigenvalue problem [6, 7]

$$Kx := \begin{bmatrix} K_s & C \\ 0 & K_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} = \lambda \begin{bmatrix} M_s & 0 \\ -C^T & M_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} =: \lambda M x. \tag{2}$$

*Address all correspondence to this author.

Here x_s is the structure displacement vector with s degrees of freedom, x_f is the fluid pressure vector with f degrees of freedom, and $\lambda = \omega^2$ denotes the eigenvalue. $K_s \in \Re^{s\times s}$ and $K_f \in \Re^{f\times f}$ are the stiffness matrices, and $M_s \in \Re^{s\times s}$ and $M_f \in \Re^{f\times f}$ are the mass matrices of the structure and the fluid, respectively, and $C \in \Re^{s\times f}$ describes the coupling of structure and fluid. Usually the order of the dimensions s and f is 10^5 or even larger, and the dynamic analysis of the structure requires hundreds or thousands of eigenmodes.

The computational solution of Problem (2) with standard sparse eigensolvers involves considerable complications. As a remedy some authors prefer an alternative modeling which additionally involves the fluid displacement potential. Then the resulting system is symmetric and efficient methods such as the shift-and-invert Lanczos method applies (e.g. [8]). As a drawback, however, the dimension of the problem increases considerably.

A common approach for solving problem (2) [6, 9] (for example in automotive industry) which works fine for weakly coupled systems is as follows: One first determines the eigenpairs of the symmetric and definite eigenvalue problems

$$K_s x_s = \omega_s M_s x_s \quad \text{and} \quad K_f x_f = \omega_f M_f x_f \tag{3}$$

by the Lanczos method or automated multi-level sub-structuring, and then projects problem (2) to $\mathrm{diag}\{X_s, X_f\}$, where the columns of X_s and X_f are the eigenmodes of problem (3) not exceeding a given cut-off level. The projected problem

$$\begin{bmatrix} X_s^T K_s X_s & X_s^T C X_f \\ 0 & X_f^T K_f X_f \end{bmatrix} \begin{bmatrix} y_s \\ y_f \end{bmatrix} = \lambda \begin{bmatrix} X_s^T M_s X_s & 0 \\ -X_f^T C^T X_s & X_f^T M_f X_f \end{bmatrix} \begin{bmatrix} y_s \\ y_f \end{bmatrix} \tag{4}$$

has the same structure as the original problem but is of much smaller dimension.

The following example demonstrates that this approach is not appropriate for strongly coupled problems: We first consider a two dimensional coupled structure (the geometry is shown in Fig. 2) consisting of steel and air portions discretized by finite elements. The resulting problem has 120473 degrees of freedom, 67616 of which are located in the solid region and 52857 in the fluid part. Tab. 1 contains in its first column the 10 smallest eigenfrequencies of the coupled problem, and its second and third column the eigenfrequencies of $K_s x_s = \omega_s M_s x_s$ and $K_f x_f = \omega_f M_f x_f$, respectively, demonstrating that each eigenfrequency of the coupled system can be assigned in an obvious way to an eigenfrequency of one of the structures portions. Hence, the coupling has only marginal influence. Column 4 shows the relative deviations in % of the eigenfrequencies of the coupled and the uncoupled system. Projecting the coupled problem to the space spanned by the eigenmodes of the individual problems in Eqn. (3) the eigenfrequencies of which do not exceed 1000 Hz, the relative errors in % are even reduced to less than 0.04% (cf. column 5 of Tab. 1).

TABLE 1. STEEL-AIR-STRUCTURE

coupled	steel	air	rel.dev.	proj.
0.00		0.00		
41.25		41.32	0.16	2.5e-4
48.67		48.71	0.08	2.7e-4
56.96	56.90		0.11	2.2e-3
75.55	75.51		0.06	3.3e-3
93.18		93.19	0.01	1.0e-4
129.99		130.04	0.05	6.1e-4
150.94	151.03		0.06	3.5e-3
158.16		158.18	0.01	1.8e-4
186.64	186.66		0.12	4.2e-3

TABLE 2. STEEL-WATER-STRUCTURE

coupled	steel	water	proj.	rel.error
0.00	56.90	0.00	0.00	
28.01	75.51	178.63	28.33	1.2
41.54	151.03	210.64	43.01	3.5
92.73	186.66	402.93	101.98	10.0
124.70	225.54		133.60	7.1
138.26	451.76		141.87	2.6
270.40	472.45		285.18	5.5
321.79			343.80	6.8
388.73			416.87	7.2
402.77			439.83	9.2

If the fluid air is replaced by water, the scene changes completely. Tab. 2 contains in its first column the 10 smallest eigenvalues of the coupled structure, and in columns 2 and 3 the smallest eigenvalues of the the steel and the water portion, respectively. Again we projected the coupled problem to the space spanned by the eigenmodes of the problems in Eqn. (3) the eigenfrequencies of which do not exceed 1000 Hz. The 10 smallest eigenfrequencies of the projected problem are shown in column 4, and its relative errors in % in column 5 demonstrating that the approximations to the correct eigenvalues are not satisfactory.

In the following we present a variational characterization of the eigenvalues of the unsymmetric problem (2) demonstrating that it has similar properties as symmetric and definite eigenvalue problems. This allows for designing structure preserving numerical methods for Eqn. (2) such that the coupling is accounted for in an appropriate way.

VARIATIONAL CHARACTERIZATION OF EIGENVALUES

It is important that problem (2) can be symmetrized. With

$$T := \begin{bmatrix} M_s^{-1}K_s & M_s^{-1}C \\ 0 & I \end{bmatrix}.$$

it holds that

$$T^T K = \begin{bmatrix} K_s M_s^{-1} K_s & K_s M_s^{-1} C \\ C^T M_s^{-1} K_s & K_f + C^T M_s^{-1} C \end{bmatrix} \quad \text{and} \quad T^T M = \begin{bmatrix} K_s & 0 \\ 0 & M_f \end{bmatrix},$$

and Eqn. (2) is equivalent to the symmetric and definite eigenvalue problem

$$T^T K x = \lambda T^T M x. \tag{5}$$

This result yields at once the following properties of problem (2) a part of which were proved directly in [10].

(i) All eigenvalues of the fluid-solid eigenvalue problem (2) are real.

(ii) Right eigenvectors of problem (2) can be chosen orthonormal with respect to $\tilde{M} := \mathrm{diag}\{K_s, M_f\}$ and left eigenvectors can be chosen orthonormal with respect to $\bar{M} := \mathrm{diag}\{M_s, K_f\}$,

(iii) If $x := [x_s^T, x_f^T]^T$ is a right eigenvector of Eqn. (2) corresponding to the eigenvalue λ, then $\tilde{x} := [\lambda x_s^T, x_f^T]^T$ is a left eigenvector corresponding also to λ.

As a further consequence of the equivalence of Eqn. (2) to the symmetric and definite problem (5) its eigenvalues can be characterized by Rayleigh's principle and the minmax characterization due to Poincaré, Fischer and Courant, and in principle can be determined by the shift-and invert Lanczos algorithm. However, this symmetrization suffers the disadvantage that one has to solve two linear systems with system matrix M_s to evaluate the Rayleigh quotient of Eqn. (5) or to apply one Lanczos step, which is very costly since the dimension s of M_s is usually very large.

In [11] we proved variational principles for the eigenvalues of Eqn. (2) which takes advantage of the relation between right and left eigenvectors, and which does not require the solution of large linear systems .

Let $[x_s^T, x_f^T]^T$ be a right eigenvector of problem (2) corresponding to the eigenvalue λ, and $[\lambda x_s^T, x_f^T]^T$ be a left eigenvector. Then it holds that

$$\lambda = \frac{[\lambda x_s^T, x_f^T] \begin{bmatrix} K_s & C \\ 0 & K_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix}}{[\lambda x_s^T, x_f^T] \begin{bmatrix} M_s & 0 \\ -C^T & M_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix}}$$

$$= \frac{\lambda x_s^T K_s x_s + \lambda x_s^T C x_f + x_f^T K_f x_f}{\lambda x_s^T M_s x_s - x_f^T C^T x_s + x_f^T M_f x_f}.$$

This equation suggests to define a Rayleigh functional for a general vector $[x_s^T, x_f^T]^T \in \Re^{s+f}$ by the requirement

$$p(x_s, x_f) = \frac{p(x_s, x_f) x_s^T K_s x_s + p(x_s, x_f) x_s^T C x_f + x_f^T K_f x_f}{p(x_s, x_f) x_s^T M_s x_s - x_f^T C^T x_s + x_f^T M_f x_f} \tag{6}$$

which is equivalent to the quadratic equation

$$p(x_s, x_f)^2 x_s^T M_s x_s + p(x_s, x_f)(x_f^T M_f x_f - x_s^T K_s x_s - 2 x_s^T C x_f) - x_f^T K_f x_f = 0. \tag{7}$$

The smaller root of Eqn. (7) is negative, and hence physically meaningless. We therefore choose the unique positive root of this equation as Rayleigh functional.

Definition

$$p(x_s, x_f) := \begin{cases} q(x_s, x_f) + \sqrt{q(x_s, x_f)^2 + \frac{x_f^T K_f x_f}{x_s^T M_s x_s}} & \text{if } x_s \neq 0 \\ \frac{x_f^T K_f x_f}{x_f^T M_f x_f} & \text{if } x_s = 0 \end{cases} \tag{8}$$

with

$$q(x_s, x_f) := \frac{x_s^T K_s x_s - x_f^T M_f x_f + 2 x_s^T C x_f}{2 x_s^T M_s x_s} \tag{9}$$

is called Rayleigh functional of the fluid-solid vibration eigenvalue problem (2).

The Rayleigh functional shares most of the properties of the Rayleigh quotient of a symmetric and definite eigenvalue problem.

Differentiating the defining Eqn. (7) of the Rayleigh functional with respect to x_s and x_s yields that every right eigenvector $(x_s^T, x_f^T)^T$ of problem (2) is a stationary point of the Rayleigh functional, i.e.

$$\nabla p(x_s, x_f) = 0. \tag{10}$$

Moreover, the well known variational characterizations of eigenvalues can be generalized to the unsymmetric eigenproblem (2). The following theorem generalizing Rayleigh's principle and the minmax characterization of Poincaré was proved in [11].

Theorem 1
Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_{s+f}$ be the eigenvalues of problem (2), and let $x_1, x_2, \ldots$ denote corresponding right eigenvectors. Then it holds that

(i) (Rayleigh's principle)

$$\begin{aligned}\lambda_k &= \min\{p(x) : x^T \tilde{M} x_j = 0,\ j = 1,\ldots,k-1\} \\ &= \max\{p(x) : x^T \tilde{M} x_j = 0,\ j = k+1,\ldots,s+f\}.\end{aligned} \tag{11}$$

(ii) (minmax characterization of eigenvalues)

$$\begin{aligned}\lambda_k &= \min_{V \in S_k} \max_{0 \neq x \in V} p(x) \\ &= \max_{V \in S_{s+f+1-k}} \min_{0 \neq x \in V} p(x),\end{aligned} \tag{12}$$

where S_k denotes the set of all k dimensional subspaces of $\Re^{s+f}$.

Once we have the minmax characterization of eigenvalues monotonicity results for orthogonal projection methods follow immediately. It is important however that the projection method preserves the structure of the eigenvalue problem.

Theorem 2
Assume that $V = \begin{bmatrix} V_s & 0 \\ 0 & V_f \end{bmatrix} \in \Re^{s+f \times k}$ has rank k. Let

$$\begin{aligned}K_V &:= V^T K V = \begin{bmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{bmatrix}, \\ M_V &:= V^T M V = \begin{bmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{bmatrix}\end{aligned}$$

be the orthogonal projections of K and M, and let $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \ldots \le \tilde{\lambda}_k$ be the eigenvalues of the projected eigenvalue problem

$$K_V y = \tilde{\lambda} M_V y. \tag{13}$$

Then it holds that

$$\lambda_j \le \tilde{\lambda}_j, \quad j = 1,2,\ldots,k. \tag{14}$$

Proof Let

$$z = \begin{bmatrix} z_s \\ z_f \end{bmatrix} \quad \text{and} \quad x := Vz = \begin{bmatrix} V_s z_s \\ V_f z_f \end{bmatrix}.$$

Then it is obvious that $\tilde{p}(z) = p(x)$ where $\tilde{p}$ denotes the Rayleigh functional of the projected problem (13). Hence, for $j = 1,\ldots,k$ it holds that

$$\begin{aligned}\lambda_j &= \min_{\dim W = j, W \subset \Re^{s+f}} \max_{x \in W, x \neq 0} p(x) \\ &\le \min_{\dim Z = j, Z \subset \Re^k} \max_{x \in VZ, x \neq 0} p(x) \\ &= \min_{\dim Z = j, Z \subset \Re^k} \max_{z \in Z, z \neq 0} p(Vz) \\ &= \min_{\dim Z = j, Z \subset \Re^k} \max_{z \in Z, z \neq 0} \tilde{p}(z) \\ &= \tilde{\lambda}_j.\end{aligned}$$

Rewriting the infinite dimensional eigenvalue problem (1) in its variational form in appropriate Sobolev spaces a Rayleigh functional can be defined in an analogous way as in (8), and Rayleigh's principle and the minmax characterization holds in this cases, too (cf. Stammberger [12]). Hence, if the discretization (2) of problem (1) is defined by a Galerkin type projection method (finite element method, e.g.), then the eigenvalues of the discrete problems are upper bounds to the corresponding eigenvalues of the infinite dimensional problem (1).

For problem (1) one can additionally prove a maxmin principal generalizing the variational characterization due to Courant and Fischer, which is in the finite dimensional case equivalent to Poincaré's principle.

STRUCTURE PRESERVING NUMERICAL METHODS

For symmetric eigenvalue problems the Rayleigh quotient iteration converges cubically to simple eigenvalues. For unsymmetric problems the convergence is only quadratic. However, if the Rayleigh quotient is replaced by the Rayleigh functional this variant can also be shown to be cubically convergent (cf. [11]).

Algorithm 1 Rayleigh functional iteration
Require Initial vector x^0
for $k = 1,2,\ldots$ until convergence **do**
  evaluate Rayleigh functional $\rho = p(x^k)$
  solve $(K - \rho M)x^{k+1} = Mx^k$ for x^{k+1}
  normalize x^{k+1}
end for

Rayleigh functional iteration converges fast, but often it is highly sensitive with respect to initial vectors. The basin of attraction can be very small, and an erratic behavior of the iteration can be observed. To avoid the possible failure of the Rayleigh functional iteration one combines it with an iterative projection

method which have proven to be very efficient if a small number of eigenvalues and eigenvectors are desired.

An iterative projection method works as follows. It starts with a subspace to which the given problem is projected. The dimension of projected eigenproblem is usually very small and therefore it can be solved by a standard eigensolver and an approximation of a wanted eigenvalue and a corresponding eigenvector can be extracted. This eigenpair is accepted if it meets specified accuracy requirements, otherwise the space is expanded by a new direction. Methods of this type are Krylov subspace methods like the Lanczos method, the Arnoldi method, or rational Krylov methods, and Davidson or Jacobi-Davidson type methods (cf. [13]). The idea of iterative projection methods is that search spaces are able to capture the essential structure of the original problem if the expanding directions are chosen appropriately, and then the dimension of the search space can be kept small.

If $V := \begin{bmatrix} V_s \\ V_f \end{bmatrix} \in \Re^{s+f\times k}$ is a basis of a search space $\mathscr{V}$ the projection of problem (2) reads

$$\begin{aligned}&(V_s^T K_s V_s + V_s^T C V_f + V_f^T K_f V_f)u \\ &= \lambda (V_s^T M_s V_s - V_f^T C V_s + V_f^T M_f V_f)u.\end{aligned}$$

Hence, the structure of problem (2) gets lost, and it is not certain that eigenvalues of the projected problem stay real. This suggests to use a structure preserving projection method $V = \begin{bmatrix} V_s & 0 \\ 0 & V_f \end{bmatrix}$, i.e. to project the mass part and the fluid part of the problem individually to search spaces. Then the projected problem receives the form (13), and the eigenvalues of the projected problem stay real. From Theorem 2 it follows that the eigenvalues of the projected problem are upper bounds of the eigenvalues of the original problem (2), and expanding the search space all eigenvalues decrease and get better approximations to the eigenvalues of problem (2).

An expansion with high approximation potential is given by the Rayleigh functional iteration, i.e. if $V \in \Re^{s+j\times k}$ is a (orthonormal) basis of the current search space, and (θ, u), is an eigenpair of the projected problem

$$V^T(K-\theta M)Vu = 0 \tag{15}$$

a reasonable expansion of the search space $\mathscr{V} = \operatorname{span}\{V\}$ is the solution v of the linear system

$$(K-\theta M)v = Mx \tag{16}$$

where $x = Vu$ denotes the Ritz vector corresponding to (θ, u).

In the course of the algorithm the approximation θ to an eigenvalue λ changes in every step, and therefore large-scale linear systems with varying system matrices have to be solved in consecutive iteration steps which is much to costly for truly large problems. We therefore keep θ fixed as long as possible, and to increase the robustness of the method (cf. [14,15]) we replace the direction of Rayleigh functional iteration $t_{Ri} = (K-\theta M)^{-1}Mx$ by the Cayley transformation

$$t_{Ct} = (K-\sigma M)^{-1}(K-\theta M)x \tag{17}$$

where θ is the current approximation of the desired eigenvalue and σ is a fixed parameter close to θ.

Notice that for t_{Ri} it holds that

$$(\theta-\sigma)t_{Ri} = x - (K-\sigma M)^{-1}(K-\theta M)x,$$

and since the current Ritz vector x is already contained in $\mathscr{V}$ the expansions by t_{Ri} and t_{Ct} are equivalent.

We obtain the following algorithm where we even replace $(K-\sigma M)^{-1}$ by a preconditioner since for truly large problems it is too costly to solve the linear system $(K-\sigma M)t_{Ri} = (K-\theta M)x$. This method was introduced in [16] for nonlinear eigenvalue problems and since it reduces for general linear eigenvalue problems to the inexact shift-and-invert Arnoldi method it was called nonlinear Arnoldi method.

Algorithm 2 Structure preserving nonlinear Arnoldi method

Require Initial basis $V = \begin{bmatrix} V_s & O \\ O & V_f \end{bmatrix}$, $V_s^T V_s = I$, $V_f^T V_f = I$;
$m = 1$; θ_m=0, σ close to first wanted eigenvalue
determine preconditioner $L \approx (K-\sigma M)^{-1}$,
while $\theta_m \leq$ maxeig
solve the projected eigenproblem

$$\begin{bmatrix} V_f^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{bmatrix} \begin{bmatrix} u_s \\ u_f \end{bmatrix} = \theta \begin{bmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_f & V_f^T M_f V_f \end{bmatrix} \begin{bmatrix} u_s \\ u_f \end{bmatrix} \tag{18}$$

choose m smallest eigval. θ_m and corresp. eigvec. $[u_s^T, u_f^T]^T$
determine Ritz vector $x = \begin{bmatrix} V_s u_s \\ V_f u_f \end{bmatrix}$ and $r = (K-\theta_m M)x$
if $\|r\|/\|x\| < \varepsilon$
while $\|r\|/\|x\| < \varepsilon$
accept approximate mth eigenpair (θ_m, x);
increase $m \leftarrow m+1$;
choose m smallest eigval θ_m and eigvec $(u_m^T, u_f^T)^T$
determine Ritz vector $x = \begin{bmatrix} V_m y_m \\ V_p u_m \end{bmatrix}$ and $r = (K-\theta_m M)x$
end while

reduce search space V if indicated
determine new preconditioner $L \approx (K-\theta M)^{-1}$ if necessary
end if
solve $Lt = r$ for $t = (t_s^T, t_f^T)^T$
orthogonalize $v_s = t_s - V_s V_s^T t_s$, $v_f = t_f - V_f V_f^T t_f$
if $\|v_s\| >$ tol expand $V_s \leftarrow [V_s, v_s/\|v_s\|]$ **end if**
if $\|v_f\| >$ tol expand $V_f \leftarrow [V_f, v_f/\|v_f\|]$ **end if**
update projected problem (18)
end while

Some comments are in order:

(i) Since the dimension of the projected eigenproblem is quite small it is solved by a dense solver and therefore approximations to further eigenpairs are at hand without additional cost.

(ii) In the inner while clause we check whether approximations to further eigenpairs already satisfy the specified error tolerance. Moreover, at the end of the while-loop an approximation to the next eigenpair to compute and the residual r is provided.

(iii) If the dimension of the search space has become too large we reduce the matrices V_s and V_f such that the columns of V_s and V_f form an orthonormal basis of the space spanned by the mass and the fluid part of the eigenvectors found so far. Notice, that the search space is reduced only after an eigenpair has converged because the reduction spoils too much information and the convergence can be retarded.

(iv) The preconditioner L is updated if the convergence measured by the reduction of the residual norm $\|r\|$ has become too slow.

A different approach for expanding the search spaces which is also based on the Rayleigh functional iteration was considered in [11]. At least close to an eigenpair the expansion $v = (K-\theta M)^{-1}Mx$ is very sensitive to inexact solves of the linear system (16).

In [15] it was shown that the most robust expansion of $\mathcal{V}$ which contains the direction v of Rayleigh functional iteration is $t := x + \alpha v$ where α is chosen such that $x^T t = 0$, i.e.

$$t = x - \frac{x^T x}{x^T (K-\theta M)^{-1} M x} (K-\theta M)^{-1} M x \tag{19}$$

It is easily seen that t solves the correction equations

$$\Big(I - \frac{Mxx^T}{x^T Mx}\Big)(K-\theta M)\Big(I - \frac{xx^T}{x^T x}\Big)t = (K-\theta M)x,\ x^T t = 0 \tag{20}$$

which demonstrates that the resulting iterative projection method is a Jacobi-Davidson type method [17].

The linear system (19) which looks even more complicated than system (16) can not be solved by a direct solver, but has to be tackled by an iterative method like the preconditioned GMRES method. A natural choice for a preconditioner is

$$P = \Big(I - \frac{Mxx^T}{x^T Mx}\Big) L \Big(I - \frac{xx^T}{x^T x}\Big) \tag{21}$$

where L is a reasonable preconditioner for $K - \theta M$. It was observed already by Sleijpen and van der Vorst [18] that including the projectors $I - Mxx^T/(x^T Mx)$ and $I - xx^T/(x^T x)$ into the preconditioner does not increase the cost of preconditioned GMRES very much. Only one additional application of the preconditioner is necessary to initialize the iteration.

AMLS FOR FLUID-SOLID VIBRATIONS

Over the last few years, a new method for huge linear eigenvalue problems

$$Kx = \lambda Mx \tag{22}$$

where $K \in \Re^{n\times n}$ and $M \in \Re^{n\times n}$ are Hermitian and positive definite, known as *Automated Multi–Level Sub-structuring (AMLS)*, has been developed by Bennighof and co-authors, and has been applied to frequency response analysis of complex structures [19–21]. Here the large finite element model is recursively divided into very many sub-structures on several levels based on the sparsity structure of the system matrices. Assuming that the interior degrees of freedom of sub-structures depend quasistatically on the interface degrees of freedom, and modeling the deviation from quasistatic dependence in terms of a small number of selected sub-structure eigenmodes the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest. Recent studies ([9], e.g.) in vibro-acoustic analysis of passenger car bodies, where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed, have shown that for this type of problems AMLS is considerably faster than Lanczos type approaches.

We first consider the component mode synthesis method (CMS method) which is the essential building block of the AMLS method. Assume that the graph of the matrix $|K| + |M|$ is partitioned into sub-structures. We distinguish only between local (i.e. interior) and interface degrees of freedom. Then K and M (after reordering) have the following form:

$$K = \begin{bmatrix} K_{\ell\ell} & K_{\ell i} \\ K_{i\ell} & K_{ii} \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} M_{\ell\ell} & M_{\ell i} \\ M_{i\ell} & M_{ii} \end{bmatrix} \tag{23}$$

where $K_{\ell\ell}$ and $M_{\ell\ell}$ are block diagonal.

Annihilating $K_{\ell i}$ by block Gaussian elimination and transforming the local coordinates to modal degrees of freedom one obtains the equivalent pencil

$$(P^T K P, P^T M P) = \left(\begin{bmatrix} \Phi & 0 \\ 0 & \tilde{K}_{ii} \end{bmatrix}, \begin{bmatrix} I & \tilde{M}_{\ell i} \\ \tilde{M}_{i\ell} & \tilde{M}_{ii} \end{bmatrix} \right) \tag{24}$$

with

$$P = \begin{bmatrix} \Phi & -K_{\ell\ell}^{-1} K_{\ell i} \\ 0 & I \end{bmatrix}.$$

Here Ω is a diagonal matrix containing the sub-structure eigenvalues, i.e. $K_{\ell\ell}\Phi = M_{\ell\ell}\Phi\Omega$, $\Phi^T M_{\ell\ell}\Phi = I$, and Φ contains in its columns the corresponding eigenvectors. Notice that $K_{\ell\ell}$ and $M_{\ell\ell}$ are block diagonal, and therefore it is quite inexpensive to eliminate $K_{\ell i}$ and to solve the interior eigenproblems.

In structural dynamics (24) is called Craig–Bampton form of the eigenvalue problem (22) corresponding to the partitioning (23).

Selecting some eigenmodes of problem (24), usually the ones associated with eigenvalues below a cut off threshold, and dropping the rows and columns in (24) corresponding to the other modes one arrives at the component mode synthesis method (CMS) introduced by Hurty [22] and Craig and Bampton [23]. Hence, if the diagonal matrix Ω contains on its diagonal the eigenvalues to keep and Φ the corresponding eigenvectors, then the CMS approximations to the eigenpairs of (22) are obtained from the reduced eigenvalue problem

$$\begin{bmatrix} \Omega & 0 \\ 0 & \tilde{K}_{ii} \end{bmatrix} y = \lambda \begin{bmatrix} I & \tilde{M}_{\ell i} \\ \tilde{M}_{i\ell} & \tilde{M}_{ii} \end{bmatrix} y \tag{25}$$

AMLS generalizes CMS in the following way. Again the graph of $|K| + |M|$ is partitioned into a small number of subgraphs, but more generally than in CMS these subgraphs in turn are sub-structured on a number p of levels. This induces the following partitioning of the index set $I = \{1, \ldots, n\}$ of degrees of freedom. I_1 is the set of indices corresponding to interface degrees of freedom on the coarsest level, and for $j = 2, \ldots, p$ define I_j to be the set of indices of interface degrees of freedom on the j-th level which are not contained in I_{j-1}. Finally, let I_{p+1} be the set of interior degrees of freedom on the finest level.

With these notations the first step of AMLS is CMS with cut-off frequency γ applied to the finest sub-structuring. After j steps, $1 \le j \le p-1$, one derives a reduced pencil

$$\left(\begin{bmatrix} \Omega_p & O & O \\ O & K_{\ell\ell}^{(j)} & K_{\ell i}^{(j)} \\ O & K_{i\ell}^{(j)} & K_{ii}^{(j)} \end{bmatrix}, \begin{bmatrix} M_{pp}^{(j)} & M_{p\ell}^{(j)} & M_{pi}^{(j)} \\ M_{\ell p}^{(j)} & M_{\ell\ell}^{(j)} & M_{\ell i}^{(j)} \\ M_{ip}^{(j)} & M_{i\ell}^{(j)} & M_{ii}^{(j)} \end{bmatrix} \right). \tag{26}$$

where p denotes the degrees of freedom obtained in the spectral reduction in the previous steps, ℓ collects the indices in I_{p+1-j}, and i corresponds to the index set $\cup_{k=1}^{p-j} I_k$ of interface degrees of freedom on levels which are not yet treated. Applying the CMS method to the south–east 2×2 blocks of the matrices, i.e. annihilating the off–diagonal block $K_{\ell i}^{(j)}$ by block Gaussian elimination, and reducing the set of ℓ–indices by spectral truncation with cut-off frequency γ one arrives at the next level. After p CMS steps and a final spectral truncation of the lower–right blocks one obtains the reduction of problem (22) by AMLS.

We already mentioned in the Introduction that in industrial praxis the unsymmetric eigenvalue problem (2) is often solved in the following way. Neglecting the coupling between the fluid and the structure one first applies an efficient solver like the shift-and-invert Lanczos method or AMLS to the symmetric eigenvalue problems (3). Thereafter the original problem is projected to a space spanned by the eigenmodes corresponding to eigenfrequencies not exceeding a predetermined cut-off frequency. We already demonstrated by an example that this approach yields unsatisfactory results in case of strong coupling.

In a recent paper [12, 24] we took advantage of the fact that the spectrum of the eigenvalue problem

$$\left[\begin{pmatrix} 0 & C & K_s & 0 \\ C^T & 0 & 0 & K_f \\ K_s & 0 & 0 & 0 \\ 0 & K_f & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} M_s & 0 & 0 & 0 \\ 0 & M_f & 0 & 0 \\ 0 & 0 & K_s & 0 \\ 0 & 0 & 0 & K_f \end{pmatrix} \right] x = 0 \tag{27}$$

is symmetric with respect to 0, that the positive eigenvalues are the roots of the eigenvalues of problem (2), and that the eigenvectors of problem (2) can be reconstructed from the ones of problem (26). The AMLS method for problem (27) of dimension $2(s+f)$ can be organized such that it raises almost the same cost as AMLS for a linear problem of dimension $s+f$. Truncating the eigenmodes the eigenfrequencies of which exceed a cut-off frequency in modulus on all levels the method can be interpreted as a shifted AMLS method with shift 0, and it is therefore not surprising that the eigenvalues close to 0 are particularly well approximated.

NUMERICAL EXPERIMENTS

To evaluate the discussed methods for fluid–solid vibrations we consider the finite element discretization of a two-dimensional model with 120473 degrees of freedom, 67616 of which are located in the solid part and 52857 in the fluid part. The solid is steel and the fluid is water. To demonstrate the coupling effects, the underlying geometry was chosen with a rather large interface between fluid and solid, cf. Fig. 2. This is the model that we already considered in the introduction.

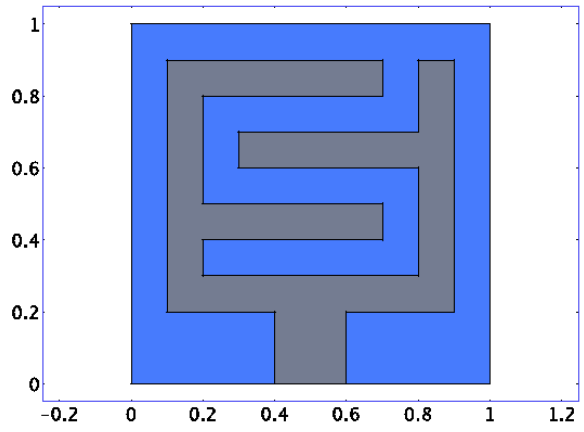

FIGURE 2. GEOMETRY OF THE EXAMPLE.

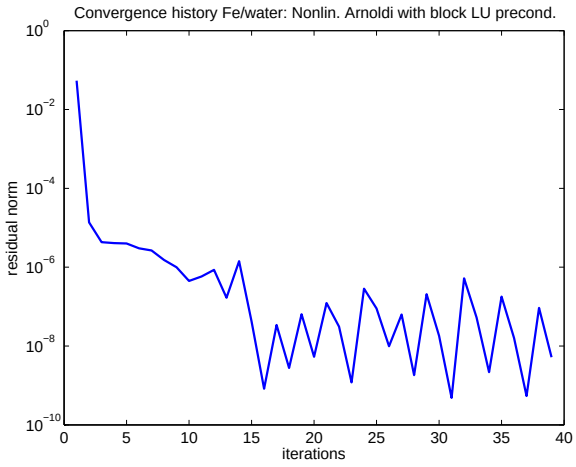

FIGURE 3. CONV. HISTORY: NONLINEAR ARNOLDI.

Tab. 2 compares the smallest eigenfrequencies of the coupled system, and the results obtained if the coupling is neglected, demonstrating that in case of strong coupling it is indispensable to take the coupling into account in the numerical method.

We determined all eigenvalues less than 500 Hz by the iterative projection methods in Section 3. Although we chose a (fixed) random initial vector such that the initial Rayleigh functional was in the middle of the spectrum the nonlinear Arnoldi method and the Jacobi–Davidson method determined the 10 wanted eigenvalues safely in all experiments demonstrating the robustness of both methods.

Using a very accurate preconditioner, namely the block matrix $\mathrm{diag}\{C_s C_s^T, C_f C_f^T\}$, where C_s and C_f denotes the Cholesky factor of K_s and K_f, respectively, the Arnoldi method turned out to be faster than the Jacobi–Davidson method. On a Pentium D processor with 3.4 GHz and 4 GB RAM under MATLAB 2009

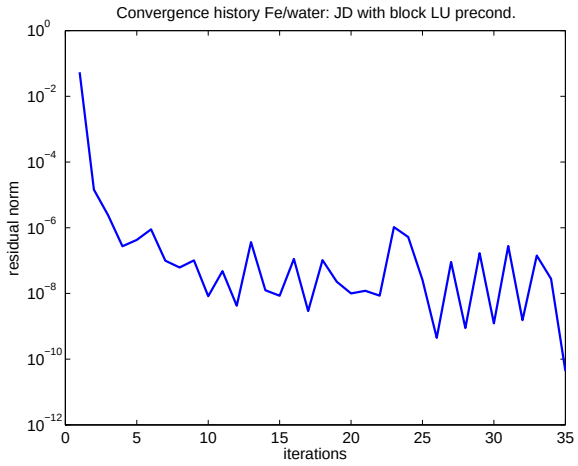

FIGURE 4. CONV. HISTORY: JACOBI-DAVIDSON.

the nonlinear Arnoldi method required 10.4 seconds, whereas the Jacobi–Davidson method needed 30.0 seconds. The convergence history is contained in Fig. 3 for the Arnoldi method and in Fig. 4 for the Jacobi-Davidson method. These figures demonstrate that both methods required quite a large number of iterations to determine the smallest eigenvalue (namely, 15 iterations for the nonlinear Arnoldi, and 10 iterations for the Jacobi–Davidson method), and then each of the following eigenvalues was found after 2 or 3 iterations. Notice that every step of the Jacobi–Davidson method requires the approximate solution of a linear system whereas in the nonlinear Arnoldi method one only has to apply the preconditioner ones. Hence, the CPU time needed by the Arnoldi method is much smaller although it requires a larger number of iteration steps than the Jacobi–Davidson method.

Replacing the block Cholesky factorization by an incomplete LU decomposition the CPU time needed by the Jacobi–Davidson method increases to 60.5 seconds if the cut-off threshold is chosen to be 10^{-3} and 113.4 seconds for 10^{-2} whereas the nonlinear Arnoldi method requires 125.6 and 3411.1 seconds, respectively. This behavior was already observed for nonlinear eigenvalue problems. If an accurate preconditioner is available, then the nonlinear Arnoldi method is usually much faster than the Jacobi–Davidson method. However, the Jacobi–Davidson method is much more robust with regard to coarse preconditioners.

To evaluate the structure preserving automated multi-level sub-structuring method We applied the AMLS variant described in Section 4 to the coupled fluid-solid problem and compared the relative errors of the eigenvalue approximations to those obtained from the standard AMLS procedure. In both cases, the algorithm was performed on 10 sub-structuring levels and 751 structures using a cut-off frequency corresponding to 10000 Hz on each partitioning level.

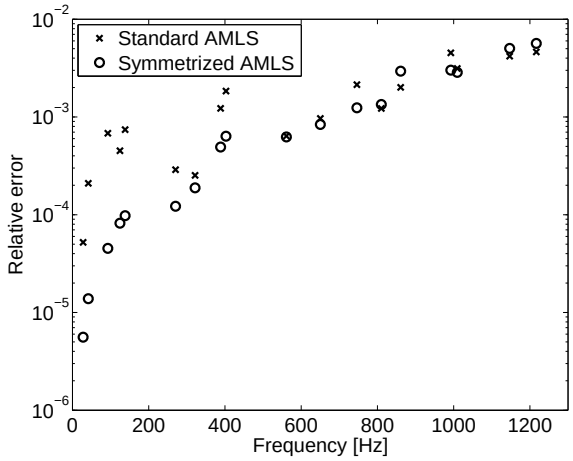

FIGURE 5. RELATIVE ERRORS FOR AMLS METHOD.

Eigenvalues with large accuracy improvements (e.g. $\lambda \approx 100$ Hz) turned out to belong to eigenforms with significant influence of the coupling. Eigenforms corresponding to larger eigenfrequencies were less influenced by the coupling and in some cases, the eigenvalue approximations are slightly worse compared with the AMLS variant neglecting the coupling effects in the reduction process. In all cases, the eigenvalue approximations were of larger magnitude than the exact eigenvalues.

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